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VOL. II.

Containing the Institutes, or Principles of

I. The *Physico-Mechanical* MATHESIS.
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III. Universal PERSPECTIVE.
IV. LOXODROMICS, or NAVIGATION.
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By BENJAMIN MARTIN.

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MDCCLXIV.

TO THE

KING,

This NEW and COMPREHENSIVE

SYSTEM

OF

MATHEMATICAL INSTITUTIONS,

Publiſhed,

By his ROYAL PERMISSION,

Under His Moſt Gracious and Auſpicious

PATRONAGE,

Is now, with all Humility,

Inſcribed,

By his MAJESTY'*s*

Moſt Loyal,

Dutiful, and Obedient

Subject and Servant,

BENJ. MARTIN.

INSTITUTIONS OF THE PHYSICO-MECHANICAL MATHESIS.

CHAP. I.

The MENSURATION *of* SUPERFICIES.

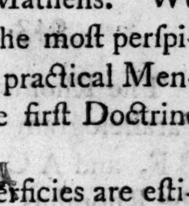
938. BEFORE we can give any rational Account of the *philosophical and mathematical Sciences*, and their practical Application, it is necessary to premise those *mechanical Principles* which are founded in the Nature of Things, and make the Basis of all that Sort of Knowledge; without these no Man can understand Philosophy, or make any Progress in the true Mathesis. We shall propose them in a natural Order, and in the most perspicuous Method we can, and begin first with the practical Mensuration of Superficies and Solids, which is the first Doctrine of practical Science.

939. The Dimensions or Content of all Superficies are estimated in that of a *Square*; as a *square Inch*; a *square Foot*; a *square Yard*, *Rod*, *Mile*, &c. and what Number of each lesser Denomination is contained the greater, is shewn in the following Table:

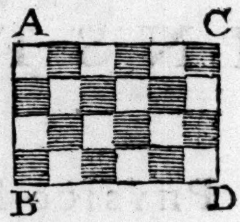
Square Inches.						
144 =	1 Foot square.					
1296 =	9 =	1 Yard square.				
39204 =	$272\frac{1}{4}$ =	$30\frac{1}{4}$ =	1 Pole square.			
1568160 =	10890 =	1210 =	40 =	1 Rood square.		
6272640 =	43560 =	4840 =	160 =	4 =	1 Acre square.	
4014489600 =	27878400 =	3097600 =	102400 =	2560 =	640 =	1 Mile square.

940. *To measure a* SQUARE A C D B.

Suppose the Length of each Side be = 4 *Inches*, *Feet*, *Yards*, &c. then A B = 4; and A B × A C = A B² = 4 × 4 = 16, square *Inches*, *Feet*, *Yards*, &c. as is evident, by Inspection, in the Figure.

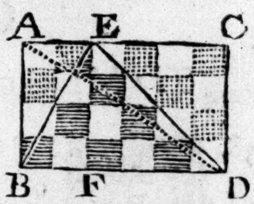


941. *To measure a* PARALLELOGRAM ABDC.



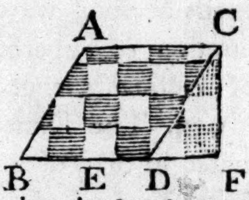
Admit the Length $AC = 6$, and the Breadth $AB = 4$; then $AC \times AB = 24$ square Inches, Feet, *&c.* according to the Measure in which the Sides were taken.

942. *To measure a* TRIANGLE BDE.



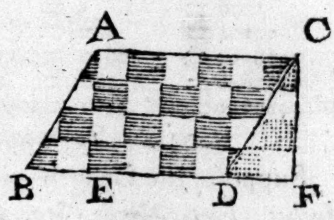
It is evident by Inspection, the Triangle ADB is equal to half the Parallelogram ABDC; but because $AC \parallel BD$, the Triangles ADB and BED are equal (635). Therefore the Triangle $BED\ (= BAD) = \frac{1}{2} AB \times AC = \frac{1}{2} EF \times BD$. Therefore if the perpendicular Altitude $EF\ (= AB) = 4$, and the Base $BD = 6$; then $\frac{1}{2} EF \times BD = 2 \times 6 = 12 =$ the Area of the Triangle BED, as required.

943. *To measure the* RHOMBUS ABDC.



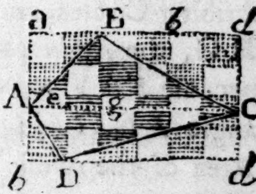
The Rhombus ABDC is equal to the Parallelogram AEFC, because $AC \parallel BF$, and $AC = FE$ (655.) Therefore the Rhombus is $= AE \times AC = AE \times BD$. Hence let the Base $BD = 4$, and the perpendicular Height $AE = 3{,}5$; then is the Area of the Rhombus $ABDC = 4 \times 3.5 = 14$.

944. *To measure the* RHOMBOIDES ABDC.



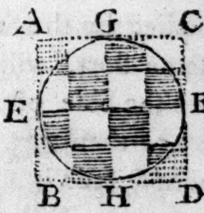
Here again the *Rhomboides* is = to the Parallelogram AEFC (655.) Therefore let the Base $BD = 6$, the Altitude $AE\ (= CF) = 4$; then is Area of the *Rhomboides* $ABDC = 4 \times 6 = 24$.

945. *To measure a* TRAPEZIUM ABCD.



Draw the Diagonal AC; on which, from the Angles B, D, let fall the Perpendiculars Bg, De; then is the Triangle $ACB = \frac{1}{2}$ Parallelogram $AaeC$, and the Triangle $ACD = \frac{1}{2}$ Parallelogram $AbdC$. Consequently the whole *Trapezium* ABCD $=$ half the Parallelogram $abcd$; $= \frac{1}{2}ab \times bd = \frac{1}{2}\overline{Bg+De} \times AC$. Thus, suppose the Diagonal $AC = 6$, and half the Sum of the Perpendiculars $= \frac{Bg+De}{2} = 2$; then $6 \times 2 = 12$, the Area of the *Trapezium*, as required.

946. *To measure a* CIRCLE EGFH.



We have shewn the Area of the Circle whose Diameter is 1, is 0,785398 (828) and the Areas of all Circles are as the Squares of their Diameters (840.) Let A, D, be the Area, and Diameter of the given Circle. Then, as $1^2 : 0.7854 : DD : A$. Therefore the Area $A = 0.7854\ DD$. Thus let $D = 4$, then $DD = 16$, and $0,7854\ DD = 12,5664 =$ Area sought.

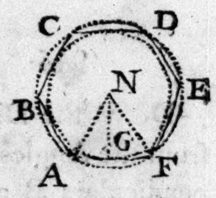
947. Let $A =$ Area, $D =$ Diameter, and $P =$ Periphery of a Circle; then any one of these being given, the other may be found by the following Equations, (see 824, 830, 840,) *viz.*

$$\text{Given}\left\{\begin{array}{l} D\text{, then } 3.1416\,D = P\text{; and } 0.7854\,D^2 = A. \\ P\left\{\begin{array}{l} P \div 3.1416 = D\text{, or } 0,3183\,P = D. \\ P^2 \div 12.5664 = A\text{, or } 0.07957\,P^2 = A. \end{array}\right. \\ A\left\{\begin{array}{l} \sqrt{A} \div 0,7854 = D\text{, or } \sqrt{1.2732\,A} = D. \\ \sqrt{12.5664\,A} = P\text{, or } \sqrt{A \div 0,07957} = P. \end{array}\right. \end{array}\right.$$

948. *To measure any regular* POLYGON.

For Example, the *Hexagon* ABCDEF. Now since these Figures consist of as many *Isosceles Triangles* ANF as they have

Sides; and the equal Sides of those Triangles $AN = NF$, are the *Radii* of the circumscribing Circles, and the perpendicular Heights NG, the *Radii* of the inscribed Circles; and because $\frac{1}{2} AF \times NG =$ Area of the Triangle ANG (942); therefore the Sum of all the Areas (or Area of the whole Polygon) will be equal to $GN \times \frac{1}{2}$ Sum of all the Bases or Sides. Now in the Hexagon, the Triangle ANF is equiangular and equilateral; and therefore putting $AN = AF = 1$, we have $NG = \sqrt{AN^2 - AG^2} = 0{,}866$; let the Sum of the Sides be $S = 6$, then $GN \times \frac{1}{2} S = 0{,}866 \times 3 + 2{,}598 =$ the Area of the Hexagon required.



949. But since regular Polygons are similar Figures, they will be to each other as Squares of their Sides (670.) Therefore the Area of any Polygon, whose Side is $= 1$, being multiplied by the Square of the Side of any other Polygon of the same Sort, will give the Area of that other Polygon. Thus, let the Side of any Hexagon be 15; the Square of which is 225, then $225 \times 2{,}598 = 583{,}05$, the Area of the Hexagon, whose Side is 15, as required.

950. And that the Area of any regular Polygon may be had in the same Manner directly, the Areas of each, supposing the Side $= 1$, are computed by the above Method, as in the Table following. (Art. 847.)

Sides.		Names.		Areas.
3	——	Trigon,	——	0.433013
4	——	Tetragon,	——	1.000000
5	——	Pentagon,	——	1.720477
6	——	Hexagon,	——	2.598076
7	——	Heptagon,	——	3.633959
8	——	Octagon,	——	4.828427
9	——	Enneagon,	——	6.181827
10	——	Decagon,	——	7.694209
11	——	Endecagon,	——	9.365675
12	——	Dodecagon,	——	11.197920

CHAP. II.

The MENSURATION *of* SOLIDS.

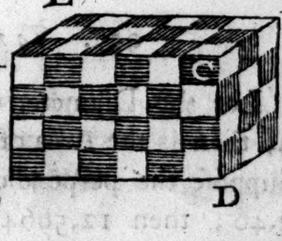
951. THE *Solidity*, or solid Content of Bodies is estimated in that of a *Cube*; as an *Inch Cube*, *Foot Cube*, *&c.* which are more generally called, a *cubic Inch*, *Foot*, &c. A Table of this Sort of Measure from the least to the greatest Denomination here follows:

Cubic Inches.						
1728 =	1 cubic Foot.					
46656 =	27 =	1 cubic Yard.				
7762329 =	$4492\frac{1}{8}$ =	$166\frac{3}{10}$ =	1 cubic Pole.			
1963885176 =	$1136507\frac{6}{10}$ =	$42092\frac{8}{10}$ =	253 =	1 cubic Rood.		
15711081408 =	9092061 =	336743 =	2024 =	8 =	1 c. Acre.	
25435806105600 =	147197952000 =	5451776000 =	32770584 =	129528 =	16191 =	1 c.M.

952. *To measure a* CUBE A G.

Suppose the Length of the Side AB = 4 Inches, then $AB \times AB = AB^2 = 16$; and $AB^2 \times AB = AB^3 = 64$ cubic Inches for the solid Content. (See 622.)

953. *To measure a* PARALLELOPIPEDON.



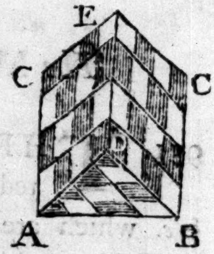
Let the Length AC = 6 Inches, the Breadth AE = 3; and the Depth AB = 4. Then

Multiply the Length	AC =	6
By the Breadth	AE =	3
And that Product $AC \times AE$ =		18
By the Depth	AB =	4

This Product is the sold Content $AC \times AE \times AB = 72$ (by 622.)

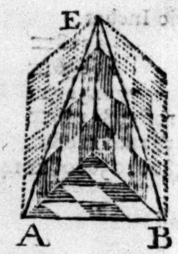
954.

954. *To measure a* PRISM AC,



Suppose it a triangular Prism, and let the Sides of the Base be equal, and A B $=$ 4 Inches. Then A B^2 $=$ 16; and 16 $\times$ 0,433 $=$ 6,928 square Inches, the Area of the Base A D B (942.) Then let the Height be also A C $=$ 4; then 6,928 $\times$ 4 $=$ 27,712 $=$ 27$\frac{1}{10}$ cubic Inches, the Solidity required.

955. *To measure a* PYRAMID A E D B.

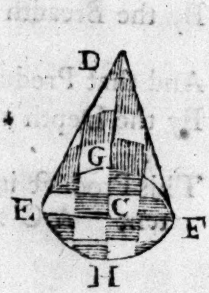


Let it be a triangular Pyramid; and each Side of the Base, *viz.* A B $=$ 4 Inches, then the Area of the Base A D B $=$ 6,928; and supposing its perpendicular Height E D $=$ 4, the Product 6,928 $\times$ 4 $=$ Solidity of a Prism, whose Base and Altitude are the same as those of the Pyramid (954), one Third of which, *viz.* $\frac{4}{3} \times 6{,}928 =$ 9,237 $=$ Solidity of the Pyramid, (by 833.)

956. *To measure a* CYLINDER CF.

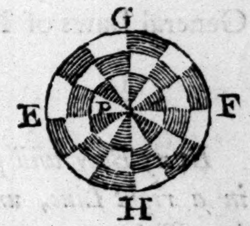
Let the Diameter of its circular Base EF $=$ 4 Inches, then the Area will be $=$ 12,5664, (by 946.) And supposing the Height of the Cylinder E C $=$ 4; then 12,5664 $\times$ 4 $=$ 50,2656 $=$ Solidity of the Cylinder required. (See 830, 831.)

957. *To measure a right* CONE E D F.



Let the Diameter of its Base be E F $=$ 4, then is the Area of the Base $=$ 12,5664. Suppose the perpendicular Height D C $=$ 3,46; then 12,5664 $\times$ 3,46 $=$ 43,478; one Third of which is the Solidity of the Cone, *viz.* $\frac{43{,}478}{3} = 14{,}492$. (See 834, 835.)

958. *To measure a* SPHERE, *or* GLOBE.



Let the Diameter of the Sphere be E F $= 4$; then is the Area of its greatest Circle $= 12{,}5664$; this multiplied by 4 is $= 50{,}2656 =$ the circumscribing Cylinder. Then two Thirds of this, *viz.* $\frac{2}{3} \times 50{,}2656 = 33{,}51$, *&c.* the *Solidity of the Sphere*, (by 873.)

N. B. $4 \times 12{,}5664 = 50{,}2656 =$ *Superficies* of the Sphere, in *Square Measure*, (by 838.)

959. Or thus; because *Spheres are as the Cubes of their Diameters* (841.) Then since the Solidity of the Sphere whose Diameter is 1, is 0,5236, (837, 847) if the Diameter of any other Sphere be given, as E F $= 4$; say, as $1 : 0{,}5236 :: 4^3 = 64 : 0{,}5236 \times 64 = 33{,}51$, *&c.* the Solidity of the Sphere, as above.

960. After the same Manner may be measured the *Spheroid* of (844.) Also the *Segment* of a Sphere may be easily measured, by the Theorem in (836,) and of the Spheroid by the Theorem in (844.)

CHAP. III.

The Philosophical PRINCIPLES *and* LAWS *of* MOTION *and* GRAVITATION.

961. MATTER, or SUBSTANCE, of which Bodies are composed, in itself considered, or in respect of its Essence is unknown to us; this is acknowledged by Sir ISAAC NEWTON, and all Mankind besides. All that we know of Matter is what relates to its various *Properties* and *Qualities* which present themselves to our Senses.

962. The first and principal of which is what Sir *I. Newton* calls the *vis inertiæ* of Matter, or its *natural Inactivity.* No Man ever yet observed any Part of Matter to have a *Principle of Action* in itself; but on the other Hand, it is absolutely *passive*, and

and ſubject to the Influence of every external Agent. And from hence is deduced the First of thoſe, which are called the General Laws of Nature, *viz.*

963. LAW I.

Every Body will perſevere in its State of Reſt or moving uniformly in a right Line, unleſs it be compelled to change its State by Forces impreſſed.

Thus a Bullet would continue at Reſt in the Gun for ever, if it was not expelled by the Force of Powder; but being thereby put into Motion, it would for ever move on in the Direction of the Axis of the Barrel, if not retarded by the Reſiſtance of the Medium, and carried downwards from that right Line by the Power of Gravity. We ſhew by Experiment, that the leſs the Friction is of the Axis of a moving Wheel, the longer its Motion continues. And we obſerve the vaſt Bodies of the Comets and Planets preſerve their Motions undiminiſhed (as to Senſe) for many 1000 Years in unreſiſting Mediums.

964. Since by Reaſon of the Inactivity of Matter (962,) there is nothing in any Body that can augment, diminiſh, or any Ways alter or vary the Action or Effect of any Force impreſſed, we thence deduce the ſecond General Law of Motion, *viz.*

LAW II.

The Change of Motion is proportional to the moving Force impreſs'd, and is made according to the right Line, in which that Force is impreſs'd.

If any Force generate any Motion, a double, or triple Force will generate twice or thrice as much. But the Alteration in reſpect of the Direction of the Motion, is a complicated Affair, which we ſhall farther conſider hereafter.

965. When Bodies act upon each other, they do it by *Contact* or *Colliſion*, and ſince in this Caſe, the Action itſelf is but one and the ſame between both; the Effects which it produces muſt of Courſe be equally divided between both Bodies, and thence an equal Mutation of their State previous to the Stroke

Stroke must needs follow. From this Consideration is deduced the *third General Law of Nature*, viz.

Law III.

Re-action is always equal and contrary to Action; or the Actions of two Bodies mutually upon each other are always equal, and directed towards contrary Parts.

966. The Truth of this is abundantly confirmed by Experiments. If the Finger presses a Stone, the Stone re-acts and presses the Finger. The Hammer striking the Anvil, receives the same Stroke from it, and is thereby made to rebound. The *Loadstone*, if fixed, attacts the moveable Iron; and the fix'd Iron equally attracts the moveable Magnet. The Horse draws the Stone, and the Stone equally draws the Horse, because the Action, or Force in the Rope which connects them is one and the same, and must act equally at each End, *viz.* upon the Horse and Stone.

967. But before we can reason well upon the Subject of Motion, we must understand by just Definitions, what Ideas we are to fix to that Word. For the Word Motion is become *ambiguous*, and is used in a *simple* and complex Sense. *Motion*, in its simple Acceptation, is only *a Change of Place* in Bodies. But in the *complex*, or *physical Sense*, it implies all the *Change that is made in the State of a Body in regard both to its* Quantity of Matter *and* Velocity of Motion. And in this Respect, it is properly called the *Momentum*, or *Quantity of Motion*, and which is always equal to the *Force* which produces it by Law II. (964.)

968. That the Quantity of Motion is as the Mass of Matter, *(cæteris paribus)* is evident from hence, that the Motion of the whole Mass is the Sum of the Motions of all the Parts, or Particles; as the Number of Particles moved, therefore, is greater or less, so will be the Sum of all their Motions, and consequently so will be the Aggregate, or whole Motion or *Momentum* of the Body.

969. Again, the *Momentum*, or *Quantity of Motion*, is *(cæteris paribus) as the Velocity* of the simple Motion. For since *Velocity* has Regard to the *Space described in the same Time*, (778) it is evident, that if a Body describes twice the Space that an-

other equal Body describes in the same Time, it is plain there is twice the Change of Place, and therefore twice the simple Motion (967) produced in the same Time. But the Velocity is also twice as great; therefore the *Motion is as the Velocity.*

970. Since then the *Quantity of Motion* is as the Mass of Matter simply, and the Velocity simply; it will be conjointly as the Rectangle or Product of both, when no Regard is had to either singly. Therefore putting $M =$ Mass of Matter, $V =$ the Velocity, and $Q =$ the *Momentum* or Quantity of Motion; then in Symbols we have $Q : M \times V$; and thence * $V : \frac{Q}{M} : Q \times \frac{1}{M}$; and $M : Q \times \frac{1}{V}$.

971. Let $S =$ Space described by a Body in Motion, and $T =$ the Time of describing it. Now since in equal Times the Spaces described will be as the Celerities of Motion (969,) therefore $S : V$; also it is manifest, that in describing the same Spaces, the Times will be *greater* as the Celerities of Motion are *less*; that is, the Time is inversely as the Celerity or $T : \frac{1}{V}$; and so also $V : \frac{1}{T}$. Whence since V is simply as S, and $\frac{1}{T}$, it will be conjointly $V : \frac{S}{T}$, or $VT : S$.

972. Because $V = \frac{Q}{M}$ (970) $= \frac{S}{T}$ (971,) therefore $QT = SM$; and $Q = \frac{MS}{T} = MS \times \frac{1}{T}$. That is the *Quantity of Motion is in the compound Ratio of the Space and Mass directly, and inversely as the Time.*

973. The Quantity of Matter (M) is (*cæteris paribus*) as the Bulk (B) of a Body; for in twice the Bulk there will be twice the Matter; therefore it is $M : B$. Again, the Quantity of Matter (M) will be (*cæteris paribus*) as the *Density* (D), that is, as the Number or Sum of the Particles in equal Bulks:

* The Reader is desired to observe once for all, that $V : \frac{Q}{M}$ denotes only, that V is always proportional to $\frac{Q}{M}$; but $V = \frac{Q}{M}$ is the same as $\frac{V}{1} = \frac{Q}{M}$, and shews that the Ratio of 1 to V is the same as the Ratio of M to Q.

Bulks; therefore M : D. Wherefore conjointly it will be M : B × D.

974. Because $M = \frac{QT}{S}$ (972) = BD; therefore $Q = SBD \times \frac{1}{T}$; or the *Quantity of Motion is compounded of the direct Ratio of the Space, Bulk and Density, and the inverse Ratio of the Time.*

975. Of Forces that actuate the Particles of Matter, we observe the following Variety. (1.) A Force or Power of one Kind causes the Particles to *adhere firmly to each other*; and hence it is called the Force or Power of COHESION. (2.) Another Sort of Power obliges the Particles of Matter, under some Circumstances, to recede and fly from each other; which Power is therefore called the *Repellent*, or *repulsive Force*. (3.) A third Power causes all large Portions of Matter, or Bodies, to tend mutually towards each other, in Directions to their Centers; and hence it is called a *Centripetal Force*.

976. What those Powers are in themselves, or how they differ, I shall not pretend to enquire; nor also what is the particular *Modus agendi*, or Manner of actuating the Particles of Matter, *viz.* whether it be by *Attraction*, *Implusion*, or otherwise; since in these Researches, there is but little Reason hitherto to expect any successful Discoveries. It will be quite sufficient, if we can make ourselves thoroughly acquainted with their *Phænomena* and Effects, and apply them in a proper Manner to the various Uses of Life.

977. The POWER OF COHESION respects the smallest Particles of Matter, and extends to but very small Distances, as is plain by numberless Experiments; it is therefore proportional to the Surfaces in contact between two Corpuscles, or their Cohesion is so much the greater, by how much the Surfaces are larger in which they touch.

978. Hence, according to the various *Figures* of Particles, they touch by different Quantities of Surface, and cohere with different Degrees of Firmness. Thus, for Instance, if all were nearly *cubical*, they would touch by a great Quantity of Surface, and constitute a very *hard* and firm Body. But on the other Hand, if we suppose Particles truly spherical, they

will touch but by a very ſmall Portion of their Surfaces, and cohere very ſlightly; and of Courſe, will conſtitute a Body whoſe Parts will be very eaſily moved among themſelves, and yield to any impreſſed Force, which Caſe we call the Fluidity of Bodies. Hence Bodies receive their various Degrees of *Hardneſs* and *Softneſs*, *Fixity* and *Fluidity*, *Firmneſs* or *Looſeneſs* of Texture, and all other Qualities depending thereon.

979. That this Power is exceeding great appears by many Experiments. Thus two Balls of Lead, having their Surfaces pared ſmooth, will cohere with a Force equal to 150 lb. tho' they touch upon no more than $\frac{1}{5}$ of an Inch ſquare. Two Braſs poliſhed Planes, 2 Inches Diameter, ſmeared over with Greaſe and put together very hot; will, when cold, cohere ſo firmly as to require 950 lb. to ſeparate them. And Wires of ſeveral Sorts of Metal $\frac{1}{10}$ of an Inch Diameter required the Weights to pull them aſunder, as in the following Table are ſpecified.

Lead	$29\frac{1}{4}$ lb.
Tin	$49\frac{1}{4}$ lb.
Copper	$299\frac{1}{4}$ lb.
Braſs	360 lb.
Silver	370 lb.
Iron	450 lb.
Gold	500 lb.

980. Though this Power acts with ſuch prodigious Force near the Surface, it decreaſes in ſuch a Manner as to become nearly inſenſible in the leaſt ſenſible Diſtance from the Surface; not only ſo, but at a certain ſmall Diſtance it is converted into another Kind of Force, or at leaſt, it acts in a Manner exactly contrary to what it did before; for it now cauſes the Parts of Matter to recede from each other, and to remain at certain equal Diſtances or Intervals among themſelves. And thus modified, it is called the *Repulſive Power* in Matter.

981. That this is Fact we are aſſured from divers Experiments, and many *Phænomena* of Bodies. Thus the Magnetic Needle will be ſtrongly attracted by either Pole of the Magnet in contact with it, but at a very ſmall Diſtance the ſame Power becomes *repulſive* in one Pole, and repels the ſaid Needle from it.

it *. So the Particles of Fluids cohere by this Power, while in a State of Contact, but when separated by Heat, they repel each other, and exist in the Form of VAPOUR or STEAM. Again, 'tis well known by numberless Experiments, that in all solid Bodies there is a certain Matter which, while in contact with the other Parts, does firmly adhere, and is strongly connected with them by this Force; but when by natural or artificial Fermentation, it is disengaged or set at Liberty, it immediately (by a repellent Power among its Particles) expands into a *fine etherial Fluid*, every Way like *common Air*. But of this we shall speak more hereafter.

982. The Consequence of such a repulsive Power among the Particles of Matter, is, that they having attained an *Equilibrium*, act mutually upon each other, and become susceptible of *Compression* and *Condensation*; and of *Expansion* and *Rarefaction*; as is well known by common Experiments on Air, Vapour, *&c.* Again, as Action and Re-action are equal (966,) it follows, that when any Force is impressed upon the Particles of such a Fluid, they all jointly resist the same; and when the impressed Force is removed, by Virtue of this Power, the Particles all retreat to their primitive equidistant Stations, with a Force equal to that impressed. And this Renitency or restituent Force, is what we call the SPRING, or ELASTICITY of such Sort of Bodies.

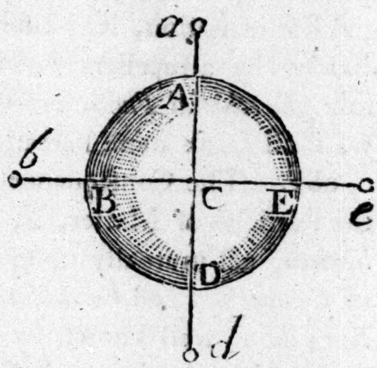
983. We also further learn by Experiments of *Thermometers*, &c. that Heat augments and Cold diminishes this elastic or expansive Force in Bodies; and that their natural Dimensions are hereby continually altering, as is evident not only in Air, but in denser Fluids, as Spirit of Wine, Water, and even Mercury itself. Yea, solid Bodies discover the same Properties in several Degrees; *Ivory* is found to be very elastic; *Metals* of all Sorts expand and contract with Heat and Cold, as we shew by the PYROMETER, and is otherwise known by common Experience.

984. The third Sort of Power, or Agent, (mentioned 975,) is

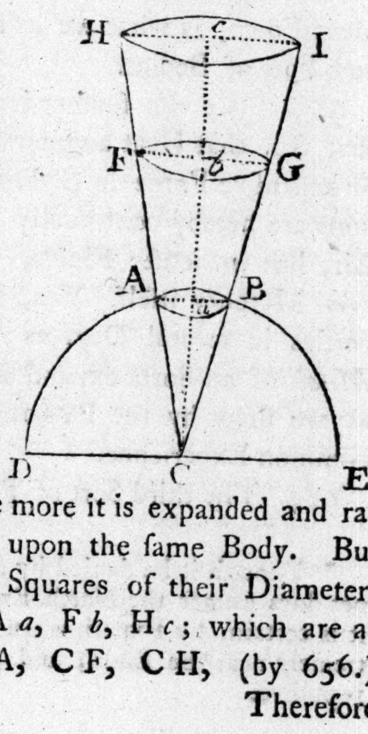
* Though it be denied by some, that the North Pole of the Magnet does attract the North End of the Needle just upon the Ends, it is certain, by Experiments, that this attracting Force appears extremely near the Ends; and there at a small Distance it becomes repulsive.

is a *centripetal Force*, and this we observe to take Place among the largest Bodies or Systems of Matter. The *Phænomena* which demonstrate such a Power most sensibly to us are those of *heavy Bodies falling to the Earth*. This Power we call GRAVITY, and this Tendency of Bodies to the Earth is called *Gravitation*.

985. That this is a *centripetal Force*, or that Bodies are thereby made to tend to the Center of the Earth is hence evident, that all Bodies are observed to fall in right Lines to the Surface of the Earth every where. But because the Earth is spherical, those Lines which are perpendicular to the Surface do all pass through the Center of the Earth, as is evident from a View of the Figure annexed. For let A B D E be the Earth, C its Center, then suppose a Body (*a*) falling in the right Line *a* A to the Earth in A, that Line if produced must go through the Center C. The same is to be said of any other Bodies at *b*, *d*, *e*.



986. But any Virtue propagated in right Lines to or from a Center, will have its Energy on Bodies every where, as the *Square of the Distance from the Center inversely*. For, suppose a Cone of this gravitating Virtue be represented by I H C, terminating in the Center of the Earth C. If then we make C A = A F = F H; the Energy of the Virtue will be at those Distances inversely as the circular Areas A B, F G, and H I; for the more it is expanded and rarified, the less will its Effect be upon the same Body. But these circular Areas are as the Squares of their Diameters (840,) or of the Semidiameters A *a*, F *b*, H *c*; which are as the Squares of the Distances C A, C F, C H, (by 656.) Therefore



Therefore *the Energy of this Power decreases as the Square of the Distance from the Center increases.*

987. But because the Semi-diameter of the Earth is near 4000 Miles, and the greatest Height to which we can elevate Bodies above the Earth's Surface being but a Mile or two, 'tis evident this Force in so small a Distance will not sensibly vary; and therefore may be esteemed *as acting uniformly* through any Spaces near the Earth's Surface.

988. The Action of this Power is *constant or perpetual*; this appears from hence, that the Velocity of falling Bodies is *constantly accelerated or increasing*, as we know by Experiments. If a Body be put into Motion by a *single* or *instantaneous Impulse*, the Velocity of that Body would be *uniform* (by Law I. 963,) and its Motion rectilineal. If the Power which puts a Body in Motion be temporary, or acts only for a certain Time, the Velocity during that Time will be accelerated, and afterwards become uniform. But if the Power acts incessantly, the Body is every Moment impelled, and its Velocity must every Moment increase. Such therefore is the Power of Gravity.

989. Since Gravity acts *constantly and uniformly*, the Velocity of Bodies will be *equably accelerated* in their Descent to the Earth; for since the Impulse communicated each Moment is the same, it will generate equal Velocities in the several equal Moments, which constantly added together, make an uniformly increasing Sum, and therefore an *equably accelerated Velocity*. For the Velocity at any Moment is equal to the Sum of all the *momentary Increments* of Velocity from the Beginning.

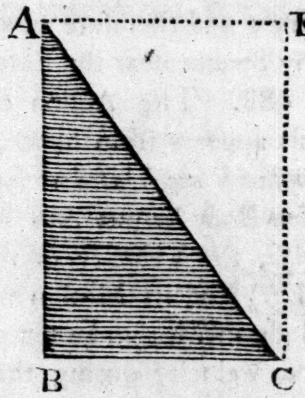
990. Hence the *Velocity* (V) of the Fall is proportional to the *Time* (T). Because since the Action of Gravity is uniform, (987), whatever Velocity is generated in one Particle of Time, a double Velocity will be generated in twice that Time, a triple Velocity in thrice that Time; and so on continually, therefore it will be always T : V.

991. The Space (S) described by falling Bodies will, in equal Times, be greater as the Velocity is so; and therefore in this Case, we have S : V. Also the Velocity remaining the same, the Space will be as the Time of describing it (971.) Therefore in this Case S : T. Consequently, when neither

the

the Velocity nor Time is given, the Space will be in the compound Ratio of both, *viz.* S = T V. But it is T : V (990.) Therefore S : T², or S : V² ; that is, *the Spaces described by falling Bodies are as the Squares of the Time or of the Velocity.* And this we prove true by Experiments.

992. To illustrate this Matter otherwise ; let A B represent the Time of the Fall, and B C the Velocity acquired at the End of that Time ; and draw A C. Then if we conceive the whole Time A B to be divided into an indefinite Number of equal Moments, the Velocity in each of those Moments will be as the right Line drawn from A B parallel to B C, corresponding to the given Moment. Now for a single Moment the Velocity may be esteemed as uniform, and so the Space described that Moment will be as the Velocity (by 971,) therefore the Sum of the Spaces described in all the Moments (or whole Time A B) will be as the Sum of all the momentary Velocities, or right Lines which represent them, but the Sum of these Lines make the Area of the Triangle A B C ; this Area, therefore, is as the whole Space described in the Time A B. And because this Area is as A B², or B C² (670,) therefore also the Space (S) is as T², or V², as above (991.)



993. Compleat the Parallelogram A B C D. Then if we suppose another Body (A) to commence Motion at the same Time with the falling Body (B,) and the Velocity of the Body (A) to be uniform and equal to the Velocity B C acquired at the End of the Fall ; then it is evident the Space described by the Body (A) in the Time A B will be represented by the Parallelogram A B C D ; *which Space is therefore double of that* (A C B) *described by the falling Body* B.

994. What has been said of descending Bodies is in the same Manner applicable to ascending Bodies. For the Motion in the latter Case will be *retarded* equably by the contrary Action of Gravity, as it was in the former Case *accelerated.* Thus let B C be the Velocity, with which any Body is projected

jected upwards from the Point B, then in every Moment of Time there will be an equal Decrement of this Velocity, so that at the End of the Time B A it will be all destroyed, and the Space it will describe in that Time will be as the Area of the Triangle A B C. Therefore, &c.

995. Hence it follows, (1.) That the Velocity every where, at equal Intervals of Time from the Moments B and A, in the Ascent and Descent is the same. (2.) That the Time of the Ascent and Descent is the same, or half the whole Time of the Flight of the Projectile. (3.) That the Body by descending, acquired a Velocity equal to that (B C) by which it was projected. All that we have hitherto said of Motion, is upon a Supposition that the Body moves *in Vacuo*, or in a Medium *without Resistance*.

996. From what has been said, it is evident, that if the Space a Body describes in any given Time *in Vacuo* be known, the Space through which it will fall in any proposed Time, will from thence be known also. Thus by Experiments very accurately made*, it has been found, that a Glass Globe filled with *Mercury*, descended through the Height of 220 Feet in four Seconds; and that a Ball of Lead fell thro' 272 Feet in $4\frac{1}{4}''$, and allowing for the Resistance of the Air, the Motion in both these Cases was at the Rate of $193\frac{1}{3}$ Inches, or 16,11 Feet in the first Second of Time. For the $4''$ in Air will make but $3{,}75''$ *in Vacuo*; then $\overline{3{,}75''}^2 : 220\text{F.} :: 1^2 : 16{,}11$ Feet, for the Space in the first Second, (by 991.)

997. Hence $1^2 : 16{,}11\text{ F.} :: T^2 : S$; therefore $16{,}11\, T^2 = S =$ the Space described in any Time T expressed in Seconds. Hence also $T = \sqrt{\frac{S}{16{,}11}} =$ Time of describing any given Space S.

998. The Power of Gravity (G) will be as the Velocity (V) generated in the same Time, because that Velocity is the whole Effect of the Power, and Effects are always as their Causes; therefore G : V. Again, Gravity (G) will always be inversely as the Time (T) in which the same Velocity is generated; for 'tis evident, a double Force (2 G) will generate the

* See the *Principia*, Edit. 3. p. 346.

the ſame Velocity in half the Time ($\frac{1}{2}$ T) that the Force (G) does; or the greater the Force, the leſs will be the Time for producing the ſame Effect; therefore $G : \frac{1}{T}$: Therefore when neither the Velocity nor Time is given, we have $G = \frac{V}{T}$, or $GT = V$, and $T = V \times \frac{1}{G}$.

999. Since when the Time T is given, G is as V, and in that Caſe V is as S (991,) therefore alſo G will be as S; *or the Power of Gravity will be as the Space deſcribed in any given Time.*

1000. Since $GT = V = \frac{Q}{M}$ (972,) we have $GTM = Q$, and when the Time T is given it is $GM = Q$; that is, the *whole Quantity of Motion in falling Bodies is compounded of the accelerating Force* (G) *of Gravity, and the Maſs of Matter* (M) *in the Body.*

1001. Since $T = \frac{V}{G}$ (998,) $= \frac{SM}{Q}$ (972,) therefore $QV = GSM$. Hence whence the Space (S) is given, it is $QV = GM$. But we have always G : V (998.) Therefore it is always Q : M; that is, *the Quantity of Motion* (Q) *is always proportional to the Quantity of Matter* (M); and ſince Gravity (G) is always the ſame near the Earth's Surface, *the Velocity* (V) *of falling Bodies will be every where the ſame too; be the Quantities of Matter in what Proportion you pleaſe.*

1002. Since Q : M, therefore the *Momentum*, or Force, by which Bodies tend towards the Earth's Center, is as the Quantity of Matter. But this Force or Tendency of Bodies, is what we vulgarly call their WEIGHT; hence it appears, *that the* WEIGHTS *of Bodies are always proportional to their Quantities of Matter.* And thus having premiſed the eſſential Principles of Philoſophy, we next proceed to the *Laws of Motion*, obſerved in *ſtriking Bodies*; for nothing uſeful can be known, 'till they are firſt aſcertained.

CHAP.

CHAP. IV.

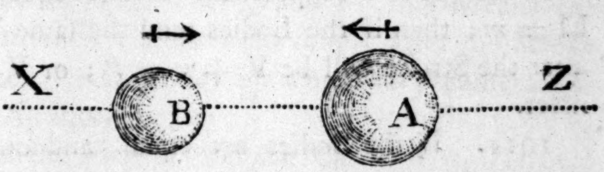
The Philoſophical THEORY *of* PERCUTIENT BODIES, *and of the* COMPOSITION *and* RESOLUTION *of* FORCES.

1003. TO determine the ſeveral Particulars relating to the *Motion*, *Velocities*, and *Direction* of *Percutient Bodies* A, and B; we repreſent the Quantities of Matter by M and m, and their Velocities by V and v; then will $Q = MV$ (970,) = *Momentum* of A, and $q = mv =$ *Momentum* of B.

1004. If the Body B ſtrikes the Body A in Motion, and

both move the *ſame Way*, or towards the ſame Parts, (as from X to Z) then the Sum of their Motion towards the Part Z, will be $MV + mv$, and the Velocity of both the Bodies after the Stroke towards the ſame Part, will be $\frac{MV + mv}{M + m} = V$. For the Velocity is always as the *Momentum*, divided by the Maſs of Matter (970.)

1005. If one of the Bodies, as A, has a contrary Direction, or tend towards



X, in which Caſe the Bodies will meet, then the *Momentum* of A will have a negative Sign, *viz.* $-MV$; and ſo the Sum of the Motions towards the ſame Part Z will be $mv - MV$; and the Velocity after Colliſion will be $\frac{mv - MV}{m + M} = V$.

1006. Becauſe in what we have hitherto ſaid, we ſuppoſe the Bodies A and B deſtitute of *Elaſticity*, therefore after the Stroke, there being nothing in the Bodies to cauſe a Reſilition or Separation, they will both go on together with the ſame Velocity V.

1007. The Sum of the Motions towards the ſame Parts is the ſame before and after the Stroke. For let them both move

 the

the ſame Way (1004,) and let B ſtrike A; then by that Impulſe, the Motion of A, *viz.* Q, will be augmented and become $Q + x$ after the Stroke; but becauſe *Action and Reaction is equal* (965,) the Body A will re-act upon B, and produce an equal Effect by the Stroke, that is, it will diminiſh the Motion of B by the ſame Quantity x, ſo that its Motion after the Stroke will be $q - x$; but the Sum of the Motions of both Bodies after the Stroke $\overline{Q + x} + \overline{q - x} = Q + q$, the Sum of the Motions before the Stroke, ſee (1004.) And the ſame is to be ſhewn, if the Bodies meet, as (1005).

1008. The *Magnitude of the Stroke* will be proportional to the Quantity $(x,)$ becauſe that is the whole Effect or Mutation produced in the Motion of each Body. The Greatneſs of the Stroke is therefore meaſured by the Loſs (x), which the moſt powerful, or percutient Body ſuſtains in its Motion.

1009. In the above Theorems (1004, 1005,) if the Body A be ſuppoſed at Reſt, then $V = 0$, and MV vaniſhes; the Velocity then after the Stroke is $\frac{mv}{M + m} = \mathit{V}$; and ſo $mv = \mathit{V}M + m\mathit{V}$. Whence $\mathit{V} : v :: m : M + m$, and $v = \frac{M + m}{m} \times \mathit{V}$.

1010. If we ſuppoſe the Bodies equal, *viz.* $A = B$, or $M = m$; then if the Bodies tend the ſame Way, the Velocity after the Stroke will be $V + v = \mathit{V}$; or $V - v = \mathit{V}$, if they meet.

1011. If the Bodies are equal, and one of them at Reſt, then $\frac{mv}{M + m} = \frac{v}{2} = \mathit{V}$; or *the Velocity after the Stroke is equal to half that of the ſtriking Body.*

1012. If A at Reſt exceed B infinitely in Magnitude; then becauſe m is infinitely ſmall in reſpect of $M + m$; therefore ſo is V in reſpect of v (1009,) conſequently V will vaniſh, or the Body B impinging againſt any firm immoveable Object, *will after the Stroke be at Reſt.*

1013. If equal Bodies moving with equal Velocities, meet; they will mutually deſtroy each other's Motions; for in this Caſe $MV = mv$; therefore $MV - mv = 0$, conſequently $\frac{MV - mv}{M + m}$

$\frac{MV - mv}{M+m} = \frac{0}{M+m} = V = 0$; or both the Bodies *remain at Rest after the Stroke.*

1014. The *Momentum* of the Body B, after the Stroke is $mV = \frac{MVm \pm m^2 v}{M+m}$ (1004,) therefore $mv - \frac{MVm \pm m^2 v}{M+m}$ $= \frac{MVm \mp mvM}{m+M} = \frac{Mm}{M+m} \times V \mp v =$ the Loss of Motion in the Body B after the Stroke; but $\frac{Mm}{M+m}$ is a constant Quantity; therefore the Loss of Motion in B is as $V \mp v$, and consequently the Magnitude of the Stroke in Bodies tending the same Way, is as $V - v$; and as $V + v$, if they meet (1008.) And if A be at Rest, then $V = 0$, the Stroke will be as v, the Velocity of the percutient Body.

1015. If the impinging Bodies A and B are *perfectly elastic*; then this *elastic Force* is ever equal in its Action to the *compressing Force* (982.) And whatever Action is exerted upon A by the Impulse of B, the same is doubled by Virtue of this renitent, or elastic Force; and the Re-action of A upon B is doubled likewise (965;) and as the Parts of each Body are mutually compressed and flatted by the Stroke, so those Parts are thrown out again by an equal Force, and by this Means the Bodies are made to recede from each other, after the Stroke with the same Forces, or *Momenta*, by which they came together, or struck each other.

1016. But since the *Force of Collision* is as $V \mp v$, (1014,) that will also be as the Force of Resilition, or that by which they separate after the Stroke. Let x and y be the Velocities of the Bodies A and B after the Stroke; and then $V \mp v = x \mp y$; and $y = V \mp v \pm x$, and so $Mx =$ the Motion of A after the Stroke, and that of B will be $my = mV \mp mv \pm mx$. And since the Sums of the Motion before and after the Stroke towards the same Parts are equal (1007,) we have $MV \pm mv = Mx + mV \mp mv \pm mx$; and thence $Mx \pm mx = MV \pm 2mv - mV$, and so $\pm x = \frac{MV \pm 2mv - mV}{M+m}$.

1017.

1017. Also the Velocity of B, after the Stroke will be $y = V \mp v \pm x = V \mp v \pm \frac{MV - mV \pm 2mv}{M + m} = \frac{2MV \mp Mv \pm mv}{M + m}$. From whence 'tis evident, if the Bodies tend the same Way, the Motion of B will be always positive; but when the Bodies meet, the Body B will proceed or recede, according as Mv is greater or lesser than $2MV - mv$.

1018. The *Momentum* of A after the Stroke, will be $\frac{M^2V - MmV \pm 2Mmv}{M + m}$: And that of B will be $\frac{2MVm \mp Mmv \pm m^2v}{M + m}$: Whence the Motion lost in A will be $MV - \frac{M^2V - MmV \pm 2Mmv}{M + m} = \frac{2MmV \mp 2Mmv}{M + m}$; and that gained in B will be found the same; whence it is evident, *that the Effect of the Stroke on each Body is equal, and double of that in non-elastic Bodies* (1014.)

1019. If the Body B be at Rest before the Stroke, then $v = 0$, and the Theorem becomes $\pm x = \frac{MV - mV}{M + m} = \frac{M - m}{M + m} \times V$. Hence it appears, the Velocity of the Body (A) after the Stroke, will be affirmative or negative, that is, forwards or backwards, as M is greater or lesser than m, or as A is greater or lesser than B.

1020. If B be at Rest and equal to A; then $v = 0$, and $M = m$, and so $x = \frac{0}{2M} = 0$; that is, the Body A will in this Case be at Rest after the Stroke; and the Body B will move on with the Velocity of A, for $y = V$ in this Case, (by 1016.) *

1021.

* After we have given the *Physical Principles* of the *Newtonian Philosophy*, we shall illustrate and confirm every Position and Doctrine by Experiments, and describe the Machines by which they are performed in the best Manner; and in that Part the Reader will see how exactly all the Cases of percutient Bodies here premised are verified by a practical Instance of each of them.

1021. Let A, B, C, be three Bodies; and let A strike B at Rest; the Velocity generated in B by the Stroke, will be $y = \frac{2MV}{M+m}$ (1017, since $v = 0$;) and therefore the *Momentum* of B will be $\frac{2MVm}{M+m} = my$. With this *Momentum* B will strike C at Rest, and contiguous to it; the Velocity generated in C will be $\frac{2my}{m+C}$, and its *Momentum* will be $\frac{2myC}{m+C}$; and if instead of y, we restore its Value $\frac{2MV}{M+m}$; the *Momentum* of C will be $\frac{2mC}{m+C} \times \frac{2MV}{M+m} = \frac{4MVmC}{Mm+MC+Cm+m^2}$.

1022. If now we make B a variable Quantity, while A and C remain the same, we may determine the Proportion of B to A and C, that shall give the *Momentum* of C the greatest possible, by making the Fluxion thereof equal to nothing, *viz.* $\frac{4M^2C^2V\dot{m} - 4MCm^2\dot{m}}{\overline{MC+Mm+mC+m^2}|^2} = 0$. Whence we get $MC - mm = 0$; and therefore $MC = mm$. Whence $M : m :: m : C$, or $A : B :: B : C$; *therefore* B *is a mean Proportional between* A *and* C.

1023. Hence if there be any Number (n) of Bodies in a geometrical Ratio (r) to each other; and the First be A, the Second will be rA, the Third r^2A, *&c.* to the last, which will be $r^{n-1}A$: Also the Velocity of the First being V, that of the Second will be $\frac{2V}{1+r}$ (for $\frac{2MV}{M+m} = \frac{2AV}{A+rA} = \frac{2V}{1+r}$) that of the Third $\frac{4V}{\overline{1+r}|^2} = \frac{2V}{1+r} \times \frac{2}{1+r}$, *&c.* to the Velocity of the last, which will be $\overline{\frac{2}{1+r}}\Big|^{n-1} V$. The *Momenta*, or Motions of the Bodies would be, of the First AV, of the Second $\frac{2AVr}{1+r}$, of the Third $\frac{4AVr^2}{\overline{1+r}|^2}$; and to the last

2 r

$\overline{\frac{2r}{1+r}}\Big|^{n-1}$ AV, as they are exhibited in the following Series, *viz.*

Bodies, A, rA, r^2A, r^3A, r^4A, &c. r^{n-1}A.

Velocities, V, $\frac{2V}{1+r}$, $\frac{4V}{\overline{1+r}|^2}$, $\frac{8V}{\overline{1+r}|^3}$, $\frac{16V}{\overline{1+r}|^4}$, &c. $\overline{\frac{2}{1+r}}\Big|^{n-1}$ V.

Motions, AV, $\frac{2AVr}{1+r}$, $\frac{4AVr^2}{\overline{1+r}|^2}$, $\frac{8AVr^3}{\overline{1+r}|^3}$, $\frac{16AVr^4}{\overline{1+r}|^4}$, &c. $\overline{\frac{2r}{1+r}}\Big|^{n-1}$ AV.

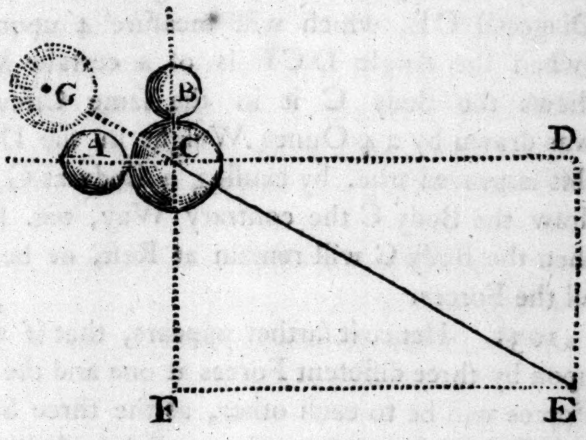
1024. If the Number of Bodies be $10 = n$, and $r = 2$; then will A : r^{n-1}A :: 1 : r^{n-1} :: 1 : 512. Also V : $\overline{\frac{2}{1+r}}\Big|^{n-1}$ V :: 1 : $\overline{\frac{2}{1+r}}\Big|^{n-1}$ :: 1 : 0.026 :: 38,45 : 1. Lastly, AV : $\overline{\frac{2r}{1+r}}\Big|^{n-1}$ AV :: 1 : $\overline{\frac{2r}{1+r}}\Big|^{n-1}$ :: 1 : $13\frac{3}{10}$.

1025. If $n = 100$, and $r = 2$, then will the first Body A be to the last r^{n-1}A, as 1 to 633825300,000000000000000000000 nearly, and its Velocity to that of the last, as 271022000000000000 to 1 nearly, and the *Momentums* will be as 1 to 233848000000, very near.

1026. Let $\frac{2r}{1+r} = R$, and let the Motion of the first Body be to that of the last, as 1 to M, that is, let $M = \overline{\frac{2r}{1+r}}\Big|^{n-1} = R^{n-1}$; if L = the Logarithm, then L, M $= \overline{n-1} \times$ L, R. Also $\frac{L, M}{L, R} + 1 = n = \frac{L, M \times L, R}{L, R}$.

1027. By these Theorems, whatever relates to the Motion of Bodies *elastic*, or *non-elastic*, acted upon by a single Impulse, may easily be determined. I shall therefore now proceed to consider the Motion of a Body, as acted upon by two or more Forces at once, and determine the Direction thereof by the foregoing Laws; as upon this Doctrine depends the whole *Rationale* of the *Mechanical Philosophy*, (which alone can be true) and therefore it is of the highest Consequence to be thoroughly and rightly understood.

Thus



Thus let C be a Body impelled in the Direction A D by a Body A, with such a Force as shall cause it to move uniformly over the Space C D in a Second of Time. At the same Instant let it receive a Stroke by another Body B, in the Direction B F, with such a Force as shall cause it to pass over the Space C F in the same Time.

1028. Now it is evident the Body C cannot move in both these different Directions; and therefore will not move in either, but in a Direction *compounded of them both*, which is thus determined. Draw D E parallel to B F; then, though the Action of B prevents the Body from proceeding in the Right Line C D, yet it can no Ways alter its Velocity of approaching to the Line D E in the given Time, by Virtue of the Force impressed by A. At the End therefore of a Second of Time, the Body C will be somewhere in the Line D E. By the same Way of Reasoning, it will at the End of the same Time be found somewhere in the Line F E, parallel to C D; and therefore in the Concourse of both, in the Point E. Its Course then is the Line C E, which is a Right Line by LAW I. (963).

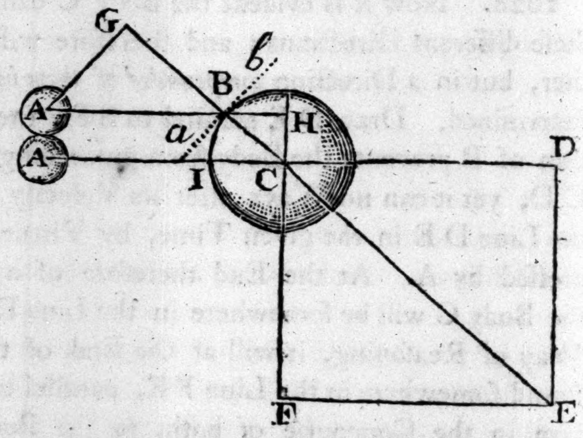
1029. Hence appears the Method of compounding a direct Force C E, out of any oblique Forces C D, and D E; and on the Contrary, of *resolving any direct Force* C E into two other oblique Forces C D and D E. Wherefore representing any two oblique Forces by the two Sides of a Parallelogram, the direct Force equivalent to them will be the Diagonal thereof. And the Truth of this Doctrine is abundantly confirmed by Experiments.

1030. Thus for Example; if the Body C be drawn with a Weight of 3 Ounces in the Direction C D, and by another of 2 Ounces in the Direction C F; then make C D to C F as 3

to 2, and compleat the Parallelogram C D E F, and draw the Diagonal C E, which will meaſure 4 upon the ſame Scale, (when the Angle D C F is of a certain Magnitude) which ſhews the Body C is in the ſame Circumſtance, as if it was drawn by a 4 Ounce Weight in the Direction C E; and this is proved true, by cauſing a Body as G, of 4 Ounces, to draw the Body C the contrary Way, *viz.* from C to G, for then the Body C will remain at Reſt, or be *in Equilibrio* with all the Forces.

1031. Hence it farther appears, that if a Body C be acted upon by three different Forces at one and the ſame Time, thoſe Forces will be to each other, as the three Sides of a Triangle C D E, which are ſeverally parallel to their Directions. This is ſo plain from what has been ſaid, that nothing more can be added.

1032. Hence we learn how to eſtimate the Quantity of any oblique Stroke. For let the Body A ſtrike the Body C in a Direction paſſing thro' its Center, as A C. Then it is certain, that it acts upon it with its whole Force; and the Stroke is ſaid to be *direct*. But if the ſame Body A ſtrikes the Body C in any Direction A B, which does not paſs through the Center C, then the Stroke is ſaid to be *oblique*; and its Force to move the Body is thus found. At the Point of impact B draw the Tangent *a b*, parallel to which draw A G to meet B C produced in G.

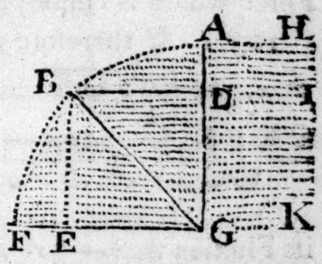


1033. Let A B repreſent the whole Force of the percutient Body A; this is reſolvable into the two Forces A G and G B (1029.) of which the Former is parallel to the Tangent *a b*, and ſo does not at all affect the Body C; but the other Force G B paſſes through the Center C, and is that alone by which the Body C is compelled to move. But A B : G B : : C B : B H (657.) That is, *The whole Force, or direct Stroke, is to the reſidual*

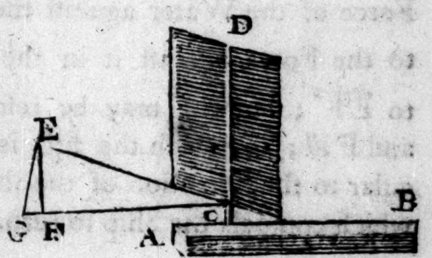
residual Force, or oblique Force, as the Radius to the Sine of the Angle of Obliquity B C H (710.)

1034. But C B : B H : : C E : E F, or C D; and therefore if in any Case C E denote the whole Force, the diminished Force or oblique Stroke will be denoted by C D; and will be greater or less, as the Angle of Obliquity F C E = B C H is so.

1035. That the extensive Use of this Doctrine of the *Composition and Resolution of Force* in *Mechanical Philosophy*, may appear, I shall subjoin the following Examples thereof in the most interesting Parts of the Science.



Thus let A G be the perpendicular Section of any Plane, as the Sail of a Windmill, &c. exposed directly to the Stream or Current of any Fluid, as Air, &c. represented by H I K; and let B G be the Section of the same Plane in an oblique Situation thereto. Then it is evident (1.) That the Number of fluid Particles which strike upon the Plane in the *direct Position* A G will be to the Number of those which fall on it in the *oblique Position* B G, as A G to D G; because all the Particles between H and I will pass by the Plane B G, and not touch it. (2.) The Force with which the Particles strike the Plane in the *direct Position*, will be to that with which they strike it in the oblique Position B G, as A G to D G (1033.) *Wherefore the whole Force of the Fluid upon the Plane in the direct Position, will be to the whole Force in the oblique one, as* $\overline{AG}^2$ *to* $\overline{DG}^2$, *or as the Square of the Radius to the Square of the Sine of the Angle of Inclination.*

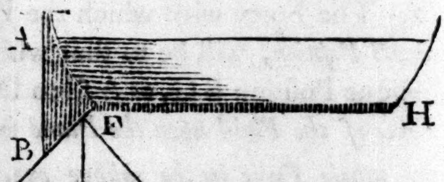


1036. Let A B be the *Axis* of the Windmill, C D one of the Sails placed in an oblique Position E C to the Direction of the Wind G C, which is parallel to the Axis A B. If then $\overline{GC}^2$ express the absolute Force of the Wind upon the Sail in a direct Position,

sition, $\overline{GE}^2$ will express the Quantity of the same Force in the oblique Position of the Sail, (because G E is the Sine of the Angle of Incidence GCE to the Radius G C;) but the Force G E is resolvable into the two Forces E F, and G F, of which the Latter being parallel to the Axis avails nothing in turning the Sail about it. But the Force E F being perpendicular thereto, is wholly spent in compelling the Sail to turn round. But the Force G E is to the Force E F as (G C to CE,) $\overline{GE}^2$ to $\frac{CE \times \overline{GE}^2}{GC}$, which therefore will express the Force which is employed to turn each Sail.

1037. If therefore we put the Radius GC $= a$, and E C $= x$, we have $\overline{GE}^2 = aa - xx$, and consequently the Force $\frac{CE \times \overline{GE}^2}{GC} = \frac{aax - xxx}{a}$; which if we make it a *Maximum*, its Fluxion $aa\dot{x} - 3xx\dot{x} = 0$, and so $aa = 3xx$ and $x = \sqrt{\frac{aa}{3}}$; which in Logarithms is $\frac{20{,}000000 - 0{,}477121}{2} = 9.761439$, the Logarithm Sine of $35° : 16'$ equal to the Angle C G E, and therefore the Angle E C G is equal to $54° : 44'$, when the Sail receives the *greatest Force* from the Wind.

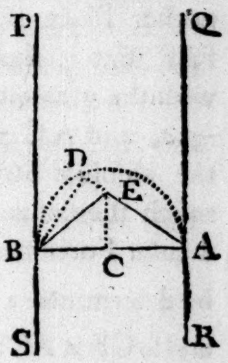
1038. If A B be the Rudder of a Ship A H, placed in the oblique Situation F C, and the Water strike against it in the Direction G C; then making C E Radius,



the Sine of the Angle of Incidence will be E F; and so the Force of the Water against the Rudder in a direct Position, is to the Force against it in the oblique Position F C, as $\overline{CE}^2$ to $\overline{EF}^2$; but E F may be resolved into the two Forces E D, and F D; of which the first is parallel, and the last perpendicular to the Direction of the Ship. Therefore F D is the Force which compels the Ship to turn. But the Force E F is to the Force F D (as C E to C F) as $\overline{EF}^2$ to $\frac{CF \times \overline{EF}^2}{CE}$; that is,

(by

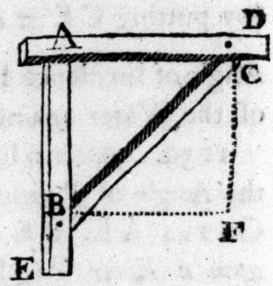
(by putting $CE = a$, $CF = x$). As $\frac{aax - x^3}{a}$; whence the Angle of Incidence $ECF = 54^\circ : 44'$ (as before) when the Force of the Water against the Rudder to turn the Ship is a *Maximum*.

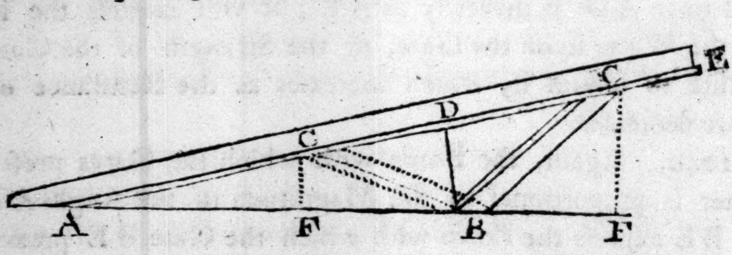


1039. After a like Manner is determined the Angle of Position $BAE = ABE$ of the Gates AE, BE of a Lock, *or* Sluice *upon a River* PQRS; such that the said Gates shall resist the Pressure of the Water with the greatest possible Force. For since the Resistance of the Gate AE diminishes in Proportion as the *Pressure of Water*, and as the Length of the Gate increases; and in the same Depth of Water, both these are as the Line AE; therefore $\overline{AE}^2$ will express the whole Resistance of the Gate AE. On the Diameter AB describe the Semicircle ADB, and continue AE to D, and draw BD and EC; then because $\overline{AE}^2 : \overline{AC}^2 :: \overline{AB}^2 : \overline{AD}^2$ by similar Triangles; and since $\overline{AD}^2$ is inversely as $\overline{AE}^2$, it will express the Force of the Water upon the Gate, or the Strength of the Gate requisite to sustain it, which increases as the Resistance of the Gate decreases.

1040. Again, the Force with which the Gates press each other is proportional to the Magnitude of the Angle AEB; let BE express the Force with which the Gate BE presses the Gate AE obliquely, this is resolvable into the two Forces DE, which is parallel to AE, and BD which is perpendicular to it; therefore that Strength of the Gate AE (equal to the Force of the Water, multiplied by the perpendicular Pressure of the Gate BE, *viz.* $\overline{AD}^2 \times BD$,) ought to be a *Maximum*. Wherefore putting $AB = a$, and $BD = x$, we have $\overline{DA}^2 = aa - xx$, and so $\overline{AD}^2 \times BD = aax - x^3$, whose Fluxion $aa\dot{x} - 3xx\dot{x} = 0$, gives $x = \sqrt{\frac{aa}{3}}$; which shews the Angle $BAE = 35^\circ : 16'$, as in the above Examples.

1041. Let A D be a Beam in the horizontal Position, supported at the End A by the upright Piece A E, and it is required to find the Angle of Position of another Piece B C, of a given Length, such that it shall support the Beam A D with the greatest Force possible. Let B C $= a$, and A C $= x$, then if B C express the absolute Strength of the Piece B C; C F will express so much thereof as supports the Beam A D, wherefore this perpendicular Force multiplied by the Distance A C, or Lever*, is to be determined a *Maximum*. Now $CF = AB = \sqrt{aa - xx}$, and so $CF \times AC = x \times \sqrt{aa - xx}$, whose Fluxion $\dot{x}\sqrt{aa - xx} - \frac{xx\dot{x}}{\sqrt{aa - xx}} = 0$; which gives $\sqrt{aa - xx} = \frac{xx}{\sqrt{aa - xx}}$; and so $x = \sqrt{\frac{aa}{2}}$, which shews the Angle A B C = 45 Degrees, or *Half a right one*.





1042. Let A E be a Beam, or Piece of Wood, so fixed in A as to make a given Angle E A B with the Horizon A B; it is required to find the Position of another Piece B C, of a given Length, such that it shall support the Beam A E with the greatest possible Force. From C let fall the Perpendicular C F; and because the Angle at A is given, the Ratio of C to A F is also given, which let be as n to m; and put $BC = a$, and $BD = x$; then if B C express the absolute Strength of the Piece B C, B D will express so much thereof as supports the Beam A E. Now as $CF : AF :: BD : AD = \frac{m}{n}x$, and D C =

* *N. B.* What relates to the Distance A C, considered as a Force derived from the Lever, will be explained in the next Chapter.

$= \sqrt{aa - xx}$, whence $AC = \frac{m}{n} x \pm \sqrt{aa - xx}$, which Distance, or Lever AC multiplied by DB must be a *Maximum, viz.* $\frac{m}{n} xx \pm x\sqrt{a^2 - x^2}$, whose Fluxion $\frac{2m}{n} x \dot{x} \pm \dot{x}\sqrt{aa-xx} - \frac{x^2 \dot{x}}{\sqrt{aa-xx}} = 0$, whence $\frac{2m}{n} x \pm \sqrt{aa-xx} - \frac{xx}{\sqrt{aa-x}} = 0$, and by Reduction we shall have $x = \sqrt{\frac{1}{2} aa \pm am}$, which will determine the Angle of Position ABC.

Note, The Equation above requires the acute Angle ABC to be the Compliment of this obtuse one ABC to two right Angles.

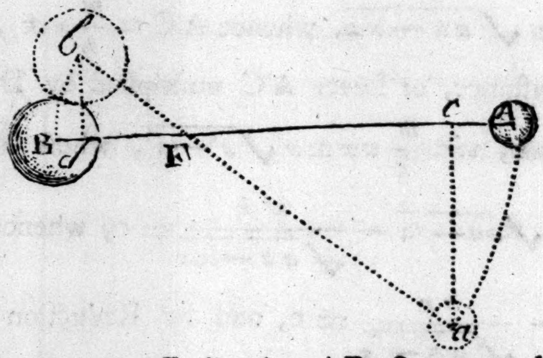
1043. If we put this Value of x into the Equation $\pm x\sqrt{aa-xx} + \frac{m}{n} xx$, we shall have $AC \times BD = \frac{aam + 2amm + 2ann}{2n}$, when the Angle is obtuse; and $\frac{aam + 2amm - 2ann}{2n}, = AC \times BD$, when the Angle is acute; whence 'tis evident, the Position of the Piece in the first Case is much more advantageous than in the last.

CHAP. V.

The Application of the foregoing PRINCIPLES *to such* MACHINES *as are called* MECHANICAL POWERS.

1044. HAVING considered the Laws of Motion and moving Forces, under the Circumstances of *Collision* and *Percussion*; we now proceed to consider those Forces which are otherwise applied; particularly in such Cases where they become subservient to all the Purposes of *moving heavy Bodies*, and *overcoming Resistances*. And as this is effected by Machines properly applied, so this Part of Philosophy has acquired the Name of MECHANICS.

1045.



1045. The Force of Bodies acting upon each other, either immediately, or by Means of a Machine, is still derived from the same Principles of *Matter* and *Velocity* (968, 969.) When therefore any two Bodies A and B act upon each other, by the Interposition of the inflexible Rod A B (the most simple of all Machines,) 'tis easy to assign the Quantity of Motion in each, while they move about any given Center of Motion F.

1046. For suppose the Rod AB to be moved out of its *horizontal Position* into the *oblique one* a F b, then will the Space described by A be the Arch A a, and that described by B be the Arch B b. These Spaces then as they are described in the same Time, by the similar Motions of A and B, will be as the Velocities of those Motions (by 971.) But since the Sectors A F a, B F b, are similar, it will be A a : B b : : A F : B F (657.) Therefore the Velocities of A and B will be denoted by their Distances from the Center of Motion, *viz.* A F and B F.

1047. If the Bodies be homogeneous, or of the same Kind, their Quantities of Matter will be as their Bulks A and B (973,) and since their Velocities are as A F and B F; therefore the *Momentum*, or Quantity of Motion in A will be as the Rectangle A × A F; and that of B will be as B × B F (by 970.) The Expressions of their Forces as required.

1048. This Theory is general, and holds good for every Sort of Motion, which Bodies so circumstanced are capable of. But that Sort of Motion which results from the Action of Gravity, ought to have the Velocity expressed by the perpendicular Spaces $a\,c$ and $b\,d$, by which one accedes to, and the other recedes from the Center of the Earth. Because Gravity acts in those Directions only (985,) and therefore its Force must be estimated thereby in given Quantities of Matter. But because of similar Triangles $a\,c$ F, and $b\,d$ F, it is $a\,c : b\,d : : a$ F : b F : :
A F

A F : B F. Whence it appears that $A \times ac$, and $B \times bd$, are the same *Momenta* as those above (1047.)

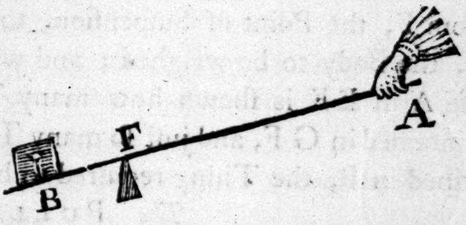
1049. Hence for Bodies in Motion there is a *threefold* Expression of their Forces, as follows:

$$\text{For A}\begin{cases}1.\ Q = A \times Aa \\ 2.\ Q = A \times AF \\ 3.\ Q = A \times ac\end{cases} \qquad \text{For B}\begin{cases}Q = B \times Bb \\ Q = B \times BF \\ Q = B \times bd.\end{cases}$$

1050. If we suppose the Forces in the two Bodies equal, that is, $A \times AF = B \times BF$, or $A \times Aa = B \times Bb$, or $A \times ac = B \times bd$, then it is, $A : B :: BF : AF :: Bb : Aa :: bd : ac$. Therefore in case of an *Equilibrium*, the *Bodies are inversely as their Distances from the Center of Motion*, or as the *circular Arches*, or the *perpendicular Spaces* described in the same Time, and this is the *fundamental Principle* of every mechanical Power, Machine, or Process whatsoever.

The Lever.

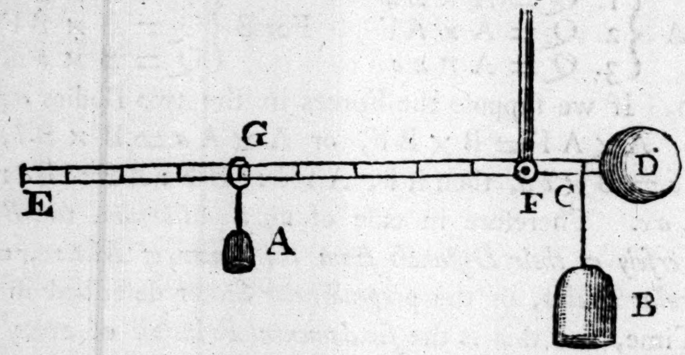
1051. To apply this Doctrine. Suppose B a very heavy Weight laid upon the End of a long Pole or Rod A B, sustained by, and moveable



upon the Prop, or *Fulcrum* F; and instead of the Weight A, suppose a Person's Hand were applied at the End of the Lever A, to raise or move the Weight B. Then the *muscular Force* of the Arm is now to be compounded with the Velocity of Motion to constitute a Force equivalent to that of the Weight B. And since, in such a Case, the Force of the Arm is to the Weight of the Body B, as B F to F A, it is evident, that though the Weight B encreases in any Proportion, and the intensive Force of the Arm remains the same, yet by taking the Distance A F to B F, in the same Proportion in which B is encreased, the Person will still have it in his Power to move the Body B. And this will be the Case in what Manner or Form soever the Lever be applied.

1052. From what has been said, the Nature of the Balance must fully appear; for this is nothing more than a *Lever*, whose *Brachia*, or Arms A F, B F are equal; in which Case the Weights A and B appended at each End will have

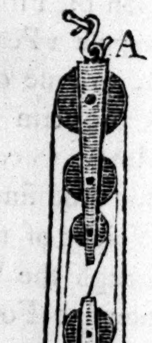
equal Velocities; and therefore, in Case of an *Equilibrium*, the Weights will be equal also (by 1047.) But the Weights of Bodies are as the Quantities of Matter (1002,) whence *equal Quantities* of any Sorts of Matter are easily determined by the *Equilibrium* of the *Balance*, which is its *only Use*.



1053. The STEELYARD is evident nothing but a *suspended Lever*, where the Weight A is applied at different Distances from F, the Point of Suspension, to make an *Equilibrium* with B, the Body to be weighed; and when this happens, then on the Arm E F is shewn how many Times the Distance C F is contained in G F, and just so many Times is the Weight A contained in B, the Thing required to be found *.

The PULLEY.

1054. A Tackle of Pullies is another mechanical Power, which Action or Force is deriv'd from the general Principles of (1045, 1046, 1047.) For let W be a Weight to be raised (by Means of the Tackle of five Pullies A B) by the Power, or Weight P. Then it is evident, that when the said Weight W is raised one Inch, each Rope belonging to the lower moveable Box of Pullies will be shortened one Inch, and the Rope to which the Power P is appended will be lengthened just so many Inches; consequently, the Spaces passed through in the same Time, and therefore the Velocities of the Bodies W and P will be to each other *as Unity to the Number of Ropes belonging to the*

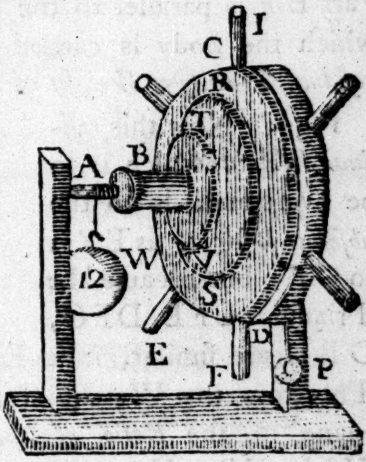


* *Note*, The Beam of the Steelyard is supposed to be without Weight in what has been said; and in Practice, the Weight of the Steelyard is compensated by a large Weight D; and further it is to be

the lower Sheave of Pullies, which is here as 1 to 5; therefore by such a Structure of Pullies the Force of P is encreased five Times. And in this Manner you compute the Force of every other Form or Construction of Pullies.

The WHEEL *and* AXLE.

1055. The *Axis in Peritrochio*, or WHEEL and AXLE, is another *mechanical Machine*, in which the Power P hanging from the Periphery of the Wheel C D, acts against the Weight W hanging from the Axis at A; and it evident when the Wheel moves, the Spaces described by the Bodies P and W, and consequently their Velocities, will be as the Peripheries of the Wheel and of the Axle, and these are as the Diameters (823;) therefore in case of an Equilibrium, we have the Power P : the Weight W : : Diameter of the Axis : the Diameter of the Wheel; which in the Fig. is as 1 to 12; and thus the Force of P is encreased 12 Times. Hence as the Peripheries R S, T V, are less, the Effect of the Machine is also less in Proportion: And on the other Hand, the Force of the Machine is encreased by Means of the Spokes I F, in Proportion as the Distance I F is greater than C D, as is extremely obvious from (1047.)



The INCLINED PLANE.

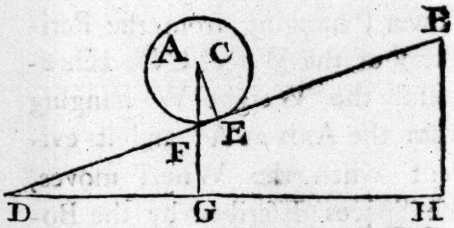
1056. The INCLINED PLANE does not (like other Machines) become a mechanical Power, by encreasing the Velocity of the Agent, but *by diminishing the absolute Weight of the Body to be moved*; which it does by its Resistance or Re-action, thus; let the Body A lie on the Inclined Plane B D; and from the Center C let fall the Perpendicular C G to the Base of the Plane H D, cutting the Plane in F; also draw C E perpendicular to the Plane B D. Now since the Body gravitates to-

 wards

be observed, that in Practice we do not regard the Resistance of the Air to the Weight B, by which its real Weight will be a small Matter diminished, as we shewed in our *Treatise on* AIR, in the YOUNG GENTLEMAN *and* LADY'S PHILOSOPHY.

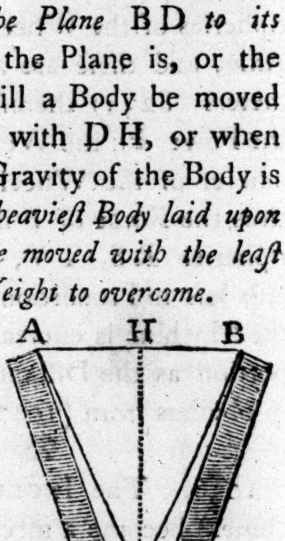
wards the Earth in the Direction C G, let C F represent the Whole, or absolute Gravity of the Body. This Force is resolvable into two others, *viz.* C E and E F (1027,) of which C E is that by which it acts on the Plane directly, and is destroyed by the equal Re-action of the Plane (965,) the other Part E F is parallel to the Plane, and consequently, that by which the Body is carried down the Plane, and is called the *residual Gravity or Weight* of the Body.

1057. Now this *residual Weight* which is to be overcome, is to the *absolute Weight*, as E F is to C F; but because the Triangles CFE, DFG, DBH are similar, it is EF : CF : : FG : FD : : BH : BD.



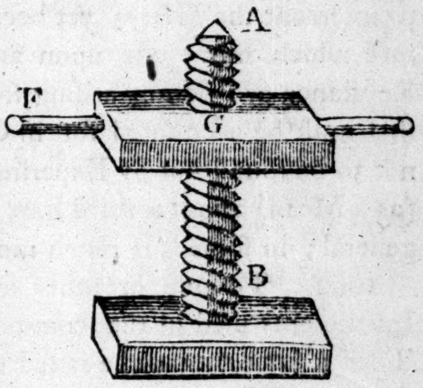
Therefore the Weight of the Body A is diminished by the Plane in the *Ratio of the Length of the Plane* B D *to its Height* H P. Hence the more inclined the Plane is, or the less the Angle at D, the more easily will a Body be moved thereon. Therefore when B D coincides with D H, or when the Plane becomes horizontal, the whole Gravity of the Body is destroyed; and hence it appears that *the heaviest Body laid upon an horizontal Plane (perfectly smooth) may be moved with the least Force*; as having in that Case *no residual Weight to overcome.*

1058. The *Wedge* has been hitherto, by all Writers, reckoned among the *simple mechanical Powers*; but when it is considered that any *Wedge* A C B is only a *double inclined Plane*, or composed of the *two single Ones* A C H and B C H, it will easily appear *that the Wedge is the same Thing with the inclined Plane*; for to *double any Thing* makes no Alteration in its Nature or Properties, and consequently cannot make it a *different Species*; this Sort of Logic does not become Mathematicians whose Characteristic it is to use the *justest Method of Reasoning.* Besides, it is idle to pretend that the Force which overcomes the Cohesion of Wood is derived from the *Wedge*, when it is so evidently



dently no other than the *Momentum* of the *Bittle*, or *Sledge* which drives it into the Wood; the *Wedge* being used as a *convenient Instrument* only, to keep the dissevered Parts of the Wood, *&c.* asunder. Therefore we take the Liberty of discharging the *Wedge* from the Office and Rank of a *mechanical Power*.

1059. We must also treat the *Screw* with the same Freedom, as it is so evidently nothing more or less than an *inclined Plane of a spiral Form* applied to a Cylinder A B, and 'tis as plain that whatever moves up and down upon a Screw moves all that while upon an *inclined Plane* in a *circular* instead of a *rectilineal Direction*.



In this *Screw-plane* the Length is the Circumference of the Cylinder, and the Height the Distance between two nearest Threads or *Helices*, which is seldom in greater Proportion than 10 to 1, a Trifle to mention for a *mechanical Power*. The Lever F G added to the *Screw-plane*, make a *compound mechanical Power* of very great Force and Use, for Motion, Power, Compression, *&c.* as is too well known to be farther insisted upon. Here the *Power is to the Force as the Distance of the Helices to the Circumference described by the End* F *of the Lever* F G, which may be encreased at Pleasure. We therefore reduce the *Number of simple mechanical* Powers to *four* only, making the most of them at the same Time; for it is certain the *Lever*, *Pulley*, and Axis *in Peritrochio*, differ only in Form, and not in their Properties whence their Power is derived, which is the same in them all, as we have shewn. We therefore refer it to the *Metaphysician* to determine if there be really any more than *two absolutely different mechanical Powers*, viz. the *Lever* and the *inclined Plane*.

1060. It is also, on the other Hand, a Wonder that we meet with nothing in our *mechanic Treatises*, on the Subject of an *Arch* considered as a *mechanical Power*, as its Nature and the common Use we make of it seems to entitle it to that Denomination, and especially that the *Catenarian Curve* should be passed

by

by in ſuch general Silence, when it is a Subject that merits in the higheſt Degree the Conſideration of every *Mechanic*, *Architect*, or *Engineer*, as we have heretofore obſerved *, and may more largely ſhew in a future Part of this Work.

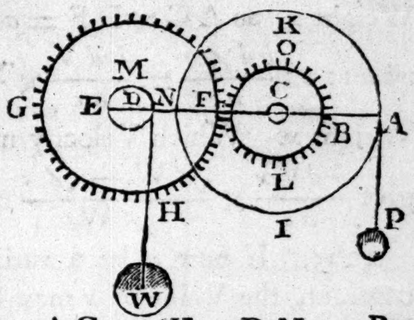
1061. In the *ſimple mechanical Powers*, I have ſuppoſed no *Friction*, or Impediment thence ariſing to the moving Power, to interrupt the *Theory*; yet becauſe there are no Bodies in Nature which move one upon another without ſome Degree of Reſiſtance or Friction ariſing from the Roughneſs of the Parts, this muſt be accounted for in Calculation of Forces, and it is not to be found but by Experiments, by which it appears to be (at a Mean) about a third Part of the Weight in Machines in general; in ſome it is much more, and in others leſs.

1062. To one or other of theſe Machines, moſt of the Inſtruments uſed in the common Affairs in Life are reducible. Thus a LADDER to be raiſed up upon one End, is one Sort of *Lever*; a WHEEL-BARROW is another; as alſo the CROW, for forcing up Shrubs and ſmall Trees by the Roots. The HAMMER applied in drawing a Nail is a third Sort; the SCISSARS, and SHEARS act on the ſame Principle. The CAPSTAN and WINDLASS, is the ſame with the *Wheel and Axle*; the JACK for raiſing up Bodies is the ſame in Effect. The KNIFE and the AX are both in the Nature of the *inclined Plane*; and every Body that acts with any Kind of Force upon another, will be found when rightly conſidered, to do it on the Principles above explained.

1063. As to compound Machines, their Force or Power may eaſily be computed, and the Reaſon of their Effects may be clearly underſtood, from the Nature of the ſimple Machines of which they are compoſed. Or thus, let the Machine be ever ſo complicated, conſider the *Velocities of the Motion of the Power and Weight*, and their Ratio will expound the Force of the Engine, as well as in any of the foregoing ſimple Machines. For the Principle of (1045, *&c.*) holds good of all Kinds of Motion whatſoever. But leſt we ſhould be thought deficient in not giving ſome further Account of compound Engines, the following general Theory is ſubjoined.

1064.

* See *Miſcellaneous Correſpondence* for *July*, 1756, and the Plate of a *Bridge* there conſtructed on ſuch Arches.



1064. Suppose the Power P by Means of the Wheel A K I and its Pinion B L O, sustain *in Equilibrio* the Weight W, hanging from the Axis E M N of the Wheel G F H; then putting A C $= a$, C B $= b$; D F $= c$, D $N = d$; we have (by 1045, 1046, 1047,) $\frac{P \times AC}{CB} = \frac{W \times DN}{DF}$, or $\frac{Pa}{b} = \frac{Wd}{c}$, and therefore $Pac = Wbd$. Or thus, let (w) be the Weight equivalent to P on the Wheel A K I, and its Axle B L O, then $P : w :: b : a$; therefore $\frac{Pa}{b} = w$. Again, let w be now considered as a Power with Regard to W on the Wheel G F H, and its Axle D N, then $w : W :: d : c$; and therefore $w = \frac{Wd}{c} = \frac{Pa}{b}$, as above. Now since the Machine is at Rest, the Point A is urged with the whole Power of Gravity or *Momentum* of P; but if the Weight W be diminished to a lesser Weight x, the Machine will move, and the Point A will have a Motion conspiring with that of P. Let $V =$ Velocity with which the Point A endeavoured to descend with the Gravity of P, and $v =$ Velocity of the Point A arising from the Diminution of the Weight W; then will $V - v$ be the relative Velocity with which it would endeavour to descend by the same Power P, counterpoising the Weight x at Rest. But the Effects of the Power P with the Velocities V, and $V - v$, in a given Time, will be as the Spaces through which it would descend in that Time, which are as the Squares of those Velocities, *viz.* as V^2 to $\overline{V - v}^2$. And these Effects are as W and x, who counter-act them; therefore since $V^2 : \overline{V - v}^2 :: W : x$, we have $V^2 x = W \times \overline{V - v}^2$, and $V\sqrt{x} = \sqrt{W} \times \overline{V - v} = V\sqrt{W} - \sqrt{W} \times v$; whence $v = \frac{V\sqrt{W} - V\sqrt{x}}{\sqrt{W}}$. But when the Machine is in Motion, the Velocity of the Power P, or v, is to the Velocity of the

Weight x, as $AC \times DF = ac$, to $CB \times DN = bd$; or $ac : bd :: v : \frac{bd \times v}{ac} = \frac{bdV}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}}$ = Velocity of the Weight x. Which Velocity multiplied by the Weight x, will give $\frac{bdVx}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}}$ = Effect of the Machine.

1065. If now x be a variable Quantitity, and all the rest constant, the Value of x may be determined, when the above Expression of the Effect of the Machine is a *Maximum*, or the greatest possible, by putting its Fluxion equal to nothing. Now the Fluxion of $\frac{bdV}{ac\sqrt{W}} \times x \times \overline{\sqrt{W} - \sqrt{x}}$, is $\dot{x} \times \overline{\sqrt{W} - \sqrt{x}} - \frac{x\dot{x}}{2\sqrt{x}} = 0$, whence we have $\sqrt{W} - \sqrt{x} = \frac{x}{2\sqrt{x}}$, and so $2\sqrt{Wx} = 3x$, and $x = \frac{4}{9}W$. *Hence if any Engine be charged with $\frac{4}{9}$ of such a Charge or Weight as will just keep it in Equilibrio, it will produce the greatest Effect possible.*

1066. Hence, if instead of x, we substitute its Equivalent $\frac{4}{9}W$, we have $\frac{bdVx}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}} = \frac{4bdVW}{27ac} = \frac{4VP}{27}$ (because $P = \frac{Wbd}{ac}$) when the Engine is in its greatest Perfection. Also putting $\frac{4}{9}W$ for x in $v = \frac{V\sqrt{W} - V\sqrt{x}}{\sqrt{W}}$, we have $v = \frac{1}{3}V$. Lastly, if we divide $\frac{4bdVW}{27ac}$ by $\frac{4}{9}W$, we have $\frac{bdV}{3ac}$ = Velocity of the Weight x, when the Machine is in its utmost Perfection.

1067. If the Power P instead of a Weight, as here represented, were a Current of Water, Air, &c. then is V the Velocity thereof $= V - v$, the Difference or relative Velocity with which the Fluid strikes the Floats of the Wheel. Whence 'tis easy to apply the foregoing Calculus to any mechanical Engine whatsoever, as we shall more particularly shew under the Subject of *Hydraulics*, in the Theory of *Mill-work*.

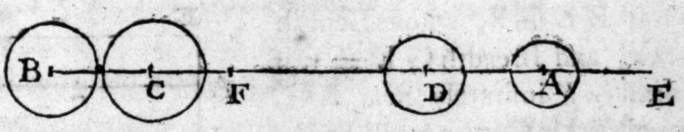
1068. The Velocity here mentioned is supposed to be that of an *uniform Motion*; for though all Machines actuated by a constant

constant Power will have their Motion at first accelerated for some Time; yet will the Increments of Velocity continually decrease by the Friction, or Resistance of the several Parts, 'till at last they become totally destroyed, and the Motion, by that Means, rendered equable and uniform.

CHAP. VI.

The Method of investigating the CENTER OF GRAVITY *in Bodies.*

1069. THAT Point F, which in the Lever was called the *Fulcrum*, (see Fig. to Art. 1051,) is otherwise called the *common Center of Gravity*; for as in every single Body there is one common Point which tends to the Center of the Earth, with the united Forces of all the gravitating Particles which compose that Body, and which therefore is called its *Center of Gravity*; so in any System of two or more Bodies, which are any how connected or depend on each other, there is one certain Point in which their whole Forces of Gravity are united, and which being suspended, keeps the Bodies *in Equilibrio*, such as is the Point F with respect to the Bodies A and B, and it is therefore their *common Center of Gravity*.

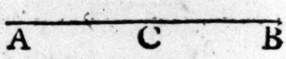


1070. If to the Bodies A and B, we suppose two others added, as C and D, at such Distances from F, that C : D :: DF : CF; then $C \times CF = D \times DF$, (by 1047,) and because it is also $A \times AF = B \times BF$; therefore these Equations added together, make $A \times AF + D \times DF = B \times BF + C \times CF$. Thus it appears the Point F is the common Center of Gravity of all the Bodies.

1071. Hence a general Rule for finding the Distance of the common Center of Gravity from any given Point E in a right

Line E B passing through all the Centers of the Bodies A, B, C, D. Thus put $EA = a$, $ED = b$, $EF = x$, $EC = c$, $EB = d$, then $AF = x - a$, $DF = x - b$, $CF = c - x$, $BF = d - x$. Whence the general Equation above (1070) will become $\overline{A\,x - A\,a} + \overline{D\,x - D\,b} = \overline{B\,d - B\,x} + \overline{C\,c - C\,x}$. And by transposing the Terms we have $A\,x + B\,x + C\,x + D\,x = A\,a + B\,d + C\,c + D\,b$. Consequently $x = \frac{A\,a + B\,d + C\,c + D\,b}{A + B + C + D}$ $= EF$ the Distance required.

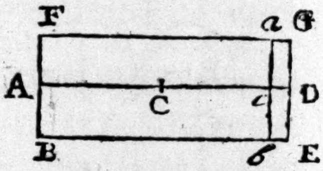
1072. The general Rule therefore for finding the Distance of the Center of Gravity from the extreme Part of any Body, is this, *divide the Sum of all the Momenta by the Sum of all the Weights, and the Quotient will be the Distance required.* Hence in a right Line A B we may consider all the Particles which compose it, as so many very small Weights, each $= \dot{x}$,



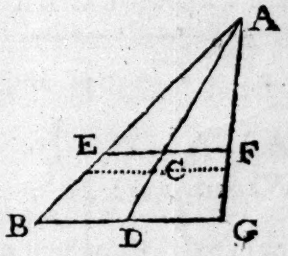
which is therefore the Fluxion of the Weights, or Line $AB = x$. Therefore the Weight $\dot{x}$ multiplied by its Distance from A, *viz.* x, is $x\,\dot{x}$, its *Momentum*; that is, $x\,\dot{x}$ is the Fluxion of all the *Momenta* in the Line A B; whose Fluent $\frac{1}{2}\,x\,x$ is the Sum of all the *Momenta*, which divided by the Sum of all the Weights x, gives $\frac{1}{2}\,x = \frac{1}{2}\,AB$, the Distance of the Center of Gravity C from the Point A.

In a PARALLELOGRAM.

1073. Thus in the PARALLELOGRAM B E G F, whose Length $AD = x$, and Breadth $GE = y$, if you draw $a\,b$ infinitely near E G, the Areola $a\,b\,E\,G = y\,\dot{x}$ will be the Fluxion of all the Weights y, which make the whole Weight of the Parallelogram, which multiplied by the Distance $AD = x$, gives $y\,x\,\dot{x}$, the Fluxion of the *Momenta*; whose Fluent $\frac{1}{2}\,x\,x\,y$ is the Sum of all the *Moments*, which divided by the Sum of all the Weights $(x\,y)$, quotes $\frac{1}{2}\,x = \frac{1}{2}\,AD = AC$, the Distance of the common Center of Gravity from A, as required.

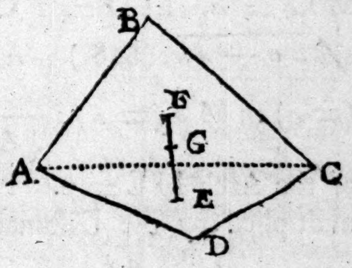


In a TRIANGLE.



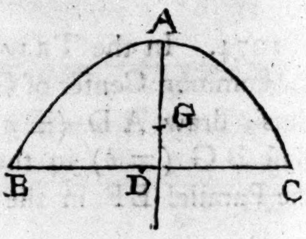
1074. In the TRIANGLE ABG, the common Center of Gravity is found thus; draw AD ($=a$) to bisect the Base BG ($=b$) in the Point D, and the Parallel EF in the Point C. Put AC $=x$, and we have $a : b :: x : \frac{bx}{a}$ $=$ EF. Which (as a *Weight*) multiplied by $\dot{x}$, gives $\frac{bx\dot{x}}{a}$ $=$ Fluxion of the Weights; this again multiplied by $x =$ AC (the Distance from A) gives $\frac{bxx\dot{x}}{a}$ $=$ Fluxion of the *Momenta*; whose Fluent or Sum of the Moments $\frac{bx^3}{3a}$ divided by the Fluent of the Weights $\frac{bx^2}{2a}$, quotes $\frac{2}{3}x = \frac{2}{3}$ AC, for the Distance of the Center of Gravity from A in the Triangle AEF; and when $x =$ AD, then $\frac{2}{3}$ AD gives the same for the Triangle ABG.

In a TRAPEZIUM.



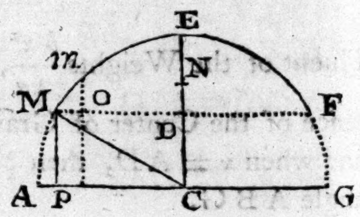
1075. To find the Center of Gravity G of the *Trapezium* BD. Let the same be divided into two Triangles ABC and ACD, and find their Centers of Gravity F, E (448.) Join EF, which divide in such a Manner in the Point G, that it may be FG : GE :: ADC : ABC, which is done by this Analogy, ABCD : ACD :: EF : FG, and the Point G is determined as required. This is evident from (1050, and 648.)

In the PARABOLA.



1076. The Center of Gravity in a PARABOLA BAC. Let AD $= x$, and BC $= 2y = z$; then will $px : zz$ (740;) and putting $p = 1$, it is $x : zz$, and $z : \sqrt{x}\,(= x^{\frac{1}{2}})$. Whence $z\dot{x} : \dot{x}x^{\frac{1}{2}}$, the Fluxion of the Weights, which multiplied by x gives $\dot{x}xx^{\frac{1}{2}}$, the Fluxion of the *Momenta*; whose Fluent $\frac{2}{5}x^{\frac{5}{2}}$ divided by the Fluent of $\dot{x}x^{\frac{1}{2}}$, *viz.* $\frac{2}{3}x^{\frac{3}{2}}$, gives $\frac{3}{5}x = \frac{3}{5}$ AD $=$ AG, for the Distance of the Center of Gravity from the Vertex A.

In the ARCH *of a* CIRCLE.



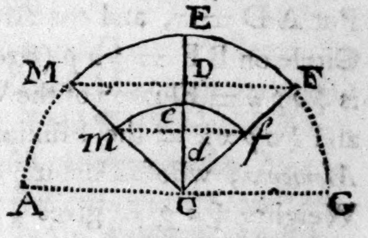
1077. *To find the Center of Gravity* N *of the Arch of a Circle* MEF *fixed to the Radius* CE. 'Tis evident the Particles M, F, equidistant from E, have their common Center of Gravity at D, in the Radius CE; and since the same is true of all the other Particles, it is manifest the common Center of Gravity of the whole Arch MEF is somewhere in the said Radius CE. Put MC $= a$, MD $=$ PC $= x$; then PM ($=$ DC) $= \sqrt{aa - xx}$ (828.) And PM $(= \sqrt{aa - xx})$: MC $(= a)$: : Mo $(= \dot{x})$: $\frac{a\dot{x}}{\sqrt{aa - xx}} =$ M$m = \dot{z}$, the Fluxion of the Arch ME $= z$. Now the Fluxion of the Weights $\dot{z}$ multiplied by the Distance of the Center of Gravity CD $=$ PM, gives $\frac{a\dot{x}}{\sqrt{aa - xx}} \times \sqrt{aa - xx} = a\dot{x} =$ Fluxion of the *Momenta*; the Fluent of which, *viz.* $ax =$ MC$\times$MD, divided by the Weights or Arch ME, gives $\frac{ax}{z} = \frac{\text{MC} \times \text{MD}}{\text{ME}}$ $=$ CN, the Distance from C required.

In a QUADRANT.

1078. Hence when the Arch E M becomes the QUADRANT A E then $\frac{ax}{z} = \frac{MC^2}{AE} = CN$, the Distance of the Center of Gravity N of the Semicircle A E G. If C M = 1, then will the Quadrant A M E = 1,57079, (see 826,) and $\frac{MC^2}{AE} = 0{,}6366 = CN$, the Distance of the Center of Gravity from the Center of the Circle C.

In the SECTOR *of a* CIRCLE.

1079. To find the *Center of Gravity N*, of the SECTOR of a Circle M C F. Describe the Arch mef, and draw the Chord mf. Put C E = a, M E = z, C e = x; then $a : z :: x : \frac{zx}{a} = me$; and



putting M D = b, we have $a : b :: x : \frac{bx}{a} = md$. Now $\frac{md \times Ce}{me} = \frac{bx}{z}$, is the Distance of the Center of Gravity of the Arch mef (1077.) And the Arch $me = \frac{zx}{a}$ multiplied by $\dot{x}$, gives $\frac{zx\dot{x}}{a}$ = the Fluxion of the Weights, which multiplied by the Distance of common Center of Gravity $\frac{be}{z}$, gives $\frac{bxx\dot{x}}{a}$ = Fluxion of the *Momenta*, whose Fluent $\frac{bx^3}{3a}$ divided by the Fluent of the Weights $\frac{zxx}{2a}$, gives $\frac{2bx}{3z}$, (or when $x = a$) $\frac{2ba}{3z}$ = Distance required.

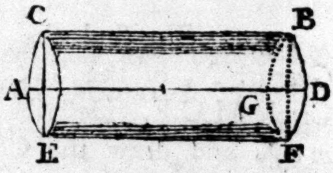
In

In a Semicircle.

1080. Hence when the Arch MEF becomes a *Semicircle* AEG, we ſhall have $\frac{2AC\times CE}{3AE}$ = Diſtance of the common Center of Gravity of the Semicircle AEG, from the Center C. If Radius AC = 1, then that Diſtance will become $\frac{2}{4.71237}$ $= 0.424 +$.

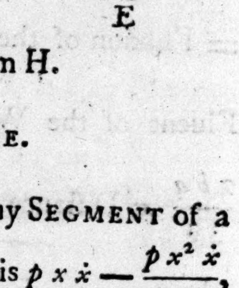
In a Cylinder.

1081. To find the *Center of Gravity* in the Cylinder ECBF. Put AD = x, and the Area of the Circle on BF = $\frac{1}{2}ap$ (830;) then is $\frac{1}{2}ap\dot{x}$ = Fluxion of the Weights; and $\frac{1}{2}apx\dot{x}$ = the Fluxion of the *Momenta*; whoſe Fluent $\frac{1}{4}apxx$, divided by the Fluent of the Weights $\frac{1}{2}apx$, gives $\frac{1}{2}x = \frac{1}{2}$AD = the Diſtance of the ſaid Center from A.



In a Cone.

1082. To find the *Center of Gravity* in a Cone AHB. The Fluxion of the Cone (or Weights) is $\frac{pax^2\dot{x}}{2bb}$ (834,) and the Fluxion of the *Momenta* $\frac{pax^3\dot{x}}{2bb}$; whoſe Fluent $\frac{apx^4}{8bb}$ divided by $\frac{apx^3}{6bb}$, will quote $\frac{3}{4}x = \frac{3}{4}$HC, the Diſtance of the Center from H.



In the Segment *of a* Sphere.

1083. To find the *Center of Gravity* in any Segment of a Sphere aDb. The Fluxion of the Segment is $px\dot{x} - \frac{px^2\dot{x}}{2a}$,

(ſee

(see Fig. to Art. 836,) and the Fluxion of the *Momenta* will therefore be $p\,x^2\,\dot{x} - \frac{p\,x^3\,\dot{x}}{2\,a}$, whose Fluent $\frac{1}{3}\,p\,x^3 - \frac{1}{8}\,\frac{p\,x^4}{a}$ divided by the Fluent (of the Weights) $\frac{1}{2}\,p\,x^2 - \frac{1}{6}\,\frac{p\,x^3}{a}$, quotes $\frac{5}{8}\,x =$ Distance of the Center of Gravity from D.

In an Hemisphere.

1084. Hence when $x = a$, or $D\,c = D\,C$, that is, when the Segment becomes the *Hemisphere* A D B, then the Distance of the Center of Gravity will be $\frac{5}{8}$ of C D from the Point D, or $\frac{3}{8}$ C D from the Center C. In like Manner you will find the Distance of the Center of Gravity in the Semi-Spheroid A B D to be $\frac{3}{8}$ of C B from the Center C.

1085. From this Method of *finding the Center of Gravity* in Bodies, there results a *general Rule or Canon* for finding the superficial and solid Content of Bodies, *viz. The Periphery described by the Center of Gravity, multiplied into the generating Line or Plane, is ever equal to the Superficies or Solid generated by the Rotation of the said Line or Plane about an Axis.* For Example; the Distance of the Center of Gravity in a Semicircle is $\frac{8}{3}\,\frac{a\,a}{3\,p}$ (1080.) And it is $a : p :: \frac{8\,a\,a}{3\,p} : \frac{8}{3}\,a =$ Periphery described by the Center of Gravity. Now the generating Plane or Semicircle is $= \frac{a\,p}{4}$ (830,) then $\frac{8}{3}\,a \times \frac{p\,a}{4} = \frac{2}{3}\,p\,a\,a =$ Solidity of the Sphere, (by 836, 837.)

CHAP.

CHAP. VII.

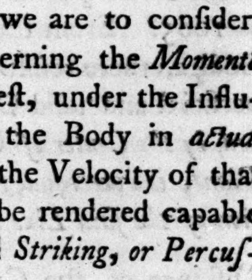
The Method of computing the CENTER *of* FORCE, *or* PERCUSSION, *in* moving *Bodies.*

1086. THERE is nothing of greater Consequence to be rightly understood in all mechanic Arts, than the Doctrine of *Momenta and Forces* of moving Bodies. The Former of which we have treated of; and the Latter, *viz.* the *Forces of Bodies* in Motion, and what is called the *Center of Force, or Percussion* in a *striking Body*, comes now to be considered.

1087. In order to render the Idea, or Notion of this percussive Force as natural and easy as possible, we are to consider, that what has been hitherto delivered concerning the *Momenta* of Bodies, respects them in a State of Rest, under the Influence of *Gravity*; but when we consider the Body in *actual Motion*, there will another Force arise from the Velocity of that Motion; by which Means the Body will be rendered capable of acting upon another, in the Mode we call *Striking, or Percussion.*

1088. This *percussive Force*, therefore, arises from three Sources, *viz.* (1.) From the *Mass of Matter* in the percutient Body. (2.) From its *gravitating Force*, in regard to its *Distance from the Center of Motion*; and (3.) From the *actual Velocity of the Motion* itself. The two first of these make the *Momentum*; and this compounded with the Latter, constitutes the *percussive Force.*

1089. Thus, suppose any uniform Rod A B, were to move or vibrate about the Point or Axis A; let it first be suspended in the horizontal Position A B at the Extremity B; then will every Particle D, C, B have a Tendency to descend in Proportion to the *Quantity of Matter in* each (968;) this Tendency will be farther augmented in proportion to the Distance from the *Center of Motion* (1047, &c.) and this makes the *Momentum* of the Particle at D, C, and B. Lastly, if the Rod



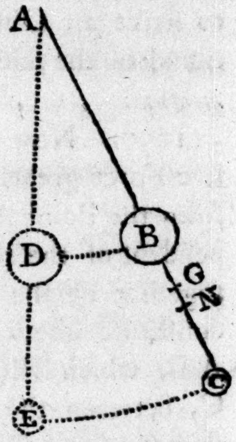
Rod A B be left at Liberty, it will commence Motion, and the actual Velocity of the Particle D, C, or B will enable it to strike an Obstacle at G, F, or E, with a Force proportional to the said Velocity, conjointly with the respective *Momentum*.

1090. Now though each Particle in the Rod has a percussive Force greater on all these Accounts, as it is more remote from the Point A, if we consider it singly in itself, and independent of the rest; yet when we consider them all connected together by the Force of Cohesion, their several Forces will constitute one compound Force of the whole Line or Rod A B, which will not be greatest at B, but in some other Point C, between the Extremes A and B; and this Point C will strike a fixed Obstacle at F in such a Manner, that the whole Force of the said Rod will be exerted upon it, and by the equal Re-action of the Obstacle, it will be destroyed (965,) and so the Rod in the Position A E, will be motionless at the Moment of the Stroke, though disengaged from the Point A. And this Point C is therefore called the *Center of Percussion* in the Rod A B.

1091. In order to this, it is necessary this Point C should have the whole percussive Force on each Side, in the Parts A C and C B, equal; for if it were greater in the Part A C, than in C B, the Part A C would, after the Stroke, move forwards, not being counteracted by an equivalent Force in the Part C B; also if the Force in the Part C B were supposed superior to that in A C, then it would continue to move forwards also after the Stroke, and so the Motion of the whole Rod would not be spent upon the Obstacle, as it is when the percussive Force is greatest of all.

1092. Now since when this percussive Force is not equal, the Rod (supposed disengaged from the Point A at the Time of the Stroke) must turn upon the Obstacle as a Center of Motion, it follows, that the Force of each Particle on each Side of the said Center, or Obstacle, will be as its *Momentum* multiplied into its Velocity, or Distance from that Point (1089,) about which it then vibrates; and that therefore *the Momenta of the Particles must be reciprocally as their Distances from the Point*

(C) *which shall be the Center of Percussion, or Point of concentrated Forces.*



1093. But to exemplify this Matter, and to investigate a *general Rule* for finding the Center of Percussion in all Bodies, let us first suppose the least compound Case, *viz.* of two Bodies B, C, fixed to an inflexible right Line A C, and call their Masses of Matter Q, q; let their Tendency to Motion, or the Distance A B, A C, be G, g; and their actual Velocity in Motion, be V, v. Then if each of these Bodies be considered by itself in Motion, the *whole percussive Force* of B, and C, upon Obstacles at D and E would be denoted by Q G V, and $q\,g\,v$. And the Sum of these Forces united, *viz.* $QGV + qgv$, is the whole conjoint Force exerted upon an Obstacle, struck by the Center of Percussion.

1094. Let G be the Center of Gravity between the two Bodies, and N the Center of Percussion to be found; put A G $= m$, and A $N = n$; then will $\frac{QG + qg}{Q + q} = m$ (1045;) also since B $N = n - G$, and N C $= g - n$; it will be $QG : qg :: g - n : n - G$ (1092.) Whence $QGn - QG^2 = qg^2 - qgn$, and so $QGn + qgn = QG^2 + qg^2$, and so $n = \frac{QG^2 + qg^2}{QG + qg}$, or (because G is as V, and g as v) it is $\frac{QGV + qgv}{QG + qg} = AN$; that is, the *Sum of the Forces of the two Bodies divided by the Sum of the* Momenta, *quotes the Distance of the Center of Percussion from the Point of Suspension* A. Which is the *general Cannon* required.

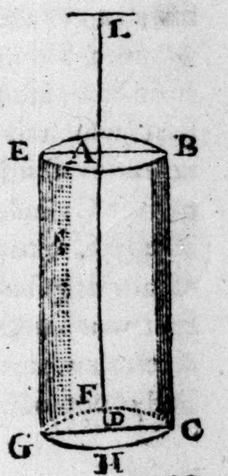
1095. *Coroll.* Put the Masses of Matter in both Bodies $Q + q = M$, and the Sum of the Forces $QGV + qgv = F$; then because $QG + qg = \overline{Q + q} \times m = Mm$; we shall have $\frac{F}{Mm} = n$, or $F = Mmn$, whence we have $\frac{M}{F} = \frac{1}{m} \times \frac{1}{n}$. Which Theorem it will be of Use hereafter to remember.

1096. Let the Line A B (in Article 1089,) be denoted by x; then will the gravitating Force of the Point B be as x, (1088;) and the Velocity of the said Point, when in Motion about A, will be as x likewise; therefore supposing the Line in its nascent State at B, it will be then as $\dot{x}$, and $\dot{x} \times x\,x = \dot{F} =$ Fluxion of the Forces; the Fluent of which $\frac{x^3}{3}$ will be as the whole percussive Force in the moving Line, or Rod A B. But the Sum of all the *Momenta* is as $\frac{x^2}{2}$ (by 1072.) Therefore $\frac{x^2}{2}\Big)\frac{x^3}{3}\Big(= \frac{2}{3}x =$ *Distance of the Center of Percussion* from the Point of Suspension A, by the general Rule at Art. (1094.)

1097. Thus also in the Parallelogram B E G F, (1073) moving about the Axis B F, the Fluxion of the *Momenta* was $y\,x\,\dot{x}$, and so the Fluxion of the Forces will be $y\,x\,x\,\dot{x}$, whose Fluent $\frac{1}{3}y\,x^3$, divided by the *Momenta* $\frac{1}{2}y\,x^2$, will give $\frac{2}{3}x = \frac{2}{3}$ A D, for the *Center of Percussion* from the Axis of Motion.

1098. Again, the Fluxion of the *Momenta* in the Triangle A B G (1074) was found $\frac{b\,x^2\,\dot{x}}{a}$, therefore the Fluxion of the Forces will be $\frac{b\,x^3\,\dot{x}}{a}$, the Fluent whereof $\frac{b\,x^4}{4\,a}$ divided by the *Momenta* $\frac{b\,x^3}{3\,a}$, will quote $\frac{3}{4}x = \frac{3}{4}$ A D, the Distance of the Center of Percussion, from the Axis of Motion at A.

1099. Thus also the Center of Percussion is found in a Cylinder E B C G (1081,) vibrating about an Axis in the Point A. For the Fluxion of the *Momenta* was $\frac{1}{2}a\,p\,x\,\dot{x}$, and therefore the Fluxion of the Forces will be $\frac{1}{2}a\,p\,x^2\,\dot{x}$, whose Fluent $\frac{a\,p\,x^3}{6}$, divided by the Moments $\frac{a\,p\,x^2}{4}$ will quote $\frac{4}{6}x = \frac{2}{3}$ A D for the Center of Force from the Axis A, the same as in the right Line (in 1072.)



1100. *To find the Center of Percussion in a* CYLINDER EBCG, *vibrating about any distant Point* L.

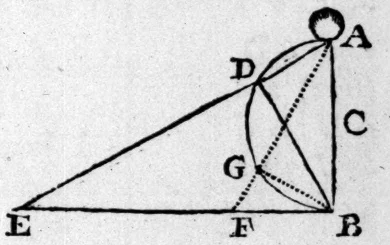
Here it must be considered, that if two or more Bodies connected together, so as to move with equal Velocities, then will the Sum of the Forces of these Bodies be equal to the Force of their common Center of Gravity. And therefore the Sum of the Forces of all the Particles in the fluxionary Circle $\left(\frac{pa\dot{x}}{2}\right.$ $=)$ CFGH, will be equal to the Force of their common Center of Gravity D, or the *Momentum* of that Point multiplied by its Distance from the Center of Motion LD. Wherefore putting AL $= b$, and AD $= x$, as before; we have $\frac{ap\dot{x}}{2} \times \overline{b+x} = \frac{abp\dot{x}}{2} + \frac{apx\dot{x}}{2} =$ Fluxion of the *Momenta*, which again multiplied by $b+x$, gives the Fluxion of the Forces $= \frac{bbap\dot{x} + 2bpax\dot{x} + paxx\dot{x}}{2}$, whose Fluent $\frac{bbpax}{2} + \frac{bpax^2}{2} + \frac{pax^3}{6}$, divided by the Sum of the *Momenta*, $\frac{bapx}{2} + \frac{pax^2}{4}$, will quote $\frac{3bb + 3bx + xx}{3b + \frac{3}{2}x}$, for the Distance required from the Point L. Or, if we put LD $= g = b + x$, then will $x = g - b$, and the Expression will become $\frac{gg + gb + bb}{\frac{1}{2} \times \overline{g+b}}$.

1101. From what has been shewn, it is evident, that since a Walking-cane is generally but little tapering, and when a Stroke is made therewith, the Center of Motion is in the Hand, the Center of Percussion will be near the Part which is $\frac{2}{3}$ of the Cane from the Hand, but somewhat nearer to the Hand, on Account of the Cane's not being a perfect Cylinder, but really the Frustum of a Cone. So likewise a Stroke made with a Sword, (because the Blade is nearly of a triangular, or rather of a pyramidal Form) will be in a Part much nearer to the Hand than in the other Case; but to determine precisely where that Point is in a Sword, (or any Body not of an *uniform Figure*) by an analytical Process, would

would prove too intricate an Affair for this Place; but it may be easily found by Means of a *Pendulum*, as shewn in the next Chapter.

CHAP. VIII.

Of the DESCENT *of* BODIES *on* INCLINED PLANES, *and the* DOCTRINE *of* PENDULUMS.



1102. THE Doctrine of PENDULUMS depends on that of the Descent of Bodies on *inclined Planes*; which therefore comes next to be considered. It has been shewn (1057) that an heavy Body (as A) upon an inclined Plane A E, has its Gravity diminished in the Ratio of the Length of the Plane A E to the Height A B. About the Height of the Plane A B, as a Diameter, describe the Semicircle A D B cutting the Plane in D, and join D B; then is the Angle A D B a right one (645,) and so the Triangles A B D and A E B are similar; and therefore A E : A B : : A B : A D : : *absolute Gravity* of the Body : *residual Gravity*, by which it descends on the Plane.

1103. Again, since Spaces described in the same Time are proportional to the accelerating Forces of Gravity (999;) the Forces which are as A B and A D, will carry the Body A through the perpendicular Space A B, and the slant Space A D in the same Time; that is, *in the Time a Body would fall freely from* A *to* B *through the Height of the Plane, another will arrive from* A *to* D *upon the Plane.*

1104. By the same Argument it is shewn, that a Body will descend on any other Plane A F (of the same Height) to the Point G in the same Time it would descend freely through the Height A B; and therefore it follows, *any two Chords* A D, A G *of the Semicircle will be described in the same Time.*

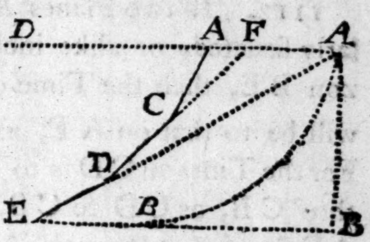
1105. Suppose A G = D B, then because the Angle F A B = A B D, we have the Angle A F B = D B F; and since both the Planes A G and D B are equal, and both alike inclined to the Horizon, it is evident a Body will describe them both in the same Time; and so the Time of describing the Chord D B will be the same as is spent in describing A D. Thus the Time of Descent in A G and G B will be the same. Consequently, *the Times of Descent thro' any Chords* D B, G B, *will be equal.*

1106. Since the Times of Descent through A D and A B are equal (1103,) the Time (t) of Descent through A D (or A B) is to the Time (T) of Descent through A E, as $\sqrt{AD}$ to the $\sqrt{AE}$ (991;) and so $t^2 : T^2 :: AD : AE$; but (because $AD : AB :: AB : AE$) we have $AD : AE : AD^2 : AB^2$, (672) and therefore $t^2 : T^2 :: AD^2 : AB^2$; and so $t : T :: AD : AB :: AB : AE$. That is, *the Time of the perpendicular Descent through* A B *is to the Time through the Plane* A E, *as the Height of the Plane to the Length.*

1107. The Velocity acquired in falling from A to D is to the Velocity acquired in descending from A to B, as A D to A B, (for since they are generated in the same Time, they will be as the Powers which produce them, (998.) Also the Velocity at D is to that at E, as $\sqrt{AD}$ to $\sqrt{AE}$ (991;) that is $v : V :: \sqrt{AD} : \sqrt{AE}$, and so $v^2 : V^2 :: AD : AE$; but because $AD : AB :: AB : AE$, it will be $AD : AE :: AD^2 : AB^2$. Therefore $v^2 : V^2 :: AD^2 : AB^2$; and so $v : V :: AD : AB$. Hence, since the Velocities at B and at E have the same Ratio to the Velocity at D, *they must be equal to each other* (198.)

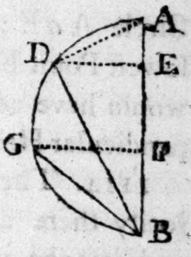
1108. In the same Manner it is shewn, that the Velocity acquired in the Point F, in descending through the Plane A F, is the same with that at B. Consequently *the Velocities acquired in Descents through any inclined Planes* A E, A F, *of the same Height, are equal to each other.*

1109. Hence a Body descending through several inclined Planes A C, C D, D E, contiguous to each other, will acquire the same Velocity in the Point E, as it would have at B in falling through the same perpendicular Height A B. For at C the Velocity is the same as it would be in descending through F C (1107;) and at D, it is the same as it would be in descending through A D; consequently it is the same at E as it would be in descending through A E, that is, the same as it would be in descending from A to B, the same perpendicular Height.

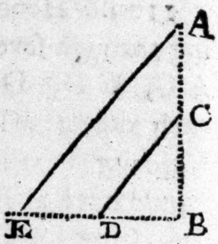


1110. If now we suppose the Number of those contiguous Planes infinite, and their Lengths infinitely small, they will then constitute a *Curve Line*; whence it follows, *that a Body descending through the Arch of any* Curve Line A *B, will acquire the same Velocity at the lowest Point* B, *as it would have at* B, *by descending through the same perpendicular Height* A B.

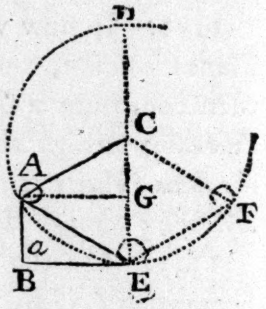
1111. On the Diameter A B describe the Semicircle A G B, and draw any two Chords D B and G B; join A D, and from D and G let fall the Perpendiculars D E, G F to the Diameter A B. *Then will the Velocities acquired in descending through the Chords be as their Lengths respectively.* For the Velocities acquired through D B and G B will be the same as would be acquired in the perpendicular Descents through E B and F B (1105,) that is, as $\sqrt{EB}$ to $\sqrt{FB}$ (991.) But since A B : D B : : D B : E B (659,) we have $AB = \frac{DB^2}{EB}$; in the same Manner it is shewn, that $AB = \frac{GB^2}{FB}$; therefore $\frac{DB^2}{EB} = \frac{GB^2}{FB}$, and so $BD^2 : GB^2 :: EB : FB$; consequently $DB : GB :: \sqrt{EB} : \sqrt{FB}$: : Velocity acquired through D B : Velocity acquired through G B. *Q. E. D.*



1112. If two Planes A E and C D similarly situated, or alike inclined to the Horizon B E, then the Time of Descent on C D will be to that on A E, as $\sqrt{CD}$ to $\sqrt{AE}$. For the Time on C D is to that of the Descent thro' C B, as C D to C B, and the Time on A E is to that through A B, as A E to A B (1111.) But the Times through C B and A B are as $\sqrt{CB}$ to $\sqrt{AB}$. But we have (by similar Triangles) $\sqrt{BC}:\sqrt{AB}::\sqrt{DC}:\sqrt{AE}$. Consequently the Time on D C is to the Time on A E, as $\sqrt{DC}$ to the $\sqrt{AE}$.



1113. A Pendulum (or *pendulous Body*) is in any Body A hanging at the End of a String A C, and moveable about the fixed Point C as a Center. Hence 'tis manifest, if the said Pendulum be in the Position A C, and there left to move freely by the Force of Gravity, it will in its Descent describe the Arch of a Circle A *a* E; and when it arrives at the lowest Point E, it will there acquire a Velocity equal to what it would have acquired, by falling freely through the same perpendicular Height G E (1108, 1109.)

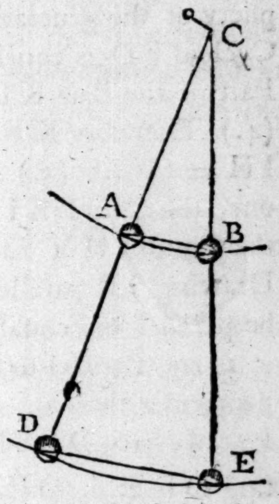


1114. The Body having descended to E, will, with the Velocity there acquired, continue its Motion forwards (963,) and describe an Arch E F in its Ascent equal to A *a* E, (supposing the Motion were in an unresisting Medium, and without Friction at C;) for there must be the same Time spent in generating and destroying any given Quantity of Motion by the same Agent acting uniformly, and consequently the same Space or Arch E F will be described in the Ascent, as was before described in the Descent.

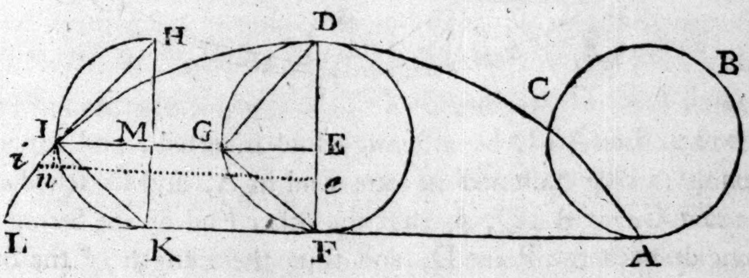
1115. Draw the Chord of the Arch A E, then will that be an inclin'd Plane, whose Height is A B = G E. Now 'tis evident, that when the Arch A *a* E is very small, the Chord A E will nearly coincide with it, and the Motion of the Body descending on the inclined Plane or Chord, and that of the Pendulum

dulum in the Arch will nearly agree in their Properties, or at least be insensibly different.

1116. Hence, if there be two Pendulums of different Lengths C A, C D, the Times in which they will describe the Arches of Descent A B and D E, will be as the Square Roots of the Lengths of these Arches, or their respective coincident Chords A B, D E (by 1112.) But because of similar Triangles A B C and D E C, it will be $\sqrt{AB} : \sqrt{DE} :: \sqrt{CA} : \sqrt{CD} :: t : T$. Therefore $CA : CD :: t^2 : T^2$. That is, *the Lengths of Pendulums are as the Squares of the Times of Vibration.*



1117. The Time in which the Pendulum would descend along the Chord A E, (see Fig. of Art. 1113,) is easily determined, being that in which it would descend through D E (1103, 1105,) = 2 A C; and the Time in which it would pass from A to F on the two Chords A E and E F, will be equal to that in which a Body would descend perpendicularly through a Space = 4 D E = 8 A C (991, &c.) But this is not the Time in which the Pendulum will vibrate through the whole Arch A E F. And therefore to determine what relates to the Time of a Pendulum's Vibration, (which is a primary Consideration) we must take to our Assistance the Properties of the Curve called the Cycloid, which must therefore be next demonstrated.

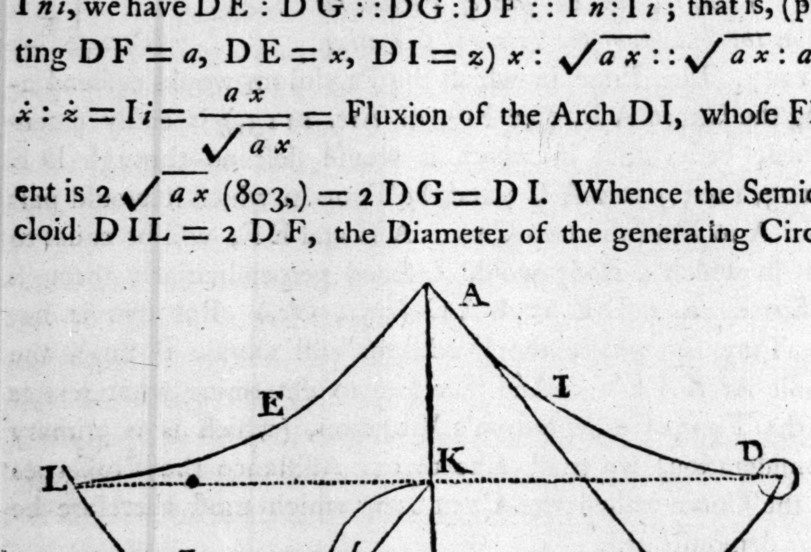


1118. If a Circle A B C insisting on a Right Line A L begin to revolve from A towards L, the Point A will by its twofold

Motion describe the aforesaid Curve or Cycloid A C D I L. Whence we observe, (1.) The Base A L is equal to the Periphery of the generating Circle A B C. (2.) The Axis of the Cycloid F D is equal to the Diameter of the Circle. (3.) The Part of the Base K L is equal to the Arch of the Circle I K. (4.) Therefore K F (= M E) is equal to the remaining Arch I H = G D. (5.) The Angle H I K being always a right one, the Chord K I will be *perpendicular* to the Cycloid, and the Chord I H a *Tangent* thereto in the Point I. (6.) Lastly, Drawing I E parallel to the Base A L, the Tangent I H will be parallel and equal to the Chord G D; and I K to G F.

1119. Parallel to E I draw $e\,i$ indefinitely near, and I n perpendicular thereto; then by similar Triangles D G E, D G F, I $n\,i$, we have D E : D G : : D G : D F : : I n : I i; that is, (putting D F = a, D E = x, D I = z) $x : \sqrt{ax} :: \sqrt{ax} : a ::$

$\dot{x} : \dot{z} = \text{I}\,i = \dfrac{a\dot{x}}{\sqrt{ax}} =$ Fluxion of the Arch D I, whose Fluent is $2\sqrt{ax}$ (803,) = 2 D G = D I. Whence the Semicycloid D I L = 2 D F, the Diameter of the generating Circle.



1120. Let A I D be a Semicycloid inverted; and suppose a flexible String fastened at one End in A, and stretched along the said Curve A I D, so that the other End of the String may coincide with the Point D, and thus the Length of the String will be equal to that of the Curve. If now a Ball or heavy Body were tied to the End D, and left freely to descend, it would describe in one Vibration the Cycloid D C L, provided there

there be placed on the other Side another Semicycloid A E L. Of this I might here give a Demonstration, but it is needless.

1121. The Velocity acquired by the Pendulum descending through any Arch R C of the Cycloid, is the same a Body would have in descending thro' the same perpendicular Height O C (1109,) and therefore is as $\sqrt{OC}$; but (because O C : S C :: S C : K C = 1,) $\sqrt{OC}$ is as the Chord S C = $\frac{1}{2}$ R C (1119.) Therefore *the Velocity in the lowest Point* C, *is ever proportional to the Space passed through, or to the Arch of the Cycloid described in the Descent.*

1122. But in all Kinds of Motions, the Space (S) is as the Rectangle of the Time (T) and Velocity (V,) that is, S : T V (by 971, &c.) therefore if in any Case (as that above 1121) it be S : V, it will be T : 1, that is, the Time of the Motion will be a given Quantity, or always the same. *Hence all the Vibrations through any Arches of a Cycloid, great or small, are performed in equal Time.*

1123. If we put $CK = a$, $KO = x$, then $2\,SC = RC = 2\sqrt{aa - ax}$. If the Descent be from L to R, the Velocity at R will be as $\sqrt{KO} = \sqrt{x}$ (1111.) Now (by similar Triangles) it is C O $(a - x)$: C S $(\sqrt{aa - ax})$ (:: CS : CK $= a$) :: R$q = (\dot{x})$: R$r = \frac{a\dot{x}}{\sqrt{aa - ax}} = \dot{z} =$ the Fluxion of the Arch L R, which divided by the Velocity $\sqrt{x}$, gives $\frac{a\dot{x}}{\sqrt{aax - axx}} = \frac{a\dot{x}}{\sqrt{ax - xx}} \times \frac{1}{\sqrt{a}} = \dot{T} =$ the Fluxion of the Time. But $\frac{\frac{1}{2}a\dot{x}}{\sqrt{ax - xx}}$ is the Fluxion of the circular Arch K S, (875.) (therefore 2 K S is the Fluent of twice that Fluxion, *viz.* $\frac{a\dot{x}}{\sqrt{ax - xx}}$). Consequently the Fluent of the Time of Descent through L R is $\frac{2\,KS}{\sqrt{a}}$. Hence when S coincides with C, L R will become L C; and so the Time of Descent thro' the Semicycloid L C is $\frac{2\,KSC}{\sqrt{a}}$.

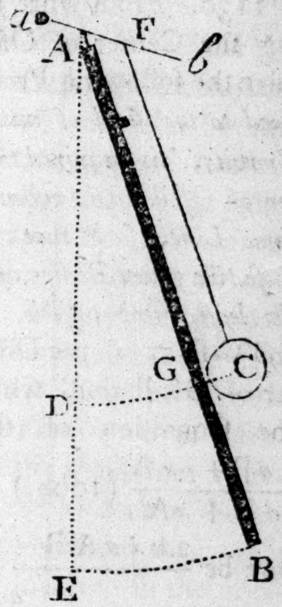
1124. Therefore the Time of a Vibration through the whole Cycloid L C D is $\frac{4\,KSC}{\sqrt{a}}$. But the Time of Deſcent thro' the Perpendicular $KC = a$, is as $2\sqrt{a}$ *; therefore we have $\frac{4\,KSC}{\sqrt{a}} : 2\sqrt{a} :: 2\,KSC : a$. That is, *The Time of Vibration in the Cycloid, is to the Time of Deſcent through Half the Length of the Pendulum, as the Circumference of a Circle to the Diameter, or as* 3,14159 *to* 1.

1125. Let the Time of Vibration of the Pendulum A C be = 1 Second, then its Length is thus found. It is known by Experiments, that a Body deſcends freely through the perpendicular Height of $193\frac{1}{2}$ Inches in a Second of Time. And ſince the Spaces deſcended are as the Squares of the Times (991,) therefore $\overline{3.14159}^2 : 1^2 :: 193\frac{1}{2} : 19{,}6 = \frac{1}{2}\,AC$; whence $AC = 39{,}2$ Inches. And ſince the Lengths alſo are as the Squares of the Times of Vibration (1116,) therefore $4 : 1 :: 39{,}2 : 9{,}8$ Inches = Length of a Pendulum vibrating in $\frac{1}{2}$ a Second.

1126. Hence ſince the Length of a *Second Pendulum* is ſo conſiderable, the Bob will deſcribe (without the cycloidal Cheeks A I D, A E L,) an Arch of a Circle *f* C *b*, which will nearly coincide with the Cycloid for a ſmall Space *h g*. Hence all the Properties of the common Pendulum vibrating through very ſmall Arches *h g*, will have the ſame Properties as though it moved in the Arch or Curve of the Cycloid.

1127.

* To make this plain, we have $S = VT$ (971.) Therefore $\dot{S} = V\dot{T}$. Therefore $\dot{T} = \frac{\dot{S}}{V} = \frac{\dot{S}}{\sqrt{S}}$ (becauſe $V : \sqrt{S}$) but the Fluent of $\frac{\dot{S}}{\sqrt{S}}$ is $2\sqrt{S} = 2\sqrt{a}$, when $S = a$. (See 803, 804.)



1127. Let A B be any Sort of Body suspended from an Axis *a b*, and moveable freely about it; then will it become a Pendulum, and vibrate in the same Manner with the common Pendulum C F. For if it be taken out of the perpendicular Position A E into any other A B, and there let go, it will by its Gravity descend, and pass the said Perpendicular A E to a Distance on the other Side equal to E B, and since every Particle in the Rod endeavours thus to descend, and all those Particles are connected together, their Forces will be all united in one particular Part as G, and that Force will so act upon the Body A B, as if all the Matter thereof were collected in that Point; and hence the Point G is called the CENTER OF OSCILLATION in such a Sort of Pendulum.

1128. Hence it follows, that any Body A B being hung upon an Axis to vibrate, the Time of a Vibration will always be equal to the Time of Vibration in a simple Pendulum F C, whose Length is equal to the Distance A G of the Center of Oscillation from the Axis of Motion in the Pendulum A B. And therefore a Method is hence obvious of finding the said Point G, or Distance A G in any Body whatsoever, *viz. by taking a single Pendulum* F C *that shall vibrate in the same Time.*

1129. From the above Definition of the *Center of Oscillation*, it is easy to understand that it is the very same with the *Center of Percussion* in the Body or Rod A B, for since the Point G is that in which the Forces of all the Particles are united to generate Motion in the Body, and the *Center of Percussion* is that in which alone the Motion of the Body can be all destroyed (1090;) it necessarily follows, they are both one and the same Point; and therefore if A B be of an uniform Figure, it will be isochronal (or vibrate in the same Time) with the common Pendulum F C = A G = $\frac{2}{3}$ A B, (1099.)

1130.

1130. From what has been premiſed concerning the Center of Oſcillation, 'twill be eaſy to ſolve the following Problem, *viz. Any Body* B *being fixed to the End of an inflexible Line* C B *void of Gravity, 'tis required to find the Diſtance of the Center of Oſcillation, when another Body* A *is fixed to the ſame Line, ſuch that the Pendulum compounded of thoſe two given Bodies, ſhall perform its Vibrations in the leaſt Time poſſible.* Let C B $= a$, C A $= x$, and C D $= n$, the Diſtance of the common Center of Oſcillation, which muſt be a *Minimum* by the Condition of the Problem ; now $n = \frac{aaB + xxA}{aB + xA}$ (1094,) whoſe Fluxion muſt therefore be $\frac{2x\dot{x}aAB + x^2\dot{x}A^2 - aa\dot{x}AB}{\overline{aB + xA}^2} = 0$,

which will give $xx + \frac{2aB}{A} = \frac{aaB}{A}$; whence by compleating the Square, and extracting the Root, we ſhall find $x = \frac{a}{A}\sqrt{AB + BB} - \frac{aB}{A}$.

1131. Having thus found $x =$ C A, if we ſubſtitute its Value in the Equation $n = \frac{aaB + xxA}{aB + xA}$, the Diſtance C D of the Center of Oſcillation of the compound Pendulum becomes known, and thus any ſingle Pendulum of the Length C D $= n$, will vibrate in the ſame Time with the compound one.

1132. Since the *Lengths* of Pendulums will alter with *Heat* and *Cold* (as we ſhall hereafter ſhew by the PYROMETER) the *Times* of their Vibrations will vary alſo on that Account (1116,) and therefore when applied to Clock-work and other Uſes, the *Compound Pendulum* will be preferable to the ſingle one, in as much as the Body A may be conſidered as a *Corrector* of the Motion of the Pendulum C B, ſince its Center of Oſcillation D may be always kept on the ſame Point by moving the Ball A up or down by Means of a Screw on the Rod C D. But in this Caſe great Skill and Caution will be required for a proper Adjuſtment.

1133.

1133. Hence it appears, the Nature of the *Pendulum* (of any Sort) constitutes it one of the best Kind of CHRONOMETERS, or Instrument for *measuring Time*; and also by this Means it is applicable to some Cases of ALTIMETRY, LONGIMETRY, *&c.* and those too, where Trigonometry is either wholly deficient, or cannot be so easily applied. It moreover serves for the *Measure of Forces* of *Percussion*, *Resistance*, *Velocity*, *&c.* and in such Cases where the common Methods of Art will fail us, and which yet make the most essential and fundamental Part of the *genuine Theory of* GUNNERY. The PENDULUM is also the only Original and *Philosophical* STANDARD of Measure of LENGTH; and if the Length of the Pendulum vibrating Seconds, had at first been made the STANDARD YARD, to all Nations, we should have had no Doubt about their other Measures, or Dimensions, which now lie involved in the greatest Uncertainty and Obscurity; all these Particulars will fully appear in the Sequel of this Work.

CHAP. IX.

The Physico-Mechanical PRINCIPLES *of* BALISTICS, *or the* Doctrine *of* PROJECTILES *applied to the Solution of all Cases in* GUNNERY.

1134. IN *uniform Motion*, or that whose Velocity is always the same, if the *Time* (T) be given, the *Space* (S) passed over will be as the Velocity (V); and if the *Velocity* be given, the Space will be as the Time; *but if neither the Time nor Velocity be given, the Space described will be as the Product or Rectangles under both, viz.* S : *s* : : T V : t *v*. This we have largely shewn (971, *&c.*)

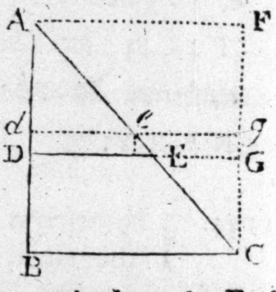
1135. If the Motion be not uniform and equable, but accelerated by a continual Action of the Force which generates the Motion; and this continual Action of the Force be equable and uniform, or in every Moment the same, then will the

Velocity

Velocity be equably and uniformly accelerated, or increaſe equally with the Time, and will therefore be proportional to the Time; and this is the Caſe of Bodies falling by their Weight or Gravity, which near the Earth's Surface is every where the ſame. Therefore in this Sort of Motion T : t : : V : v.

1136. Now though in reſpect of any large Interval of Time, the Motion of falling Bodies is accelerated, yet in the fluxionary Moments of Time the Acceleration is ſo ſmall or inconſiderable, that with reſpect to any proximate Moments the Motion may be eſteemed equable; and conſequently the Space deſcribed in thoſe Moments will be as the Rectangle under the Moments or Fluxion of the Time and Velocity, that is, $\dot{s} = \dot{t}\,v$ (1134.) Alſo ſince T : t : : V : v; we have $t = \frac{T}{V} v$; and ſo $\dot{t} = \frac{T}{V} \dot{v}$, which if ſubſtituted for $\dot{t}$ in $\dot{s} = \dot{t}\,v$, we ſhall have $\dot{s} = \frac{T}{V} v \dot{v}$; and therefore $S = \frac{T}{2V} v v$. *Whence the Space deſcribed by a Body falling freely by its Gravity, is always proportional to the Square of the Velocity,* ſince T and V are given Quantities. This agrees with what was ſhewn in (991, *&c.*)

1137. Since $t = \frac{T}{V} v$, 'tis plain the Motion is rectilinear, and the Space deſcended ſuch as may be repreſented by a Triangle, one of whoſe Sides repreſents the *Time*, the other the *Velocity*. Thus, if the Triangle A B C repreſent the Space deſcended through in any given Time A B, and B C be the Velocity acquired at the End of that Time; then will the Triangle A D E be the Space deſcribed in the Time A D (992,) and D E (parallel to B C) will be the Velocity acquired in that Time. And ſo the Triangle A B C (S) : A D E (s) : : $\overline{AB}^2$ (T^2) : $\overline{AD}^2$ (t^2) : : $\overline{BC}^2$ (V^2) : $\overline{DE}^2$ (v^2). And therefore A B (T) : A D (t) : : B C (V) : D E (v); (671) which gives the Equation, expreſſing the Nature of a Triangle, as above.

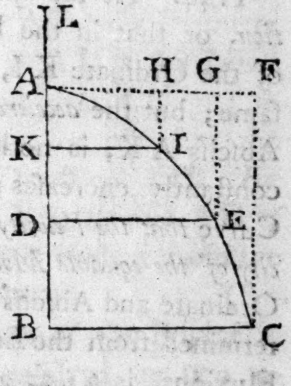


1138.

1138. Hence we may easily compare the Spaces pass'd thro' by a Body moving with an accelerated Velocity in the Time A B, and afterwards with an uniform Velocity, equal to the last acquired Velocity B C, in the same Time. For the Fluxion of the latter Space is $\dot{s} = \dot{t}\ V = D\,d\,g\,G$; and the Fluxion of the Former, or that described by the accelerate Velocity, is $\dot{s} = \frac{T}{V}\, v\dot{v} = D\,d\,e\,E$, whose Fluents t V and $\frac{T}{2V}\, v\,v$ are the Spaces themselves; but since by the Supposition $T = t$, and $V = v$; therefore $t\,V : \frac{T}{2V}\, v\,v :: 2\,T\,V\,V : T\,V\,V :: 2 : 1$. That is, *the Space described by the equable Motion is to that by the accelerate, in the same Time, as* 2 *to* 1; *or as the Rectangle* A B C F *to the Triangle* A B C. (See 993.)

1139. If the Times are not in the simple Ratio of the Velocities, but as any Power t^n to any Power v^m; that is, if $t^n : v^m$, and so $t : v^{\frac{m}{n}}$; then since this is a Ratio expressing the Nature of a *Paraboliform Figure*, 'tis evident the Motion will now be *Curvilinear*; and the Tract which the Body describes will be the Curve of some Kind of *Parabola*. If $n = 1$, and $m = 2$, then $t : v^2$; or $AD : \overline{DE}^2$, then will the Curve A E C which the Body will describe, be that of the *common Parabola*. If $n = 1$, and $m = 3$, then will $t : v^3$; and the Curve will be that of the *cubical Parabola*. If $n = 2$, and $m = 3$; then $t^2 : v^3$, or $t : v^{\frac{3}{2}}$, and the Curve will be that of the *Semicubical Parabola*; and so of others *in infinitum*.

1140. Here also, the Space described is represented by the *Parabolic* Area A B C, which is to the Space A B C F described by an equable Motion in the same Time A B, and with the last Velocity B C, as 2 : 3, or as $n : n + m$, as we have before shewn (827.)



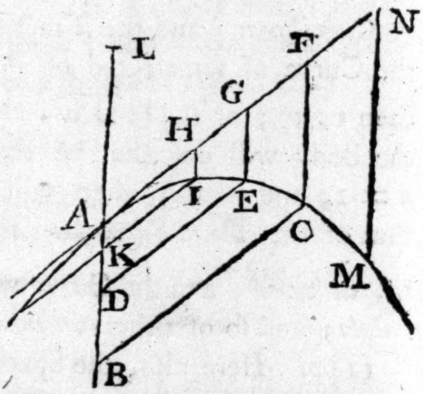
1141. *This accelerated Motion in the Curve of a Parabola is the same with the Motion of a* PROJECTILE. For if the Body be projected (as

 from

from a *Cannon*) from the Point A, in the horizontal Direction A F, it will proceed with an equable Motion; and therefore the Spaces A H, A G, A F will be as the Times of describing them; but while the Body is going from A towards F, it is constantly acted upon by Gravity, and drawn downwards towards the Earth's Center, and therefore in a Direction A B, perpendicular to A F; by which Means the Motion towards F is not at all impeded.

1142. Therefore suppose in the Time it would by the equable Motion arrive at F, it descends by its Gravity through the Space A K, then drawing K I parallel to A F, and H I to A B, 'tis evident by this compound Motion the Body will be found at the End of the Time A H in the Point I. But A H, the Time, is as $\sqrt{AK}$, the Space described by the accelerated Motion, or A K is as $\overline{AH}^2$, or $\overline{KI}^2$, consequently the Point I is in the Curve of a *common Parabola* (740, &c.) and the same may be proved of all the other Points E, C. *Therefore the Path of a Projectile is the Apolonian Parabola.*

1143. If the Projection be not in the *Horizontal Direction*, but in oblique Direction *A N*, then also will the Curve or Path of the Projectile A E M be that of a Parabola, as is demonstrable in the very same Manner, as before in the other Case.

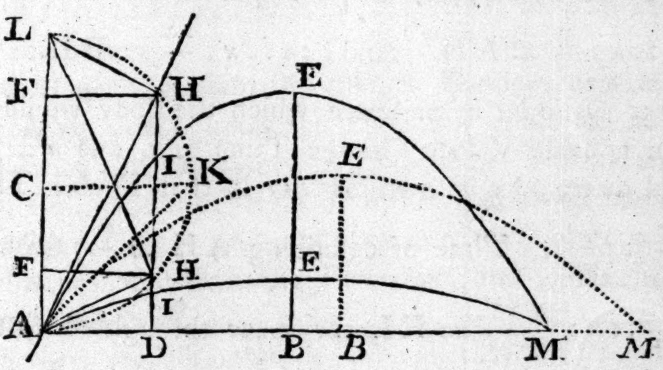


1144. As the *equable Motion*, or that in the Direction of the Ordinate K I, is in the Point A and every where the same; but the *accelerate Motion*, or that in the Direction of the Abscis A K, is nothing in the Point A, but there begins and constantly encreases; it must happen at some Point in the Curve *that the Velocity of the accelerate Motion is equal to the Velocity of the equable Motion, viz.* there where the Fluxion of the Ordinate and Abscis are equal. Which Point is thus easily determined from the Equation of the Curve $px = yy$; which, in Fluxions, is $p\dot{x} = 2y\dot{y}$, and making $\dot{x} = \dot{y}$, we have $p = 2y$, or

or $y = \frac{1}{2}p$; therefore $yy = \frac{1}{4}pp = px$; hence $x = \frac{1}{4}p =$ A K; whence K is the Focus of the *Parabola*.

1145. Hence, if in the Axis produced, we take A L $= x =$ $\frac{1}{4}$ of the Parameter of the Point A, then will the Point L be the Altitude, from which if a Body fall, it will in the Point A acquire ſuch a Velocity, that if it be from thence reflected in the Direction A F, it will deſcribe the given Parabola A E C.

1146. Now ſince it requires the ſame Force to project a Body from the Point A to the Altitude L, as the Body will acquire in the Point A, by falling from the ſaid Altitude L, it follows, that this Altitude is given from the Time of the Aſcent and Deſcent of the Ball projected perpendicularly upwards from the Muzzle of the Cannon; and conſequently the Trajectory or Curve which the Ball will deſcribe, when projected with the ſame Charge of Powder in any Direction A F whatſoever, will be given alſo; and the Solution of the *common Problems of Gunnery* very eaſy.



1147. Thus, for Example, if about A L as a Diameter you deſcribe the Semicircle A K L; then as A L is the Meaſure of the Effect of the *direct Force* of the given Charge; ſo the Chord of the Circle A H will meaſure or repreſent the Effect of the ſame Charge in the oblique Direction A H. *For the direct is to the oblique Force, as Radius* A L *to the Sine* A H *of the Angle of Incidence* A L H $=$ H A D; as is evident from the Reſolution of compound Forces, (1027, *&c.*)

1148. Now ſince A H is as the Force of the Engine in that oblique Direction, ſo this may be reſolved by two o-

 thers,

thers, *viz.* H D perpendicular to the Horizon, and A D parallel to the same. Therefore will H D = B E, be the greatest Height to which the Ball will rise in that Projection; and 4 A D = A M, the horizontal Distance, Random, or Amplitude thereof. For since A L : A H : : A H : H D; the Height to which the Ball will rise with the Celerity H D is to the Height it will rise with the Celerity A H, as $HD^2 : AH^2 :: HD : AL$; but A L is the Height with the Celerity A H; therefore H D is the Height with the Celerity H D. Again, the horizontal Velocity A D, being uniform, will carry the Body through twice the horizontal Distance A D in the Time it will rise to the Altitude H or E; and therefore thro' four times A D, while it ascends and descends thro' the Altitude D H or B E.

1149. But these Matters are best investigated by Fluxions, in the Method *de Maximis & Minimis*, by finding the Relation of the Abscifs A D to the Ordinate D I. In order to which let *Radius*, *Sine*, *and Cosine*, of the Angle of Elevation H A D be called 1, *s*, *&c.* and put $AD = x$. Then will $c : 1 :: x : \frac{x}{c} = AH$. And $c : s :: x : \frac{sx}{c} = DH$.

1150. Also let m = Space which the Body would describe by an equable Velocity in the Time $= t$, and n = Space it would descend by its own Gravity in that Time. Then $m : t :: \frac{x}{c} : \frac{tx}{mc}$ = Time of describing $AH = \frac{x}{c}$. And as $t^2 : \frac{t^2 x^2}{m^2 c^2} :: n : \frac{n x^2}{m^2 c^2} = HI$, the Space through which the Body descends in the same Time. (1136.)

1151. Hence $DH - HI = \frac{sx}{c} - \frac{nx^2}{m^2 c^2} = \frac{cm^2 sx - x^2 n}{m^2 c^2}$ $= DI$, which is to be determined to a *Maximum*, by making its Fluxion $\frac{csm^2 \dot{x} - 2nx\dot{x}}{m^2 c^2} = 0$, (818.) which gives $csm^2 = 2nx$; and therefore $x = \frac{csm^2}{2n} = AB$, because now D I is a *Maximum*, and $= BE = \frac{s^2 m^2}{4n}$ = the Height of the Projection.

1152.

1152. If we make $DI = \frac{c m^2 s x - n x^2}{m^2 c^2} = 0$; then $c s m^2 = n x$; and therefore $x = \frac{c s m^2}{n} = AM$, the Amplitude of the Projection. And then $\frac{t x}{m c}$ will become $\frac{s t m}{n} =$ the Time of the whole Projection.

1153. If we make the Angle HAD *a Right* one, then will AH coincide with AL, and AL will be the Height, or *Impetus* of the perpendicular Projection, and will be equal to $\frac{m^2}{4n}$, because in this Case $s = 1$ in the Expression of the Height $\frac{s^2 m^2}{4n}$.

1154. If now we put $AL = \frac{m^2}{4n} = a$; then will the Height of the Projection in the Direction AH be $BE = a s^2$, and the Amplitude $AM = 4acs$. And since $1 : s :: a : sa = AH$; therefore $1 : c :: sa : acs = AD = \frac{1}{4} AM$, as was shewn before, (1148.)

1155. Hence 'tis plain the greatest Random or Amplitude will be AM, made upon the Elevation AK, of an Angle $KAD = 45^\circ$, because then AD becomes a *Maximum*, or equal to CK, and the Height of the Projection will be $BE = AC = \frac{1}{2} AL = \frac{1}{4} AM$; and therefore the horizontal Random AM is in this Case the *Parameter*, and B the Focus of the Parabola AEM.

1156. In this Case only the Perpendicular HD touches the Circle in the Point K; in every other Case it will intersect it in two Points H, H, which therefore give two Elevations AH, AH, for striking the same Object M on the Horizon, either of which may be taken as the Exigence of the Case shall require.

1157. There is, besides the above, another Method very concise and simple, and yet general for all Cases in the Science of Balistics, by Means of the *Tangent* instead of the *Sine* and *Cosine* of the Angle of Elevation as before. Thus, put $AL = a$, the Velocity of the Projection will be as $\sqrt{a}$ (1145.)

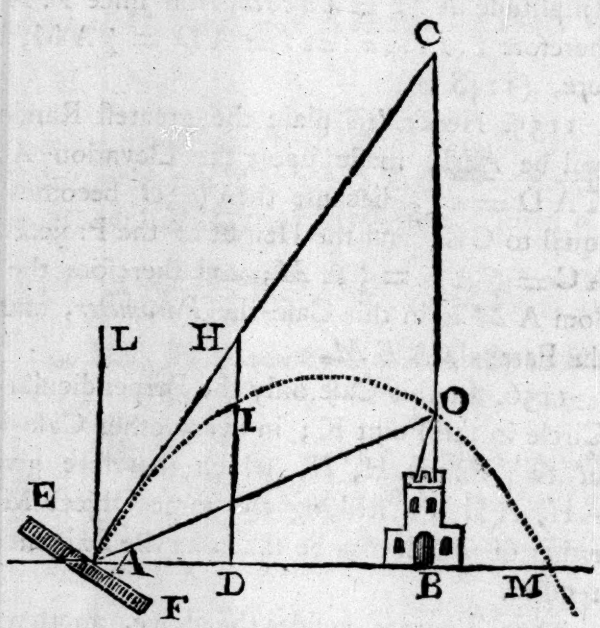
(1145.) Alſo let A H $= t$, and H I $= z$. Then for the Direction A H we have $t : 2z :: \sqrt{a} : \sqrt{z}$ (992.) Whence $t^2 = 4az$.

1158. Again in the Triangle A D H, we have Radius : Tangent : : 1 : n : : A D : D H, and putting A D $= x$, and D I $= y$, we have D H $= nx$; and H I $=$ D H $-$ D I $= nx - y = z$.

1159. Alſo A H² $=$ H D² $+$ A D²; that is, $t^2 = x^2 + n^2 x^2$. Hence $x^2 + n^2 x^2 = 4az$, (1156;) and therefore $z = nx - y = \frac{x^2 + n^2 x^2}{4a}$ (1157;) from whence we get $4anx - 4ay = x^2 + n^2 x^2 = x^2 \times \overline{1 + nn}$. From hence the principal Caſes of *Gunnery* are eaſily ſolved.

1160. Case I.

With a given Impetus (*or Charge of Powder*) *a*, *to ſtrike a given Point* O, *at a given horizontal Diſtance* A B.



Put A B $= b$, and O B $= c$; then will $x = b$, and $y = c$, and the Equation above (1159,) will become $\overline{nn + 1} \times bb = 4nab - 4ac$, from whence we get the Tangent of the required Angle of Elevation $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 - 4ac - b^2}$. Whence it appears there are two ſuch Angles or Poſitions of the Cannon that may be taken, agreeable to (1156.)

1161.

1161. If the Point O be in the Horizon, then $OB = c = o$, and $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 - b^2}$; and if O be below the Horizon, it will be negative, or $-c$; and then $n = \frac{2a}{b} \pm \frac{1}{b}$ $\sqrt{4a^2 + 4ac - bb}$. In every Case $4aa$ must be equal to, or greater than $4ac + bb$, else the radical Quantity will be negative and render the Solution impossible.

1162. CASE II.

To find the Elevation that shall produce the greatest horizontal Random possible, with a given Charge of Powder.

Here we have $AM = x = \frac{4n}{nn+1} \times a$, whose Fluxion made $= o$, will give $n^2 = 1$, or $n = \sqrt{1} = 1$, and shews that the Angle sought is 45°, (as in 1155.)

1163. CASE III.

To find the Charge of Powder requisite to strike the given Point O on a given Elevation of the Piece.

In this Case we have $\frac{nn+1}{4nb-4c}bb = a$, which gives the Velocity of the Ball at the Muzzle of the Gun, (1146.)

1164. If we make the Fluxion of this Value of $(a) = o$, it will give $n = \frac{c}{b} \pm \frac{1}{b}\sqrt{bb+cc}$; and if this be substituted for n, in the foregoing Equation, we shall have $a = \frac{1}{2}c + \frac{1}{2}$ $\sqrt{bb+cc}$, *the least Charge of Powder that will throw the Ball on the given Object* O.

1165. Join $AO = \sqrt{bb+cc}$; then since the least Impetus $a = AL = \frac{1}{2}BO + \frac{1}{2}AO$; if in OB continued out we take $OC = AO$; then BC is the *Tangent of the Angle* BAC, and AC the *Elevation* of the Mortar, for striking the given Object O with the least Charge. For Radius : Tangent : : 1 :

$n : : b\,(= AD) : c + \sqrt{bb + c}\,(= BO + OC)$ whence $n = \frac{c + \sqrt{bb + cc}}{b}$, as before.

1166. In this Caſe, ſince $AO = OC$, the Angle $OAC = OCA$, and ſince B C and A L are parallel, the Angle $ACB = CAL$; and therefore the Angle $CAL = CAB$, whence it is evident, the Line of Direction A C biſects the Angle L A O.

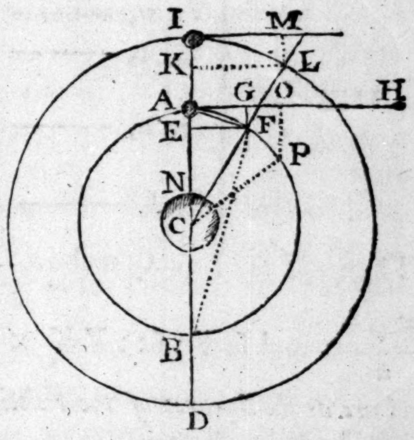
1167. Therefore if E F be a reflecting plain Surface, placed on the Cannon, perpendicular to its Axis, and the Cannon be moved up and down, 'till an Eye placed over the Glaſs, ſees the Object O in the Pependicular A L (conſidered as a *Plumb-Line*) then that will be the required Elevation of the Piece for ſtriking the Object O with the leaſt Charge. And this *mechanical Method* was firſt invented, or obſerved from the Theory by the late Dr. HALLEY.

1168. We might here largely inſiſt on ſtating the Ratios, and finding the Values of the *Time*, *Direction*, *Height*, *Amplitude*, *Impetus*, *&c.* of Projections; but as they are eaſily deducible from the foregoing Theorems, and we are not now on the *practical Part* of Gunnery, it will be beſides our Purpoſe to dwell any longer on this Head.

1169. And eſpecially, if it be conſidered, that this Theory of Projectiles *in Vacuo* is only of uſe in two Caſes, *viz.* of *Swift Motions in Vacuo*, ſuch as thoſe of the *Planets*; and *Slow Motion* in a *Reſiſting Medium*, as that of *Spouting Fluids*, *&c.* but for *Swift Motion* in a *Reſiſting Medium*, ſuch as is that of a *Bullet or Bomb* in the *Air*, it can be of little or no Service; as we purpoſe to ſhew further on, when we ſhall treat profeſſedly on the ELEMENTS of MILITARY PHILOSOPHY, and from thence deduce a NEW and GENUINE THEORY of GUNNERY, which though the moſt neceſſary, has hitherto been the leaſt improved Part of the MECHANICAL PHILOSOPHY.

CHAP. X.

Of the PROJECTILE *and* CENTRAL FORCES *by which Bodies describe* CIRCLES, *or any Species of* Circular MOTION; *a Comparison and Computation of the* FORCES, VELOCITIES *and periodical* TIMES.



1170. LET C be any large attracting Body, A any lesser Body attracted by it at the Distance A C; again, suppose a *projectile Force impressed* on the Body A in the Direction A H, perpendicular to A C, and in such Proportion to the *attractive Force* of C, as that the Body being at Liberty, and urged by both, may describe the Circle A F P B A about the central Body C.

1171. Let A F $= \dot{z}$, be the fluxionary Arch described in the first Moment; and from F let fall the Perpendicular F E, F G, on the Right Lines A C, and A H; then will A G be to A E (= G F,) as the *Projectile Force* to the *Attractive Force*, at the Distance A C. Which latter Force is called the *Centripetal Force*, as being thereby carried towards the Center of Attraction C. Draw the Radius C F and the Chords A F and B F, and put A C $= x$, and A B $= d$. Then by the Nature of the Circle, $AB : AF :: AF : AE = \frac{\overline{AF}^2}{AB}$: Now because the Sine F E, the Arch A F, the Chord A F, and the Tangent A G are all equal to each other in their *first*, or *nascent State*; therefore $\frac{\overline{AF}^2}{AB}$ will in that Case be $= \frac{\dot{z}\dot{z}}{d} = \dot{x}$; whence $\dot{x} : \dot{z} :: \dot{z} : d ::$ the Centrifugal Force : the Projectile Force; and be-

cause $\dot{z}$ is infinitely less than d, therefore $\dot{z}$ the Projectile Force does infinitely exceed $\dot{x}$, the Central Force in *circular Motion*.

1172. In any other Circle ILD, the same Things are represented by Roman Symbols $\frac{\dot{\mathrm{z}}\,\dot{\mathrm{z}}}{\mathrm{d}} = \dot{\mathrm{x}}$; then we have $\dot{x} : \dot{\mathrm{x}} :: \frac{\dot{z}\,\dot{z}}{d} : \frac{\dot{\mathrm{z}}\,\dot{\mathrm{z}}}{\mathrm{d}}$; that is, *the Central Forces in two different Circles, are as the Squares of the Projectile Forces, or circular Velocities applied to the Diameters of the Circles.*

1173. Let $\left\{\begin{array}{ll} \text{F, f,} & \text{represent the } \textit{Central Forces } \dot{x}, \dot{\mathrm{x}}. \\ \text{V, } v\text{,} & \text{——— the circular Velocities } \dot{z}, \dot{\mathrm{z}}. \\ \text{T, t,} & \text{——— the periodical Times or Revolutions.} \\ \text{D, } d\text{,} & \text{——— the Diameters of the Circles, or } \textit{Radii.} \\ \text{P, } p\text{,} & \text{——— the Peripheries of the Circles.} \end{array}\right.$

Then will the Equation above $\frac{\dot{\mathrm{x}}\,\dot{z}\,\dot{z}}{d} = \frac{\dot{x}\,\dot{\mathrm{z}}\,\dot{\mathrm{z}}}{\mathrm{d}}$ become $\frac{\mathrm{f\,V\,V}}{\mathrm{D}} = \frac{\mathrm{F}\,v\,v}{d}$; and so $\mathrm{F : f :: V\,V} \times \frac{1}{\mathrm{D}} : v\,v \times \frac{1}{d}$; that is, *the Forces are as the Squares of the Velocities directly, and as the Diameters reciprocally,* as before.

1174. If the Bodies I and A so move as to describe equal Areas in equal Times, (*viz.* ILC = ACP,) that is, if $d\,\dot{z} = \mathrm{d}\,\dot{\mathrm{z}}$, then $\dot{z} : \dot{\mathrm{z}} :: \mathrm{d} : d$; and so $\dot{z}^2 : \dot{\mathrm{z}}^2 :: \mathrm{d}^2 : d^2$; consequently $\dot{x} : \dot{\mathrm{x}} :: \left(\frac{\dot{z}^2}{d} : \frac{\dot{\mathrm{z}}^2}{\mathrm{d}} ::\right) \frac{\mathrm{d}^2}{d} :: \frac{d^2}{\mathrm{d}} :: \mathrm{d}^3 : d^3$. *That is, the Centrifugal Forces are reciprocally as the Cubes of the Diameters,* (*or* Radii) *of the Circles.*

1175. If $\mathrm{D} = d$, then $\mathrm{F : f :: V^2} : v^2$; *or the Forces are directly as the Squares of the Velocities in the same Circle.* If $\mathrm{V} = v$; then $\mathrm{F : f ::} \frac{1}{\mathrm{D}} : \frac{1}{d} :: d : \mathrm{D}$; *or the Forces are inversely as the Distances or Diameters, when the Velocities are equal.*

1176. Since the Motion in a Circle is equable, the Spaces will be as the Times; and therefore as $\mathrm{V : P :: 1 : T}$; hence $\mathrm{T\,V = P}$, and so $\mathrm{V} = \frac{\mathrm{P}}{\mathrm{T}} = \sqrt{\mathrm{DF}}$ (because $\dot{z} = \sqrt{d\,\dot{x}}$

1171;)

1171;) therefore $P^2 = T^2DF = \overline{3,1416}^2 DD$; whence $T^2F = \overline{3,1416}^2 D$; and since $\overline{3,1416}^2$ is a constant Quantity, we shall have F always as $\frac{D}{T^2}$; or $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$; *that is the central Forces in different Circles will be as the Diameters directly, and reciprocally as the Squares of the periodical Times.*

1177. When $T = t$, then $F : f :: D : d$; that is, *when the periodical Times are equal, the central Forces are as the Distances from the Center directly.*

1178. When $D = d$, $F : f :: t^2 : T^2$; or *the Forces are reciprocally as the Squares of the periodical Times in the same Circle.*

1179. If $F = f$, then $\frac{D}{T^2} = \frac{d}{t^2}$, and so $T : t :: \sqrt{D} : \sqrt{d}$; that is, *the periodical Times are in the subduplicate Ratio of the Distances, when the Forces are equal.* Also then we have $\frac{V^2}{D} = \frac{v^2}{d}$; and so $V^2 : v^2 :: D : d$; or the *Squares of the Velocities are then as the Distances directly from the Center.* Hence also $T : t$ $(:: \sqrt{D} : \sqrt{d}) :: V : v$; that is, *when the cent. Forces are equal in two different Circles, the periodical Times will be as the Celerities.*

1180. Again; when $V = v$, we have $T : t :: (P : p ::) D : d$; that is, *when the Velocities are equal, the periodical Times will be as the Diameters directly.* And when $D = d$, then $T : t :: v : V$; that is, *in the same Circle the Times will be inversely as the Velocities.*

1181. Since when $F = f$, we had $T : t :: \sqrt{D} : \sqrt{d}$; and when $V = v$, we had $T : t :: D : d$; therefore when neither F, nor V are given, it will be $T : t :: D\sqrt{D} : d\sqrt{d}$, or $T^2 : t^2 :: D^3 : d^3$; that is, *the Squares of the periodical Times are universally as the Cubes of the Distances of Bodies circulating about the same Center* C.

1182. Or thus more generally; let $F : f :: D^m : d^m :: \frac{D}{T^2} : \frac{d}{t^2}$; whence $d^m Dt^2 = D^m dT^2$; or $d^{m-1}t^2 = D^{m-1}T^2$, and so

T : t (: : $d^{\frac{m-1}{2}}$: $D^{\frac{m-1}{2}}$: :) $D^{\frac{1-m}{2}}$: $d^{\frac{1-m}{2}}$. Now if $m = 0$; then T : t : : $D^{\frac{1}{2}}$: $d^{\frac{1}{2}}$: : $\sqrt{D}$: $\sqrt{d}$; as before when F = f. When $m = 1$, then T : t : : D° : d° : : 1 : 1. *viz. when the Forces are as the Distances, the Times of the Periods will be equal, in any Circles whatsoever.* Lastly, if $m = -2$; then T : t : : $D^{\frac{3}{2}}$: $d^{\frac{3}{2}}$, or $T^2 : t^2 :: D^3 : d^3$; but in this Case F : f : : D^{-2} : d^{-2} : : d^2 : D^2; that is, *when the centripetal Forces are inversely as the Squares of the Distances, then will the Squares of the periodical Times be as the Cubes of the Distances.* And this is that *universal Law of Nature* which is found to obtain *in the Motions of all the heavenly Bodies.*

1183. In the above Theorems and Analogies, we have exhibited the Ratios of the Forces, Velocities, and Times of Revolution in different Orbits, but to express them in their proper Measures for any given Orbit whose Radius is r, we must proceed in the following Manner: The Force is measured by the Velocity that may be uniformly generated in a given Time, 1. which let us expound by the Power r^n (the n Power of the Radius (r.) Then the Distance through which a Body will freely descend in the same Time will be express'd by $\frac{1}{2} r^n$ (993.) Therefore if A F be an Arch described in the Time (t,) the Distance AE descended in that Time, will be found by this Analogy; as $1^2 : T^2 :: \frac{1}{2} r^n : \frac{T^2 \times \frac{1}{2} r^n}{1^2} = AE$ (992.)

1184. Now from the Nature of the Circle, we have $AF^2 = AB \times AE = 2\,AC \times AE = 2\,AE \times AC = \frac{T^2 r^n}{1^2} \times r = \frac{T^2 r^{n+1}}{1^2}$, therefore $AF = \frac{\sqrt{T^2 r^{n+1}}}{1^2} = \frac{T \times r^{\frac{n+1}{2}}}{1} =$ Space described in the Circle in the Time t.

1185. But in equable Motions, the Spaces described with a given Velocity (r^n) will be as the Times (971;) therefore T : $T r^{\frac{n+1}{2}}$: : 1 : $r^{\frac{n+1}{2}}$ = the true Measure of the Celerity in the Circle. This might have been deduced from (1173,) where $v = \sqrt{rf} = \sqrt{r \times r^n} = \sqrt{r^{n+1}} = r^{\frac{n+1}{2}}$.

1186.

1186. Then by putting $q = 3{,}14159$, &c. we shall have $r^{\frac{n+1}{2}} : 2rq :: 1 : 2qr^{\frac{1-n}{2}}$ = the true Measure of the *periodical Time.*

1187. To find the Space S through which a Body must descend, to acquire the Velocity $r^{\frac{n+1}{2}}$, we have $r^{2n} : r^{n+1} :: \frac{1}{2}r^n : S = \frac{\frac{1}{2}r^{2n+1}}{r^{2n}} = \frac{\frac{1}{2}r \times r^{2n}}{r^{2n}} = \frac{1}{2}r$ (992.) *The Velocity in the Circle therefore is acquired by a Descent through half the Radius.*

1188. To accommodate these Expressions for Use; suppose the Force we speak of be that of GRAVITY; then its Measure $r^n = 2s = 32\frac{1}{6}$ Feet, because it is known by Experience, that $\frac{1}{2}r^n = s = 16\frac{1}{12}$ Feet for the Descent in the first Second of Time. Therefore $r^{n+1} = 2sr$, and $r^{\frac{n+1}{2}} = \sqrt{2sr}$ = the Velocity *per Second*, in any given Circle whose Radius is (r.)

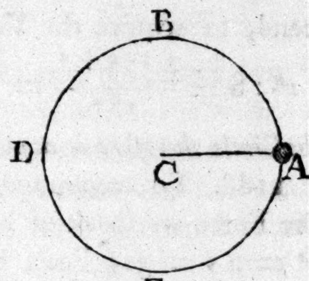
1189. Again, $\sqrt{2rs} : 2rq :: 1 : \frac{2rq}{\sqrt{2rs}} = q\frac{\sqrt{2r}}{s} =$ T, the *periodical Time*, as is evident from (1186); because $2r = D$, the Diameter, and $2rq = C$ = the Circumference of a Circle, therefore $\sqrt{Ds}$ = Velocity, and $\sqrt{\frac{qC}{s}}$ = periodical Time; which Expressions are now in the most simple Forms.

1190. Suppose a Circle equal to 8000 Miles, or 42000000 Feet in Diameter, which is nearly equal to that of our *Earth's* Circumference, and the centripetal Force be equal to that of *Gravity*; then will $D = 42000000$, and $s = 16{,}083$; and so $\sqrt{Ds} = 26000$ Feet the Velocity; and $\sqrt{\frac{qC}{s}} = 5075'' = 1^h : 24' : 35''$ nearly the periodical Time, at the Distance of the Earth's Surface.

1191. In any Circle, whose Diameter is d, and t the Time of Revolution in Seconds, then the central Force in such a Circle may be compared with Gravity. For since $\sqrt{\frac{qC}{s}} =$ T,

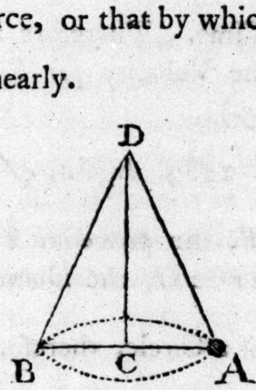
T, we have by (1176,) $\frac{D}{T^2} : \frac{d}{t^2} :: F : f$, that is, (ſuppoſing Gravity, or $F = 1$) $\frac{Ds}{qC} : \frac{d}{tt} :: 1 : \frac{qCd}{Dstt} = \frac{q^2 d}{stt} = f$, the central Force required.

1192. For Example, ſuppoſe A a Ball of *one Ounce*, whirled about the Center C, ſo as to deſcribe the Circle A B D E, and each Revolution be made in *Half* a Second; and let the Length of the Cord A C be two Feet. Then $t = \frac{1}{2}$, and $d = 4$, and $\frac{q^2}{s} = 0{,}6136$;



therefore $\frac{q^2 d}{stt} = 9{,}818 =$ the central Force, or that by which String A C is ſtretched, *viz.* 10 Ounces nearly.

1193. If the String and Ball be ſuſpended from a Point D, and deſcribes in its Motion a conical Surface A D B; then putting $DC = h$, $AC = r$, and $AD = g$; and putting $F = 1$, the Force of Gravity, as before; then will the Body A be affected with three Forces, *viz.* Gravity acting in the Direction DC, a *centrifugal Force*, in the Direction C A, and the *Tenſion of the String*, or Force by which it is ſtretched in the Direction D A; hence theſe three Powers will be as the three Sides of the Triangle C A D reſpectively, (1031) and therefore as CD $(= h) : AD (= g) :: 1 : \frac{g}{h} =$ Tenſion of the String compared with the Weight of the Body.



1194. And $CD (= h) : CA (= r) :: 1 : \frac{q^2 d}{stt}$ (1191,) that is, $h : r :: 1 : \frac{2q^2 r}{stt}$, whence $\frac{2q^2 rh}{stt} = r$, therefore $2h \times q^2 = stt$; and ſo $t = q\sqrt{\frac{2h}{s}}$, or becauſe $\frac{2q^2}{s} = 1{,}2272$, we have $t = 1{,}108\sqrt{h} =$ to the periodical Time.

1195.

1195. We farther observe, that $s\,t\,t = S =$ the Space any Body descends in the Time t, since $1^2 : t^2 :: s : S$ (992,) therefore since $2\,h \times q^2 = s\,t\,t \times 1^2$, we have $1^2 : q^2$ ($= \overline{3,14159}^2$) :: $2\,h$: ($s\,t\,t =$) S; that is, *as the Square of the Diameter is to the Square of the Periphery of any Circle, so is twice the Height of the Cone to the Space through which a heavy Body will descend in the Time of one Revolution.*

1196. Since the mean Distance of the Moon is nearly 60 Semidiameters of the Earth, or 1257696000 Feet; if C be the Earth and A the Moon, (Fig. to 1170) then will $AB = 251539200$; and the Circumference of her Orbit $A\,F\,B\,A = 7897834380$, which is described in $27^d : 7^h : 43' = 2360580''$. Hence the Feet passed over in one Second will be $3346 = A\,F = V$. Therefore $A\,F^2 = 11128976$; whence $\frac{11128976}{2515392000} = 0.00443 = A\,E$, the Distance through which the Moon would descend in the Time of *one Second*, if the circular Motion were to cease. Now on the Earth's Surface Bodies descend in the same Time $16\frac{14}{100}$ Feet; but $0,00443 : 16,14 :: 1 : 3643$; which is very near as 1 to 3600, the Squares of the Distances from the Earth's Center reciprocally; and therefore confirms the Law of central Forces above laid down (1182.)

1197. Again, since $T^2 : t^2 :: D^3 : d^3$, we have as $D^3 = 60^3 : d^3 = 1^3 :: T^2 = 2360580''^2 : \frac{2360580}{60^{\frac{3}{2}}} = t = 83' : 53''$, the periodical Time at the Earth's Surface, nearly the same as in (1190.)

1198. And since when $D = d$, we had $T^2 : t^2 :: f : F$; therefore as $24^h = \overline{1440'}^2$, the Square of the Time of the Earth's diurnal Rotation is to $\overline{84'}^2$, the Square of the periodical Time when the Force is equal to Gravity, so is 1 to $\frac{1}{293,8}$, or so is 293,8 to $1 =$ the central Force on the Surface of the Earth arising from the Rotation. Hence the centrifugal Force, *viz.* that by which Bodies endeavour to fly off from the Earth's Surface directly from its Center, is to Gravity, nearly as 1 to 293,8 under the *Equator*.

1199. But at all Distances from the Equator to the Pole, this centrifugal Force continually diminishes in the Ratio of the Velocities of the Places (1177); but these Velocities are as the Distances from the Earth's Axis, or as the Co-sines of the Latitude; *therefore the centrifugal Force is every where to Gravity as the Co-sine of the Latitude to* 293,8; *the Radius being* = 1.

1200. Let AFB, and ILD be the Orbits the Moon describes about the *Earth* and *Sun* (Fig. to 1170); then will the Semidiameters of those Orbits be AC = 240000, and IC = 80000000; and if we put the Earth's annual Revolution, or periodical Time = 1, then will that of the Moon be $\frac{27,3}{365} = 0.0748$; the Squares of which Times will be 1, and 0,005594. And therefore since $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$; we have $\frac{82000000}{1} : \frac{240000}{0,005594} ::$ 80000000 : 43800000 : : 800 : 438 : : F : f. That is, the Gravity of the Moon to the Sun is to its Gravitation to the Earth, as 800 to 438, or as 1,82 to 1 nearly.

1201. Lastly, we may compare in this Manner the centrifugal Forces arising from the diurnal and annual Motions of the Earth, for the Times are, as 1 to 365, whose Squares are 1 and 133125; and the Distances in Semidameters of the Earth are, as 1 to 20500; therefore $F : f :: \frac{D}{T^2} : \frac{d}{t^2} :: \frac{20500}{133125} : 1 :: 15 : 100$. And such are the Forces arising from the annual and diurnal Motions of the Earth, by which Bodies endeavour to fly off from its Surface.

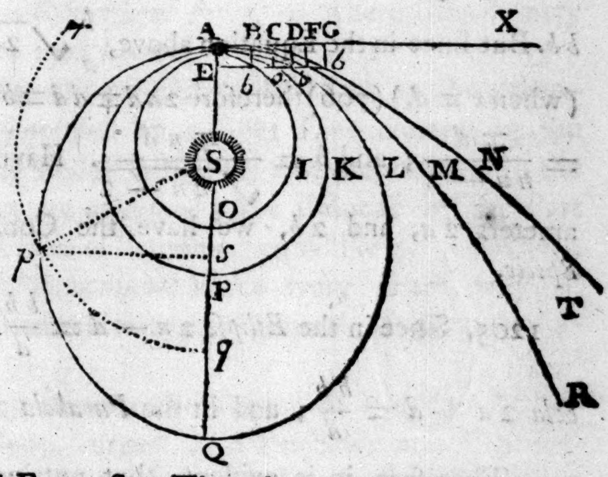
1202. What we have said may suffice for the Doctrine of circular Motion and central Forces; in which, I presume, we have been as full and as plain as the Nature of the Subject will admit; for this is a Species of Knowledge of great Delicacy as well as Use; and is to be reckoned among the *first Principles* of *Astronomy*, and some other Sciences. We next proceed to consider the Curves, or Trajectories, that will be described by a Body urged with different projectile Forces, compounded with a given centripetal Force.

CHAP. XI.

The general DOCTRINE *of* TRAJECTORIES, *or* ORBITS *described about a* CENTER, *by a* Given FORCE *tending thereto, and compounded with a variable* PROJECTILE FORCE: *Also of the* Proportionality *of* AREAS *and the* TIMES *in which they are described.*

1203. HAVING dispatched the Doctrine of *Circular Motion*, where both the *Projectile* and *Centripetal* Forces were given, or constant Quantities, in every Part of the Curve; we will now consider one of them, (*viz.* the Projectile Force) as variable, while the other remains the same, and observe from thence the different Kinds of *Trajectories*, or *Curves*, that a Body impelled by such a Composition of Forces will describe about a given Center. And since at any Distance from the Center the Projectile Force, by which it may describe a Circle, is given (1187.) therefore that Force may be made the Standard for Comparison, by which the Quantity of other Forces whereby other Curves are described may be discovered; as we shall shew in the following easy and perspicuous Method.

1204. Let a Body be projected from the Point A, with such a Velocity AC = 1, as compounded with the centripetal Force A E, may cause it to describe a Circle AKP about the Sun, or Center of Force S. Then let it be projected from the



the ſame Point, and in the ſame Direction A X, with any other Degree of Velocity $AD = n$, to determine the *Conic Section*, or *Curve* it will deſcribe.

By Suppoſition A X is perpendicular to A S, let E a b be drawn parallel to A X, cutting the Circle and Curve in the Points a and b. Put $AS = d$; the Semi-tranſverſe Diameter $= a$, the Semi-conjugate $= b$; and $AE = x$. Then will $\sqrt{2dx - xx} = y = Ea$ in the Circle; and $\frac{b}{a}\sqrt{2ax \mp xx} = y = Eb$, in the Conic Section. Now the Fluxions of the Ordinates E a and E b, viz. $\frac{\overline{d - x} \times \dot{x}}{\sqrt{2dx - xx}}$, and $\frac{b}{a} \times \frac{\overline{a \mp x} \times \dot{x}}{\sqrt{2ax \mp xx}}$ will be as the Velocities in every Point of Curves in the Direction A X; but theſe Fluxions are as $\frac{d - x}{\sqrt{2d - x}}$ and $\frac{b}{a} \times \frac{a \mp x}{\sqrt{2a \mp x}}$, (dividing each by $\frac{\dot{x}}{\sqrt{x}}$;) and therefore when $AE = x = 0$, then the Ratio of theſe Fluxions will become $\frac{d}{\sqrt{2d}} : \frac{b}{a} \times \frac{a}{\sqrt{2a}}$, or as $\sqrt{d} : \frac{b}{\sqrt{a}}$ in the Point A. Conſequently $\sqrt{d} : \frac{b}{\sqrt{a}} :: 1 : n$; and ſo $n\sqrt{d} = \frac{b}{\sqrt{a}}$, and therefore $nnd = \frac{bb}{a}$. Whence $nnad = bb$. But ſince in the Equation above, $\frac{b}{a}\sqrt{2ax \mp xx} = y = \frac{bb}{a}$ (when $x = d$,) (766) therefore $2ad \mp dd = bb = nnad$; whence $a = \frac{\mp d}{nn - 2}$; and $b = \frac{\mp nd}{\sqrt{nn - 2}}$. Having therefore the Diameters $2a$, and $2b$, we have the Conic Section given *in Specie*.

1205. Since in the *Ellipſe* $2a - d = \frac{bb}{d}$; and in the *Hyperbola* $2a + d = \frac{bb}{d}$; and in the *Parabola* $2a = \frac{bb}{d} =$ Infinity. Therefore it is evident, that putting $nn = \overline{AF}^2 = 2$,

the

the Equation above for (a), will be $a = \frac{\mp d}{0}$; whence a is infinite; and therefore the Curve described with a Velocity in the Vertical Point, which is as $\sqrt{2} = n$, will be a PARABOLA A M R.

1206. If nn be less than 2, then it is plain that a will be affirmative, and b negative, or $+ a$, and $- b$; and therefore the Curve described will be an ELLIPSIS. If n be greater than 1, that is, if E b = A D, be greater than E a = A C, then will the Ellipsis A L Q be without the Circle, and the Sun, or Center of Force in the upper Focus. But if n be less than 1, or E b = A B less than E a = A C = 1, then will the Ellipsis A I O be described within the Circle, and have the Sun or Center of Force S in its lower Focus. Lastly, when $n = 0$, the Body A will descend in a right Line A S to the Sun S.

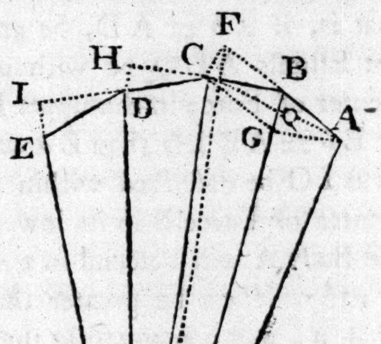
1207. If nn be greater than 2, then will d be affirmative, or $+ d$; and consequently the Curve A N T described will be an HYPERBOLA.

1208. Hence it appears, that the Curves described with all Degrees of Velocity n, between 0 and $\sqrt{2}$, will be ELLIPSES, (for the *Circle* described with the Velocity $n = 1$, is really one Species of Ellipses) and all the Velocities from $\sqrt{2}$ to Infinity will produce Trajectories of the *hyperbolic Kind*, by a Centrifugal Force. The PARABOLA with the Velocity = $\sqrt{2}$ being the common Limit between them; and where the *Centripetal* becomes a *Centrifugal Force*, or where *Gravity* may be said to be changed into *Levity*.

1209. In any determinate Orbit, described as abovementioned, the Areas generated by a right Line connecting the Center and revolving Body, will always be proportional to the Times in which they are described; the Velocity in any Part of the Orbit; the Times of describing equal Parts, or Arches; and the Expression of centripetal Force every where, may all be determined and demonstrated in a general Manner, as follows.

1210. Let S be an immoveable Center of Force or Attraction, and A any Body, urged by a Projectile and Centripetal Force conjointly. In the Point A let the Projectile Force

be ſuch, as would carry the Body through the Space A B in an indefinite ſmall Particle of Time; then in the ſame Time it would deſcribe B F = A B, were it not drawn from the Tangent A F to ſome other Point C, by the Centripetal Force F C, acting in a Direction parallel to B S. In like Manner, in the Third equal Particle of Time, it would deſcribe C H = C B, but is drawn to the Point D by the Centripetal Force H D in a Direction parallel to C S; and ſo on.



1211. Now all theſe triangular Areas A S B, B S C, C S D, D S E, &c. deſcribed about the Center S in equal Times, are equal to each other. For joining S F, the Triangle A S B is equal to the Triangle B S F, as being upon equal Baſes and of the ſame perpendicular Altitude. (635) Alſo the Triangles B S F = B S C; as being both on the ſame Baſe B S, and between the ſame Parallels B S and C F; therefore the Triangles A S B = B S C. In the ſame Manner the Triangle C S D is proved equal to B S C, and therefore to A S B; and ſo of all the Reſt; whence the Propoſition is evident.

1212. Hence, when the Particles of Time, or the Spaces A B, B C, &c. are taken infinitely ſmall, the Polygon A B C D E, &c. will become a Curve; and *the Areas deſcribed by any Radius* A S *will be always proportional to the Times of their Deſcription.*

1213. Alſo, becauſe in equal Triangles $\frac{1}{2}$ H B = $\frac{1}{2}$ *h b*, whoſe Baſes B, *b*, are unequal, thoſe Baſes are reciprocally as their Heights H, *h*, (*viz.* B : *b* : : *h* : H;) it follows, *that the Velocities in every Part of the Orbit of the revolving Body are reciprocally as the Perpendiculars let fall from* S *to the Tangents of the Orbit in thoſe Points.*

1214. And because Triangles upon equal Bases are as their Altitudes, (*viz.* $\frac{1}{2}$ H B : $\frac{1}{2}$ *h* B : : H : *h*;) it follows, *that the Times in which equal Parts, or Arches of the Orbit are described, are directly as those Perpendiculars to the Tangents.*

1215. Draw A G, G C, parallel to B C, B F; then will B G = F C, or be as the Centripetal Force in the Point B, and because the Diagonal A C bisects B G in Q, and this is every where the Case; *therefore in every Part of the Orbit the Centripetal Force will be as the* Sagitta B Q *of the indefinitely small Arch* A B C.

1216. From what has been said, it follows, that every Body which moves in a Curve about an immoveable Point S, so that by a Radius drawn to that Point, it describes Areas proportional to the Times, I say, such a Body is urged by a Centripetal Force tending to that Point. For such a Body in the Point B is drawn from the Tangent by a Force which acts in the Direction of a Line parallel to C F, that is, in the Direction B S; and in the Point C it acts in some Line parallel to H D, *viz.* in the Line C S; and therefore in every Point, it acts in Lines tending to the Center S.

1217. But if the Areas are not proportional to the Times, 'tis evident that C F, H D, *&c.* are not parallel to B S, C S, *&c.* and therefore the Direction of the Centripetal Force is not to the fixed Point S; but to some other uncertain and variable Point, on this Side or that, as the Areas described in equal Times increase or decrease, as is evident from the Construction of the Figure.

CHAP.

CHAP. XII.

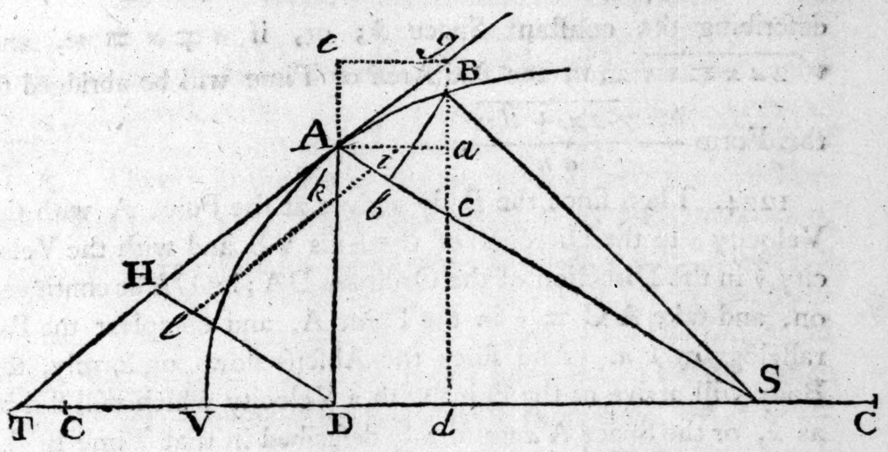
A Determination of the Law *of the* Centripetal Force, *tending to a given Point any where placed in the* Axis *of a* Conic Section, *by which the Curve pertaining thereto may be described.*

1218. To determine what the Law of the Centripetal Force muſt be, that if the Point to which it tends be placed any where in the Axis of a Conic Section, the Body ſhall be made to deſcribe the Curve proper thereto, is a Problem of the greateſt Importance, and may be ſolved either by *Lineal Geometry*, or by *Fluxions*. In the firſt Way it has been often done, but it is moſt general and expeditious in the Latter, which here follows.

1219. Let V A B be a Part of the Curve, T V C its Axis, C the Center, S a Point in the Axis to which the revolving Body tends, A the Place of the Body, A B the Arch deſcribed in an indefinite ſmall Part of Time; A D $= y$, an Ordinate, V D $= x$, the Abſciſs belonging thereto, A T a Tangent, and A S the Diſtance of the Body; D H a Line drawn parallel to A S. Let V S $= d$, A S $= z$; the *Semi-tranſverſe* Diameter (a,) the *Semi-conjugate* (b;) and let the Abſciſs V D ($= x$) flow uniformly, ſo that its Fluxion $\dot{x} =$ A a, may be conſtant; while the Fluxion of the Ordinate A D, *viz.* B a $= \dot{y}$, is variable.

1220. Then the Equation of the Conic Section being $2ax \mp xx = \frac{aa}{bb}yy$ (767,) we have $2a\dot{x} \mp 2x\dot{x} = \frac{aa}{bb} 2y\dot{y}$; and ſo $\dot{x} = \frac{a^2 y\dot{y}}{ab^2 \mp b^2 x}$. Alſo A e : $e\,g$: : A D : T D; that is, $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} =$ T D, the Sub-tangent. Therefore

$$TD = \left(\dot{x} \times \frac{y}{\dot{y}} = \frac{a^2 y\dot{y}}{ab^2 \mp b^2 x} \times \frac{y}{\dot{y}}\right) \frac{a^2 y^2}{ab^2 \mp b^2 x} = \frac{a^2}{b^2} y^2 \times \frac{1}{a \mp x}$$
$$= \frac{1}{a \mp x} \times 2ax \mp xx = \frac{2ax \mp xx}{a \mp x}.$$



1221. Again, $DT - VD = VT = \left(\frac{2ax \mp xx}{a \mp x} - x =\right)$ $\frac{ax}{a \mp x}$; therefore $TS = \frac{ax}{a \mp x} + d = \frac{ax + ad \mp dx}{a \mp x}$; and by Reason of the similar Triangles TSA, and TDH, we have $TS = \frac{ax + ad \mp dx}{a \mp x} : AS = z :: DT = \frac{2ax \mp xx}{a \mp x}$ $: DH = \frac{z \times 2ax \mp x^2}{ax + ad \mp dx}$.

1222. And lastly, $DS = d - x : DA = y :: Aa = \dot{x} :$ $ac = \frac{y\dot{x}}{d - x}$; and so $Bc = \dot{y} + \frac{y\dot{x}}{d - x} = \frac{\dot{y}d - \dot{y}x + y\dot{x}}{d - x}$; now the Triangle Bbc is similar to Scd, and therefore to SAD, whence $SA : SD :: Bc : Bb$; and so $\frac{SD \times Bc}{2} =$ $\frac{SA \times Bb}{2} = \frac{\dot{y}d - \dot{y}x + y\dot{x}}{2} =$ Area of the Triangle ASB, which is the Fluxion of the Area SVA, or the Space that is uniformly described in the same Time with $\dot{x}$.

1223. And because $y = \frac{b}{a}\sqrt{2ax \mp xx}$, we have $\dot{y} =$ $\frac{b\dot{x} \times \overline{a \mp x}}{a\sqrt{2ax \mp xx}}$; and $\ddot{y} = \frac{ab\dot{x}\dot{x}}{\overline{2ax \mp xx}^{\frac{3}{2}}}$; therefore these Values of y and $\dot{y}$ substituted in the above found Area, will bring it to this Expression $\frac{b\dot{x} \times \overline{ax + ad \mp dx}}{2a\sqrt{2ax \mp xx}}$ for the Time of describing

describing the constant Space $\dot{x}$; or, if $a \mp x = m$, and $\sqrt{2ax \mp xx} = n$, the said Area or Time will be abridged to this Form $\frac{b\dot{x} \times \overline{ax + dm}}{2an}$.

1224. Then since the Body arrives at the Point A with the Velocity $\dot{x}$ in the Direction of the Axis V S, and with the Velocity $\dot{y}$ in the Direction of the Ordinate DA; let DA be continued on, and take $AC = \dot{y}$ in the Point A, and compleat the Parallelogram $e\,a$. And since the Abscifs flows uniformly, the Body will arrive at the Point with a Velocity which will still be as $\dot{x}$, or the Space A a uniformly described in that Time in the Direction V S: But the Velocity with which it arrives at B in the Direction dB or A D is different from $\dot{y}$, or A e; and is $\dot{y} \mp q = Ba$, and so $\dot{y} - Ba = \mp q = Bg$, which differential Quantity being the Variation of the Fluxion $\dot{y}$, will be the *second Fluxion* of y, or $\ddot{y} = \mp Bg = \frac{ab\dot{x}\dot{x}}{\overline{2ax \mp xx}^{\frac{3}{2}}} = \frac{ab\dot{x}^2}{n\sqrt{n}}$, and therefore will be as the Space passed over in the Direction A D or dB, in the same Time with $\dot{x}$ by a uniform Velocity, generated in that Time in the Fluxion $\dot{y}$.

1225. Since then we have the Space $\frac{ab\dot{x}^2}{n\sqrt{n}}$ passed over in the indefinite Particle of Time $\frac{b\dot{x} \times \overline{ax + dm}}{2an}$, we can find the Space passed through in the constant Particle of Time $= 1$, in the Direction D A, by the following Analogy of the Squares of the Times to the Spaces, *viz.* As $\frac{b^2\dot{x}^2 \times \overline{ax+dm}^2}{4a^2n^2} : 1^2 :: \frac{ab\dot{x}^2}{n\sqrt{n}} : \frac{4a^3}{b\sqrt{n} \times \overline{ax + dm}^2} =$ Space described in the given Particle of Time $= 1$, by the Velocity that is uniformly generated in that Time in the Body A, in the Direction A D.

1226. Lastly, since the Spaces run over in equal Times are as the Velocities, and the Velocities as the Forces by which they are generated; 'tis evident, the Force by which the Body is urged in the Direction A D must be every where, or in any given Particle of Time, proportional to the Space

4 a^3

$\frac{4a^3}{b\sqrt{n} \times \overline{ax+dm}^2}$, which Distance or Force let be represented by A k, and through k draw $i\,l$ parallel to the Tangent T A. Then will the Force A k be resolved into the two Forces $k\,i$, and A i; of which the Latter is in the Direction of A S, and therefore the same with the Centripetal Force by which the Body A tends to S, in every equal or given Moment of Time. And because the Triangle A $k\,i$ is similar to D $k\,l$, and therefore to A D H, we have the following Analogy for the Expression of the Force A i, *viz.* As A D $=$

$$\frac{b\sqrt{n}}{a} : \mathrm{DH} = \frac{zn}{ax+dm} :: \frac{4a^3}{b\sqrt{n} \times \overline{ax+dm}^2} :$$

$\frac{4a^4 z}{b^2 \times \overline{ax+dm}^3} =$ A i, the *Centripetal Force* required.

1227. Because $\frac{4a^4}{bb}$ is a constant Quantity, the said Centripetal Force will be every where as $\frac{z}{\overline{ax+dm}^3} = \frac{z}{\overline{ax+ad\mp dx}^3}$

$= \frac{\mathrm{AS}}{\overline{\mathrm{DC}}^3 \times \overline{\mathrm{TS}}^3}$; because D C $= a \mp x$.

1228. If now we suppose $d = a$, or V S $=$ V C, then will the Center of Force be in the Center of the Curve, and in the *Ellipsis* we have $\frac{z}{\overline{ax+da-dx}^3} = \frac{z}{\overline{ax+aa-ax}^3} = \frac{z}{a^6} = z$, (because we may put $a = 1$.) Also in the *Hyperbola*, because a is negative, or $-a$, we have $-ax - da + dx = ax - aa^2 + ax = -a^2$; and so the Force will in this Case also be as $-z$. In the *Parabola*, a will be infinite, and ax and $\pm dx$ will vanish out of the Equation, and leave only $\frac{z}{\overline{ad}^3} : \frac{z}{1} : z$, as before. And therefore in every Section the Force will be directly as the Distance S A.

1229. If the Center of Force be in the Vertex of the Curve V, then $d = 0$; and the Force will be every where as $\frac{z}{x^3}$. When the Ellipsis becomes a Circle, then $x = \frac{z^2}{2a}$, (660,)

 and

and the Force is every where as $\frac{8a^3}{z^5}$, or inversely as the fifth Power of the Distance from the Vertex. But if the Center S be removed to an infinite Distance, then will the Force be as $\frac{z}{a\,\overline{d \mp d\,x}^3} = \frac{1}{\overline{a \mp x}^3} \times \frac{z}{d}$, but $\frac{z}{d} = 1$, as being each infinite and equal; whence, in this Case, the Force will be every where as $\frac{1}{\overline{DC}^3}$.

1230. If the Center of Force be placed in the Focus of the Section; then (*per Conics*) we shall have $AS = \sqrt{AD^2 + DS^2} = \frac{a\,x + a\,d \mp d\,x}{a} = z$; and therefore $\frac{z}{\overline{a\,x + a\,d \mp d\,x}^3} : \frac{z}{z^3} : \frac{1}{zz}$: Force A i; *That is, the Centripetal Force is every where inversely as the Square of the Distance, when the Center of Force is in the Focus of the Section.* And this is the Case with respect to the *Sun* and *Planets*, and especially the *Comets*, whose Orbits are very eccentric Ellipses.

1231. From the above Demonstrations, we may observe the following Particulars, *viz.* that when the Center of Force S was supposed in the Center of the Curve, then the Force was every where as the Distance z, which was demonstrated also of the Circle; when, in different Circles, the periodical Times are equal; *whence the periodical Times also in similar Ellipses are all equal, if made about the same Center* C.

1232. Or thus, more universally of all Ellipses. Let A = Area of an Ellipse, T = periodical Time. V = Velocity at the Vertex; therefore since A = (S =) T V, we have $\frac{A}{V} = T$, and so in any other Ellipse $\frac{\mathcal{A}}{\mathcal{V}} = \mathcal{T}$; whence $T : \mathcal{T} :: A \times \frac{1}{V} : \mathcal{A} \times \frac{1}{\mathcal{V}} :: ab \times \frac{1}{V} : \mathit{ab} \times \frac{1}{\mathcal{V}}$; and when the Ellipses have the same common *Transverse*, that is, when $a = \mathit{a}$, it will be $T : \mathcal{T} :: b \times \frac{1}{V} : \mathit{b} \times \frac{1}{\mathcal{V}} :: b\,\mathcal{V} : \mathit{b}\,V$; but $V^2 = \frac{bb}{a\mathit{d}}$, and

$\mathcal{V}^2$

$V^2 = \frac{bb}{ad}$ (1204); and therefore since a$d = ad$, it is V : V : : b : b; and so V $b = V$b; consequently T $= T$. *So that universally Bodies revolving in similar elliptic Orbits about the same common Center* C, *will all perform their Periods in equal Time.*

1233. Since T : T : : ab $\frac{1}{V}$: $ab\frac{1}{V}$; and ab = (a $\sqrt{n^2 ad}$ $= n d^{\frac{1}{2}} a^{\frac{3}{2}} =$) V $d^{\frac{1}{2}} a^{\frac{3}{2}}$, and $ab = n d^{\frac{1}{2}} a^{\frac{3}{2}}$; therefore T : T : : $d^{\frac{1}{2}} a^{\frac{3}{2}} : d^{\frac{1}{2}} a^{\frac{3}{2}}$; and so $T^2 : T^2 : : d a^3 : d a^3$. Now let r and r be the Radii of two Circles, and t, t, the periodical Times of those Circles; then it will be $T^2 : t^2 : : d a^3 : d r^3 : : a^3 : r^3$, (when d = r); and in like Manner $T^2 : t^2 : : (d a^3 : d r^3 : :)\ a^3 : r^3$, (when $d = r$.) But since $t^2 : t^2 : : r^3 : r^3$; therefore $T^2 : T^2 : : a^3 : a^3$; *That is, The Squares of the periodical Times are as the Cubes of the transverse Axes, in Bodies revolving in elliptic* (*as well as circular*) *Orbits about a Center of Force, posited in the Focus.*

1234. Hence the *periodical Times in Ellipses are the same as in Circles, whose Diameters are equal to the greater Axes of the Ellipses*; and therefore when the conjugate Diameter (or projectile Force) is = 0, the Curve will become a strait Line, which the Body will describe in $\frac{1}{2}$ the Time it would describe a Circle, whose Diameter is equal to that Line.

1235. Since the Velocity of a Body is every where as the Perpendicular (= p) let fall from S to the Tangent to the Curve in the Point A (1213); and when the Point A is the Extremity of the conjugate Axis, then will the Velocity be V : $\frac{1}{p} : \frac{1}{b} : \frac{1}{\sqrt{al}} : \frac{1}{\sqrt{a}}$ (by putting the *Latus Rectum* $l = 1$). Let a = Semidiameter of a Circle; then since the periodical Times in such a Circle, and the Ellipsis are equal; and in the Circle when T^2 is as a^3, we have $V : \frac{1}{\sqrt{a}}$, every where; it follows, *that the Velocity of a Body revolving in an Ellipsis, is at its mean Distance from the Focus, or Center of Force, equal to the Velocity of a Body revolving in a Circle, whose Semidiameter is equal to that mean Distance.*

1236. The *greatest*, *least*, and *mean Velocities* of a Body revolving in an Ellipsis A *p* Q, and *s p*, will be as S Q, S A, and (see Fig. to Article 1204.) because those Lines are the Perpendiculars to the Tangents in the Points Q, A, and *p*. In the *Parabola*, the *Velocity will always be as the Square Root of the Distance from the Center of Force*; because the Perpendicular to the Tangent is always as the Square Root of the Distance in that Curve (1205.)

1237. As in the *Ellipsis*, the Force is *Centripetal*, so in the *Hyperbola* it will be *Centrifugal*; but in the *Parabola* it will be neither one or the other; for since the Center of Force is there at an infinite Distance, the Body cannot be properly said to move to or from such a Center. And in this Case the Directions in which the Power acts are all parallel; and therefore, *è converso*, when the Directions in which a Power acts upon a Body are parallel, that Body will describe in its Motion the Curve of a *Parabola*. Whence it follows, that since the Center of the Earth is not at an infinite Distance, the Directions in which Bodies near its Surface are attracted towards it are not quite parallel, and therefore the Curves which Projectiles describe are not truly (tho' very nearly) *Parabolas*, but really the Arches of very eccentric *Ellipses*.

CHAP. XIII.

The ELEMENTS *of a* PLANET'S MOTION, *deduced from the foregoing* PRINCIPLES.

1238. WE have hitherto been considering such Physico-Mathematical Principles of Motion, particularly, with regard to revolving Bodies, as will enable us to account very naturally and easily for the Motion of a Planet, or Comet, in its Orbit about the Sun; and therefore the preceding Institutions are to be regarded as the first, or elementary Principles in Astronomy, without which, the *Rationale* of that celestial Science can by no Means appear.

1239. The Motion of a Planet is known to result from a Projectile Force, in the Direction of a Tangent to its Orbit, and a Centripetal Force directed to the Center of the Sun, both which are so compounded together, that the Curve which the Planet describes, in Consequence thereof, approaches very nearly to the Form of a Circle; because in the Case of the Planet, the Projectile Force is almost infinitely greater than the Force of Gravity, by which the Planet tends to the Sun; in which Case, an Orbit nearly circular must be described, as was shewn (1171.)

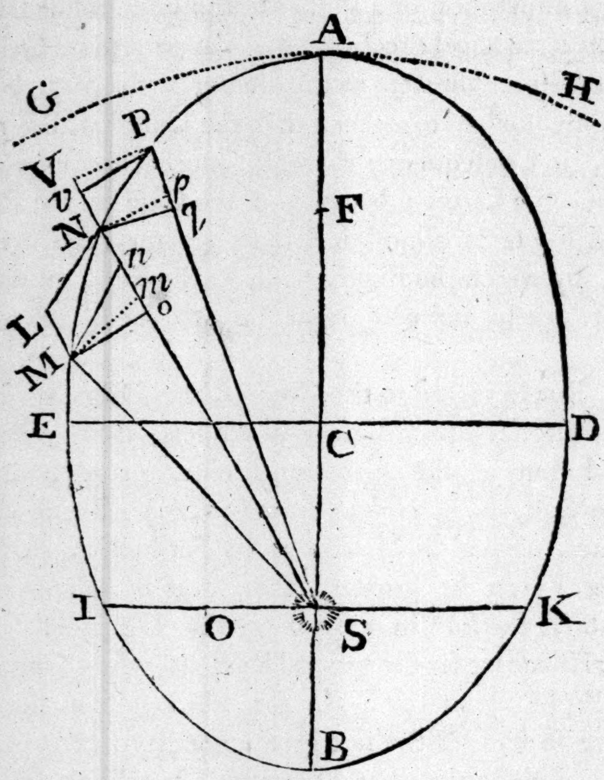
1240. But in regard to the Comets, this Disproportion of the Projectile and Central Forces is much less, if we consider it, as put in Motion at the Aphelion Point, or greatest Distance from the Sun, as is evident from (1206.) But if we consider the Comet, as put in Motion at its Perihelion, then will the Projectile Force be greater, than that by which it would be carried about the Sun in a Circle at that Distance.

1241. Hence we see the general Reasons, why a Comet revolves in an Ellipsis about the Sun; for when it is in the Aphelion, the Force in its Orbit is not great enough to carry it in a Circle about the Sun at that Distance. It will therefore descend from that Point towards the Sun with a variable Velocity in its Orbit always increasing (by 1213,) till at Length it arrives to the Perihelion Point, where its Velocity is greatest of all.

1242. But, as in this Situation its Projectile Force, or Velocity in its Orbit, is greater than that by which it can describe a Circle about the Sun, it will necessarily fly off, and recede from the Sun to greater and greater Distances, but with a Velocity always decreasing, all which is evident, from the variable Ratio of these two Forces above and below what is necessary to produce a circular Motion.

1243. But farther, it has been shewn, that a Body, actuated by a Centripetal Force, which is every where *inversely, as the Square of the Distance*, must describe a conic Section about that Body, or Point, supposed to be placed in the Focus of the Section (1230,) and in all Cases where the projectile Force is to that which would carry it in a Circle, at the same Distance in a Ratio less than that of $\sqrt{2}$ to 1, the Section will be an *Ellipsis* by (1206.)

1244.



1244. When the projectile Force is nearly equal to that of the Circle at the same Distance, the Ellipsis will differ but little from a Circle; and this we find is the Case of all the *planetary Orbits.* And, on the other Hand, when the Projectile Force compared with that of a Circle ($=1$), is nearly equal to $\sqrt{2}$, then will the Orbit be extremely elliptical, and its extreme Parts, or Apsides will nearly coincide with a *Parabola*, and such are the Orbits we find pertaining to all the *Comets*, and this is the Reason why Astronomers generally make use of the Curve of a Parabola for calculating the Motion, Places, and other Phænomena of Comets, in the perihelion Parts of their Orbits, which admits of greater Ease, Expedition, and hardly any sensible Error.

1245. But to be more particular, with regard to the Motion of the Comet in every Part of its Orbit, it must be remembered, that the *Forces*, *circular Velocities* and *Distances*, were all defined by the following Equation $F\,D\,v\,v = f\,d\,V\,V$ (1172.) And since the Comet is actuated by the Force of Gravity, which

which every where increases in Proportion, as the Square of the Distance decreases; that is, $F : f :: d^2 : D^2$; this will give $V : v :: \sqrt{d} : \sqrt{D}$, or the circular Velocity will be every where inversely as the Square Root of the Distances; but the Velocity of the Comet in its Orbit is every where inversely as the Perpendicular to the Tangent.

1246. From hence it will follow, that the Comet descends from the Aphelion A, towards the Sun S, because its Velocity is there less than the circular Velocity; as it descends to lesser Distances P S, its Velocity in its Orbit increases in a higher Proportion than the circular Velocity at the Distance P S; and this will be the Case every where, till the Comet arrives at the lowest Point, or Perihelion B, where the Proportion of its Velocity to that which it had at A, will be as A S to B S, by (1213,) but the Proportion of the circular Velocities at B and A will be, as $\sqrt{AS}$ to $\sqrt{BS}$. Suppose $SB = 1$, and $SA = 4$, then will the Velocity of the Comet at B be four Times greater than at A, but the Velocity in the Circle at B will only be twice as great as that in a Circle at A.

1247. Whence it easily appears, that the Velocity in the Orbit, getting the better of that in the Circle, will carry the Comet off again from the Sun, when it has attained the lowest Point B, since there it is greater than the Velocity of the Circle at the same Distance; and, as it recedes from the Sun, the Velocity in the Orbit will decrease much faster than the circular Velocity; the Latter will prevail by Degrees, and cause the Comet to describe in its Ascent a Semi-ellipse B D A, equal to that in its Descent A E B, till at last, having attained the highest Point A, its Motion is then in the Direction of the Circle A G, but for want of a sufficient Projectile Force, to continue in that Circle, it will descend again from thence towards the Sun as before. Therefore the Velocity in the Circle prevailing in the higher Apsis A, and the Velocity in the Orbit in the lower Apsis B alternately, will occasion the Comet to describe the same Ellipsis perpetually about the Sun. We here suppose the Planet, or Comet, is not affected by any other Force than that of Gravitation to the Sun.

1248.

1248. To make this Matter ſtill more evident, we may conſider the Proportion of the *Centripetal* and *Centrifugal* Forces. We have ſhewn (1230,) that the Centripetal Force every where encreaſes, as the *Squares of the Diſtances decreaſe*; alſo it has been ſhewn (1174,) that the Centrifugal Force ariſing from the circular Motion about the Sun S, does increaſe in Proportion, as the *Cubes of the Diſtances* decreaſe; ſo that, when the Comet arrives to the Perihelion B, its Gravity is but 16 Times greater than at A; but the Centrifugal Force, or that by which it endeavours to fly off from the Sun, is 64 Times greater than in the Aphelion A, ſuppoſing, as before, that A S be 4 Times greater than B S.

1249. Hence it will follow, that tho' Gravity prevails in the higher Part of the Orbit, the Centrifugal Force (as it increaſes much faſter) will prevail over it in the lower Part, and ſo prevent the Comet from approaching any nearer to the Sun. But at B, as the Comet recedes from the Sun in its Aſcent, this Force will be conſtantly checked by Gravity, which, as it decreaſes in a much lower Proportion than the Centrifugal Force, will, at length, prevail over it at the higheſt Apſis A, and there put a Stop to any farther Receſs from the Sun. Here the Comet again begins to deſcend, by Virtue of a ſuperior Gravity, and ſo alternately deſcends and aſcends, according as the Action of theſe two Powers prevails.

1250. Such are the mechanical Laws and Principles of a *planetary* or *cometary Motion*; and it may be worth while to obſerve, that were the Centripetal Force to be in any other Ratio than that of the Squares of the Diſtances reciprocally, ſuch a regular and beautiful Order could not have obtained in the Syſtem. Thus, for Inſtance, ſuppoſing that the Centripetal Force as the Cubes of the Diſtance inverſely, (or that $F:f::d^3:D^3$) then from the foregoing Equation (1245,) we ſhall have $D:d::v:V$, or the circular Velocities will be in the inverſe Proportion of the Diſtances. But that is the very ſame Proportion that the Velocities in the Orbit have at A and B. And therefore, ſince the Velocities in the Circles and in the Orbit at A and B, vary in the ſame Proportion, it is evident, that the ſame which prevails at one Diſtance, muſt prevail at the other;

others; and therefore, if the Velocity in the Orbit at A be less than the circular Velocity, there the Comet will begin to descend, and it must always continue to descend, for the same Reason that it first of all began to do so, and consequently will, after an infinite Number of Revolutions, fall into the Sun. But if, on the other Hand, the Comet be supposed at B, there the circular Velocity, being greater than that in the Orbit, will carry it off from the Sun; and because it continues always in the same Proportion greater, the Comet must ever keep rising in spiral Revolutions from the Sun. Therefore, in this Law of Gravity, the present Frame of Nature could not in the least exist.

1251. The same Thing would also appear from what we have said of the Centrifugal Force; for as that Force every where increases in the reciprocal Ratio of the Cube of the Distance, which is the very same Ratio as that in which Gravity is supposed to increase, it must follow, that, if Gravity once prevail, as in the higher Apsis A, it must ever prevail over the Centrifugal Force, and cause the revolving Body constantly to descend in a spiral Orbit toward the Sun. But if, on the contrary, the Centrifugal Force prevail in any Point, as at B, then that Force will ever prevail over Gravity, and not only make the Body begin, but cause it continually to recede from the Sun.

1252. If the Gravity increases in a higher Proportion than as the Cube of the Distance decreases; then will the circular Velocity increase in a higher Proportion than the Distances decrease, and consequently, in a higher Proportion than the Velocity in the Orbit increases from A to B; so that, as the circular Velocity exceeds the Velocity in the Orbit at A, it will much more exceed it at B, and consequently, the Body will every where continue to descend to the Sun with an accelerated Velocity; and the higher the Power of the Distance is, to which the Gravity is reciprocally proportional, so much the quicker, or in a less Number of Revolutions, will the Body descend to the Center of Force. On the other Hand, if once the Body recede from the Center, it must continue to do so for ever.

1253. Again, if Gravity increase in the reciprocal Proportion of some Power of the Distance between the Square and

Cube, the Body will take more than half a Revolution to descend from the higher to the lower Apsis; for it takes half a Revolution, when the Gravity is reciprocally as the Square of the Distance, and it has no lower Apsis, when it is reciprocally as the Cube of the Distance, whence the above Proposition is evident.

1254. If the Gravity increase in Proportion as some Power of the Distance less than the Square decreases, the circular Velocities will increase in a lower Ratio than that, in which the Velocity in the Orbit increases, and consequently, the latter will more easily prevail; also the Centrifugal Force will sooner exceed the Gravity, and therefore the Body will descend to the lower Apsis in less than half a Revolution, and return to the higher Apsis, in less than a complete Revolution.

1255. From all that has been said, it appears, that were the Planets, or Comets, affected in their Motions by one attracting Body only, whose Power is reciprocally as the Squares of the Distances, then they would describe what one might call a fixed Ellipsis, whose Apsides have no Motion at all. But if it happens, that any foreign, attractive Force be added to that of the Sun, so as to make the Sum, or Difference of those Gravities, vary in a higher or lower Proportion than that of the Squares of the Distances inversely, it will occasion the Apsides to move forward or backward, and the elliptic Orbit to become, as it were, moveable. The Excentricity of such an Orbit will also be changed, and the periodical Times considerably varied; and this is really the Case with regard to the primary and secondary Planets, and Comets; but particularly in the Case of our own Moon, the gravitating Force of the Earth, added to, or substracted from that of the Sun, makes her Phænomena very variable in all the abovementioned Circumstances. Also the Forces of *Jupiter* and *Saturn* will sensibly disturb the Motions of the Comets, in regard to their Velocity, Aphelion Distance, periodical Times, *&c.*

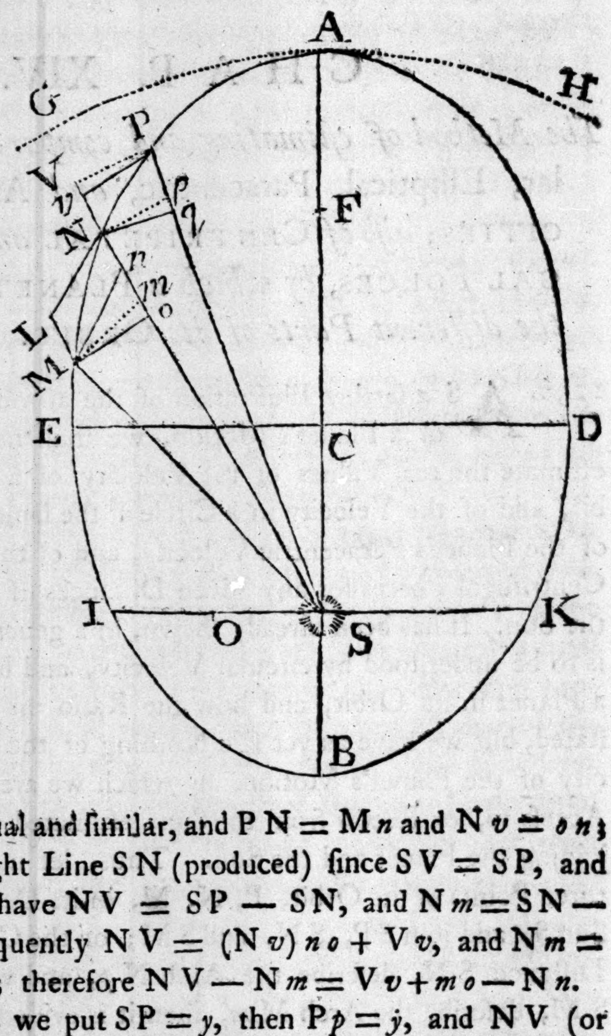
CHAP.

CHAP. XIV.

The Method of estimating and comparing the Circular, Elliptical, Paracentric, *and* Angular VELOCITIES; *also of* CENTRIPETAL *and* CETRIFUGAL FORCES, *by which a* PLANET *is affected in the different Parts of its* ORBIT.

1256. AS a farther Illustration of the astronomical Elements of a Planet's Motion, we shall next shew how to estimate the real Values of the Velocity of a Planet in its Orbit, and of the Velocity in a Circle at the same Distance; also, of the Planet's Paracentric Velocity, and of the Centripetal and Centrifugal Forces for any given Distances of the Planets from the Sun. It has been already shewn, in a general Manner, what is to be understood by circular Velocity, and by the Velocity of a Planet in its Orbit, and how the Ratio may at all Times be stated, but we have as yet said Nothing of the *Paracentric* Velocity of the Planet's Motion, by which we are to understand its Access to, or Recess from the Sun, estimated in a right Line, joining the Planet and the Sun. Thus, let a Planet be in the three Points of its Orbit, P, N, M, in its Descent towards the Sun S, and join S P, S N, and S M; on the Center S, with the Distance S N, describe the Arch N *p*, and with the Distance S M, describe the Arch M *m*, then it is evident, when the Planet comes to N, it will be nearer the Sun than it was at P, by the Distance P *p*; and at M it is nearer the Sun than at N, by the Distance N *m*; now these two Distances, or Lines, P *p* and N *m*, are called the Paracentric Velocities of the Planet's Motion in those Parts of its Orbit.

1257. On the Center S, describe the Arch P V, and from P let fall the Perpendicular P *v*; also, from N and M draw the Perpendiculars N *q*, M *o*. Lastly, draw M *n* parallel to the Tangent L P, and L M parallel to S N, then V *v*, or *p q* denotes the Planet's Centrifugal Force at the Points P or N; also, *m o* denotes the same Thing at M: And L M is the Expression of the Centripetal Force, as is evident from what we have said in the XIIth Chapter.



1258. From the Points P and M, draw the Lines P v, M o, perpendicular to S N; then because the Triangles P S N, N S M are equal, (the Times being supposed equal) therefore (because the Base S N is common to both) the Altitudes P v, M o are equal; take N n = L M, then the Triangles P N v, M n o will be equal and similar, and P N = M n and N v = o n; again, in the right Line S N (produced) since S V = S P, and S m = S M, we have N V = S P — S N, and N m = S N — S M; and consequently N V = (N v) n o + V v, and N m = N n + n o — o m; therefore N V — N m = V v + m o — N n.

1259. If now we put S P = y, then P p = $\dot{y}$, and N V (or P p) — N m = $\ddot{y}$; then because V v = p q = m o, therefore $\ddot{y}$ = 2 m o — N n, or Fluxion of the Paracentric Velocity. Now it is evident, that while the Paracentric Velocity $\dot{y}$ increases, its Fluxion $\ddot{y}$ will be Negative or — $\ddot{y}$ = N n — 2 m o (932) till at Length it becomes Nothing, or $\ddot{y}$ = 2 m o — N n = o, in which Case 2 m o = N n. After this, the Paracentric Velocity $\dot{y}$ decreases, and its Fluxion is affirmative, or + $\ddot{y}$ = 2 m o — N n; till the Planet arrives at B, where it entirely vanishes. From hence we learn

1260. First, that in the Descent of the Planet from its Aphelion Distance A, toward the Sun, its Paracentric Velocity, begins and increases till it arrives at a certain Point; after which it de-

decreases continually, till it vanishes in the Aphelion Point B, or its Velocity of Access to the Sun is accelerated in the first Part of its Orbit, and retarded in the last; and *vice versâ* in regard to its Recess in the other Part of its Orbit.

1261. Secondly, the Fluxion of the Paracentric Velocity $-\dot{y} = \mathrm{N}n - 2mo$ shews, that so long as the said Velocity continues to increase, or be accelerated, *Twice the Centrifugal Force* ($2mo$) *will be less than the Centripetal Force* N n.

1262. Thirdly, when the Paracentric Velocity is a *Maximum*, or greatest of all, *the Centripetal Force* (N n) *is equal to twice the Centrifugal Force* ($m o$.)

1263. Fourthly, from the Time the Paracentric Velocity continues to decrease, till it vanisheth at B, *the Centripetal Force, or Gravity* (N n,) *will be less than twice the Centrifugal Force* ($m o$.) But that Gravity N n can never decrease so far, as to become equal to the Centripetal Force $m o$, even in the Perihelion Point B, is what we shall shew by and by.

1264. Let the Triangles P S N and M S N be equal, or described in equal Times; then will $\mathrm{SN} \times \mathrm{N}p = \mathrm{SM} \times \mathrm{M}m$, then it will be $\mathrm{N}p : \mathrm{M}m :: \mathrm{SM} : \mathrm{SN}$. Let I K be the *Latus Rectum*, or Parameter of the elliptic Orbit, and $\mathrm{L} \times a$ be a constant Rectangle, equal to $\mathrm{SM} \times \mathrm{M}m$. Put $\mathrm{SM} = \mathrm{D}$, and $\mathrm{IK} = \mathrm{L}$. Then the Arch $\mathrm{M}m = \frac{\mathrm{L}a}{\mathrm{D}}$, and $\mathrm{M}m^2 = \frac{\mathrm{L}^2 a^2}{\mathrm{D}^2}$, and $mo = \frac{\mathrm{M}m^2}{2\,\mathrm{SM}}$ (1171) $= \frac{a^2 \mathrm{L}^2}{2\mathrm{D}^3} =$ Centrif. Force.

1265. Again, the Centripetal Force, or Gravity L M, is inversely as D^2, or directly, as $\mathrm{M}m^2$, (1264,) or as $\frac{a^2 \mathrm{L}^2}{\mathrm{D}^2}$; that is, (dividing by the constant Quantity $\frac{1}{2}$ L,) $\frac{2\mathrm{L}a^2}{\mathrm{D}^2}$, is as the Force of Gravity. Wherefore Gravity is to the Centrifugal Force, as $\frac{2\mathrm{L}a^2}{\mathrm{D}^2}$, to $\frac{\mathrm{L}^2 a^2}{2\mathrm{D}^3}$, or as D to $\frac{1}{4}$ L. That is, *The Force of Gravity is to the Centrifugal Force every where, as the Distance of a Planet to a fourth Part of the* Latus Rectum *of the Ellipsis.*

1266. Let $\mathrm{IO} = \frac{1}{2}\mathrm{IS} = \frac{1}{4}\mathrm{L}$, then in the Point I, Gravity is to the Centrifugal Force, as I S is to I O, or as 2 to 1, and consequently, the Paracentric Velocity in the Point I, will be the greatest of all, or a Maximum (1262.)

1267.

1267. That the Centrifugal Force is always leſs than Gravity, will be evident, when we conſider, that the Perihelion Diſtance S B, does, in an Ellipſis, always exceed S O, or $\frac{1}{4}$ of the Parameter, becauſe in a *Parabola*, which lies without the Ellipſis, the ſaid Line S B is then but juſt equal to $\frac{1}{4}$ of that larger Parameter (742.)

1268. Hence it will again appear, that a Planet in the Aphelion Point A, will deſcribe a Circle A G, when the Force of Gravity is there equal to twice the Centrifugal Force. But if it be greater, the Planet will deſcend in an Ellipſis towards the Sun, and in the loweſt Point B, Gravity, being leſs than double the Centrifugal Force, can carry the Planet no nearer to the Sun from that Point; therefore, it muſt of Courſe begin to aſcend with an increaſing Paracentic Velocity of Receſs, till it arrives to the Point K, where Gravity becomes equal again to twice the Centrifugal Force. After this, the Centrifugal Forces leſſening much faſter than Gravity, the Latter will prevail, and the Paracentric Velocity of receding from the Sun will conſequently decreaſe, till the Planet arrives to its Aphelion A, where it will entirely vaniſh.

1269. And thus we ſee, more particularly now, by the Difference of theſe two Forces, how the Planet is made conſtantly to revolve to and from the Sun, and in ſo conſtant and regular an Order, as to give us the cleareſt Ideas of a Uniformly Variable, and Perpetual Motion.

1270. Beſides the Velocity of a Planet's Motion hitherto mentioned, there is one other, which is called the *Angular Velocity* of a Planet in its Orbit. In order to eſtimate this, it muſt be conſidered, that any Angle is greater in Proportion, as the Arch deſcribed with a given Radius is ſo. And alſo, when the Arch is given, the Angle will be leſs in Proportion, as the Radius is greater; and therefore every Angle will be in a Ratio compounded of the direct Ratio of the Arch, and reciprocal Ratio of the Radius, and farther we have juſt now ſhewn (1264,) that in the Caſe of deſcribing equal Areas, the Arches are inverſely as the Radii; therefore, in this Caſe of a Planet's Motion, *the Angles deſcribed in equal Times will be inverſely as the Squares of the* Radii:

1271. But it has been ſhewn, that the Force of Gravity is

every

every where inversely as the Radius, and consequently, directly as the angular Motion of a Planet in its Orbit; and therefore the Latter will be an adequate Measure of the Former.

1272. Therefore the *Impetus*, or Sum Total of all the Impressions of Gravity, which the Planet acquires in moving from A to P, is to the Impetus acquired at M, as the Angle A S P is to the Angle A S M. Hence likewise it appears, that the Impetus acquired in descending from A to I, is just Half that which is acquired in descending from A to B; and therefore the Impressions of Gravity upon the Planet, as it passes from I to B, are equal to all it receives before, in its Passage from A to I.

1273. Having thus stated the Ratios of the Velocities and Forces concerned in a Planet's Motion, we shall next proceed to illustrate the same by a familiar Instance, where all those Quantities will be expressed in proper Numbers, and therefore be more easily comprehended and understood. In order to this, let A S represent the Distance from the Center of the Earth to the Circumference H A G, a Circle on the Surface of the Earth; it was shewn, that a Body revolving in this Circle (1190) with a Centrifugal Force, equal to that of Gravity, must be projected from A with a Velocity of 26000 Feet *per* Second.

1274. Now, suppose it was required to find the Velocity, with which a Body should be projected from the said Point A, to describe the Ellipses A E B D; so that S B may be equal to 1000 Miles. Then as A S is equal to 4000 Miles, the whole transverse Diameter A B will be equal to 5000, and E D will be equal to 4000; and I K will be 3200; such are the Dimensions of the Ellipsis. Then by the Theory in

$$(1204,)\ n = \sqrt{\frac{bb}{ad}} = \sqrt{\frac{2000 \times 2000}{4000 \times 2500}} = 0{,}63246.$$

Therefore $1 : V :: n : v$, or $1 : 26000 :: 0{,}63246 : 16444$ Feet *per* Second, the Velocity required.

1275. Thus, the Velocity in the Circle and the Ellipsis, at the Point A, is known, and the Velocity at B in the Circle is to the Velocity at A in the Circle as $\sqrt{AS}$ to $\sqrt{BS}$, or as 2 to 1, that is, the Circular Velocity at B will be at the Rate of 52000 Miles *per* Hour (1245.)

1276. But the Velocity in the Orbit at B will be to that by

which it was projected at A, in Proportion as AS to S B, or as 4 to 1. Therefore the Velocity of the Planet in B will be 16444 × 4 = 65776 Feet per Second, which, as it greatly exceeds the Velocity in the Circle, will prevent the Planet from revolving in a Circle about the Sun, and carry it off in its own proper Orbit, in the Manner before-mentioned (1247.)

1277. Then, as to the Forces, that in the Circle at A is equal to Gravity, which suppose to be 100 Pound Weight, and we have shewn, that Gravity is every where to the Centripetal Force of revolving Bodies, as the Distance to $\frac{1}{4}$ of the *Parameter*, that is, in the Point A, it will be as A S to S O, or as 40 to 8, or 5 to 1. Therefore the Centrifugal Force of the Body in A is but 20 Pound.

1278. In the Point B, Gravity is 16 Times greater than at A, or equal to 1600 Pound Weight, and since there the Gravity is to the Centrifugal Force as S B to S O, or as 10 to 8, the Centrifugal Force at B will be equal to 1280 Pounds, which tho' it be considerably less than Gravity, will yet prevent any nearer Approach of the Planet to the Sun.

1279. Lastly, it appears, that the angular Velocity of the Planet at A, is 16 Times less than that at B, or to an Eye placed at S, the Space described in one Second at A will appear 16 Times less than that which is described in the same Time at B, agreeable to (1270.) Since A S = 4 S B, (1274.)

1280. Thus we have applied the Physico-Mathematical Principles of Motion to the Theory of Astronomy, as far as it can be done without the Assistance of OPTICS; but, as the greatest Part of Astronomy, both theoretical and practical, depends entirely upon optical Principles, nor can by any Means be understood without them, it will be necessary here to desist, and proceed to the Elements of the Science of Vision; nor will the Principles of common Optics be sufficient to answer our Purpose, with regard to finishing a complete Treatise of Astronomy. The Doctrine of PERSPECTIVE must be well understood, as also the general Principles of the *Projection* of the SPHERE in *Plano*, and that too in a different Manner from that in which they have usually been treated, these will all be found necessary in the various Branches of that Science. We shall therefore, in the next Place, proceed to lay down a Series of Institutions, containing the Principles of *universal Optics*.

INSTI-

INSTITUTIONS OF Universal OPTICS:

CONTAINING

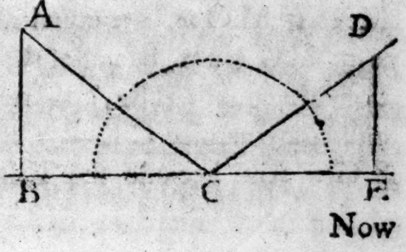
The ELEMENTS of CATOPTRICS, DIOPTRICS, and PERSPECTIVE.

CHAP. I.

The Physico-Mathematical ELEMENTS *of* CATOPTRICS, *or* VISION, *by* reflected LIGHT, *from all Sorts of polished* SPECULUMS, *or* MIRRORS.

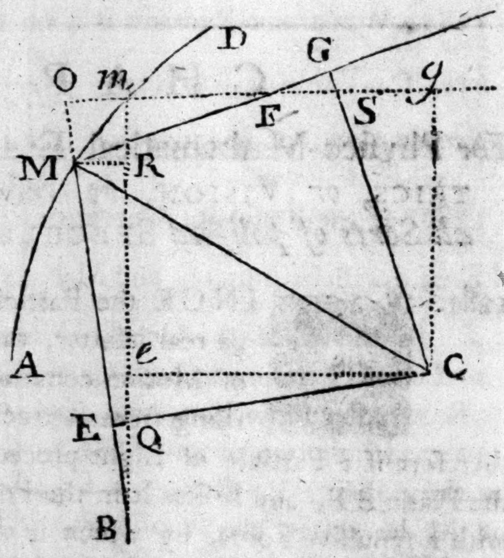
1281. SINCE the Particles of Light are found to be real Matter, they will observe the Laws of Motion common to all other Bodies arising from Attraction and Repulsion; and therefore if a Particle of Light proceeds from the Point A to the Plane BE, and strikes it in the Point C, it will there meet with a repulsive Force, by which it will be reflected from the said Plane in the Direction CD, making therewith the Angle ECD. Now it is required, to find the Point C in the Plane BE, such, that the Ray of Light impinging thereon shall be reflected to the Point D, so that its Passage from the given Point A to another given Point D shall be the least possible.

1282. Let the Perpendicular $AB = a$, and the Perpendicular $DE = b$; $BE = c$, and $BC = x$. Then $CE = c - x$, and $AC = \sqrt{aa + xx}$; also $CD = \sqrt{bb + cc - 2cx + xx}$.



Now, ſince A C + C D, that is, $\sqrt{aa+xx}+\sqrt{bb\times cc-2cx+xx}$ muſt be a Minimum, and therefore its Fluxion $\frac{x\dot{x}}{\sqrt{aa+xx}}+\frac{x\dot{x}-c\dot{x}}{\sqrt{bb+cc-2cx+xx}}$ $=0$, conſequently $x+\sqrt{bb+cc-2cx+xx}+\overline{x-c}+\sqrt{aa+xx}=0$. Therefore $x\times\sqrt{bb+cc-2cx+x^2}$ $=\overline{c-x}\sqrt{aa+xx}$, or B C + C D = C E × A C, and ſo B C : A C : : C E : C D. Hence the Triangles A C B and D C E are equiangular (657;) *and ſo the Angle of Incidence* A C B = D C E, *the Angle of Reflection*; and that this is really the Caſe in regard to the Reflection of Light, we are well aſſured from Experiments.

1283. The Nature of the Curve A M D, the Diſtance of the luminous Point B, and the Poſition of the incident Ray B M being given, it is required to find in the reflected Ray M F, the Point, or Focus F, where all the Rays iſſuing from the Point B will be united. In order to this, let C M be the Radius of Curvature to the Point of Incidence M, and take the Arch M *m* infinitely ſmall, and draw the right Lines B *m*, *m* F; on the Centers B and F deſcribe the little Arches M R, M O; and draw the Perpendiculars C E, C *e*, G G, C *g*, to the Rays of Incidence and Reflection; and ſuppoſe the Diſtance B M = *d* and M E or M G = *a*; then 'tis evident, that the Triangles M R *m*, M O *m*, are equal and ſimilar, and conſequently M R is = M O; and becauſe the Angles of Incidence and Reflection are equal (1282,) therefore C E = C G, and C *e* = C *g*, and conſequently C E — C *e*, or E Q is = to C G — C *g*, or S G: And becauſe the Triangles B M R, B E Q, F M O,



FGS

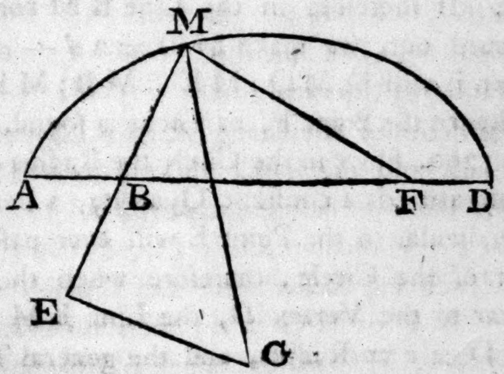
FGS are similar, it is, $BM + BE\ (2d - a) : BM\ (d) :: MR + EQ$, or $MO + GS : MR$, or $MO :: MG\ (a)$: $MF = \frac{ad}{2d - a}$.

1284. If the radiant Point B fall on the other Side of the Point E in respect of M, or (which is the same Thing) if the Curve AMD be *convex* towards the Radiant B, then d will be negative, or $-d$; and therefore $MF = \frac{-ad}{-2d - a} = \frac{ad}{2d + a}$. Hence in this Case, the Focus F will always be positive, or on the same Side the Curve with the Point C; and the Rays after Reflection will diverge.

1285. When the Radiant B is on the concave Side of the Curve, the Value of MF $(= \frac{ad}{2d - a})$ will be positive when d exceeds $\frac{1}{2}a$; but negative when d is less than $\frac{1}{2}a$; and infinite, when $d = \frac{1}{2}a$.

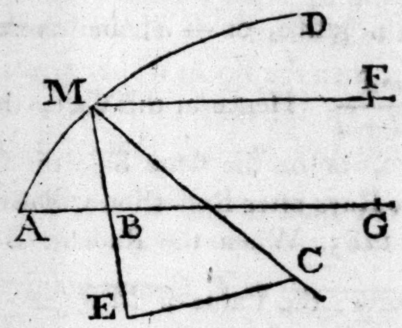
1286. If the Radius of Curvature MC be infinite, then also $ME = a$ will be infinite, and $MF = \frac{ad}{2d \mp a}$ will become $\mp d$. In this Case the Curve, or small Arch M*m* becomes a *strait Line*. Therefore in both Cases, when it is $-d$, or $+d$, the Rays, after Reflection, will diverge.

1287. If any two of the three Points B, C and M be given, the Third may be found. Thus, if the Curve AMD be an ELLIPSIS, and the Radiant Point B be in one Focus, 'tis evident all the reflected Rays MF will be united in the other Focus F (772.) Whence $MF = \frac{ad}{2d - a} = f$, and $a = \frac{2df}{d + f}$; hence $a \times \frac{d + f}{2} = df$; and therefore $d : a :: \frac{d + f}{2} : f$; that is, $BM : ME :: \frac{1}{2}AD : MF$.



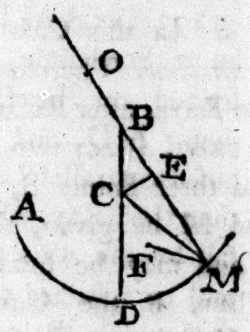
M F. Hence likewise, when the Points B and M are given, the Point E or C may be found, for then $a=\frac{2df}{d+f}$, or $\frac{1}{2}$ A D : M F : : B M : M E; and a Perpendicular on the Point E gives the Point C in a right Line M C, bisecting the Angle B M F. *Note*, The same Demonstration serves for the *Hyperbola*, where $a=\frac{2df}{d-f}$.

1288. In the PARABOLA A M D, the Focus F is removed to an infinite Distance; and the Radiant being in the Focus B as before, 'tis evident the reflected Rays M F will be parallel to the Axis A G; and therefore since M F



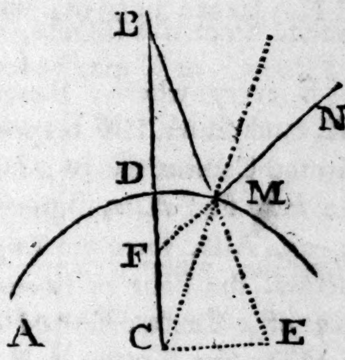
$=f=\frac{ad}{2d-a}$ is infinite, we have $2d=a$, or 2 B M = M E. Therefore a Perpendicular E C erected on the Point E will assign the Point C for the Center of Curvature to the Point M, in the Perpendicular M C.

1289. If A M D be a CIRCLE, since $f=\frac{ad}{2d-a}$, we have $2d-a:a::d:f$. If therefore in the Line B M continued out, we make $MO=2d-a$; then it will be M O : M E : : M B : M F, whence the Point F, or Focus is found.



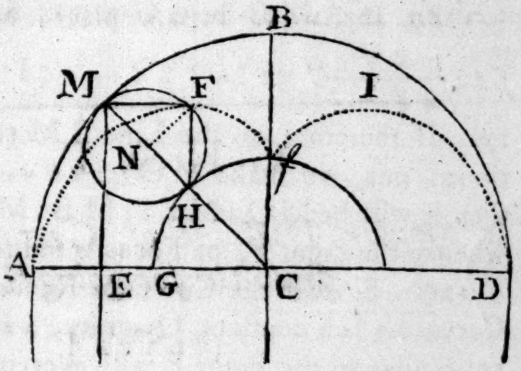
1290. Since in the Circle the Radius of Curvature is a constant Quantity, a Perpendicular to the Point E will ever pass through C the Center of the Circle; therefore when the Point M is infinitely near to the Vertex D, the Line $EM=a$, will be equal to $CD=r=$ Radius, and the general Theorem for finding the Focus F, *viz.* $MF=\frac{ad}{2d-a}$, will become $\frac{dr}{2d-r}=f$, in this particular Case; and the Focus F will be in the Axis of the Curve.

1291.



1291. If the Radiant B be at an infinite Distance, then d being infinite, gives $\frac{dr}{2d-r} = \frac{1}{2}r = f$, or the Focus is then equal to Half the Radius. If $2d$ be greater than r, the Focus f is *positive*, or on the same Side with the Radiant. If $2d$ be less than r, then the Focus will be *negative*, or on the contrary Side, in regard to the Radiant. If d be negative or $-d$, that is, if the Radiant be placed on the convex Side of the Curve, the Theorem becomes $\frac{-dr}{-2d-r} = \frac{dr}{2d+r} = f$, always positive, as before (1284.)

1292. Since $dr = 2df + rf$, and consequently $df + rf = dr - df$, we have $d + r : d :: r - f : f$; that is, B C : B D :: C F : D F. And therefore the Axis of the Curve, or Line B C is *harmonically* divided in the Points C, F, D, B, as will appear, when we treat of *Harmonical Proportions*.



1293. Let A M B D be a Semicircle described on the Diameter A D, and exposed to parallel Rays; then those Rays which fall by the Axis C B will be reflected to f, the middle Point of C B (1291,) and those which fall at A, as they touch the Curve only, will not be reflected at all; and any intermediate Ray E M will be reflected to a Point F, somewhere between A and f; and also, since every different incident Ray will have a different focal Point, therefore those various focal Points will constitute a Curve-line A F f in one Quadrant, and f I D in the other, which Curve is called the *Caustic by Reflection*.

1294. Since d is infinite, we have $\frac{d\,a}{2\,d - a} = MF = \frac{1}{2}a = \frac{1}{2}ME$ every where; therefore if we bisect the Radius C M in H, and drawn H F perpendicular to M F, the Point F will be in the Caustic Curve; for the Triangles M F H and M E C are always similar, and give M H : M C : : M F : M E.

1295. Also, since the Angle M F H is a right one by Construction, the Point or Focus F will ever be in the Circumference of a Circle described on the Diameter M H $= \frac{1}{2}$ M C. Therefore the Caustic A F f is a *Semi-epicycloid*, described by the Revolution of a Circle M F H, on the Periphery of a Circle G H f about the Center C, whose Radius C H = M H, the Diameter of the generating Circle.

1296. If the Angle A C M be Half a right one, than because the Angle E M C = C M F = M C E, the reflected Ray M F will be parallel to A C, and will therefore touch the Caustic A F f in the highest Point F.

1297. This Theory, with respect to the *Caustic* by Reflection, is most evidently confirmed by Experiment; for if a cylinder Bowl, or Glass, be exposed to the Sun-beams, or Candle-light, this Curve A F f l D will appear very strongly delineated on any white Surface placed horizontally within the same.

CHAP. II.

The popular Doctrine of CATOPTRICS, *deduced from the foregoing Theory.*

(PLATE I.)

1298. ACCORDING to the different Modification of the Rays of Light, they receive a threefold Distinction. (1.) They are said to be *parallel* when they proceed in Directions equidistant, or parallel to each other; as at A in Fig. 1. (2.) *Converging Rays* are such as tend to *one Point* F, as those at B. (3.) *Diverging Rays* are such as proceed from a Point F, in different Directions, as represented at C.

1299.

1299. With regard to reflecting Surfaces, usually called SPECULUMS, or MIRRORS, there are likewise three different Forms, *viz.* (1.) A *Plane Speculum*, or Looking-glass, as A B in Fig. 2. (2.) *Concave Speculum*, which is the Segment of a hollow Sphere, foliated on the Outside A V B, or polished on the Inside, as Fig. 3. (3.) *Convex Speculum*, the same Segment of a Sphere, but foliated on the Inside A C B, or polished on the Outside, as in Fig. 4.

1300. If we consider a single Ray of Light D C, falling on these three Surfaces, as it respects but one single Point C in each of them, so the Law of Reflection will be the same in all, *viz. That the Angle of Incidence* D C E *is equal to the Angle of Reflection* E C F (1282,) E C being supposed perpendicular to the several Mirrors in the Point C.

1301. Let A V B (Fig. 5.) be a *Concave Mirror*, E the Center, V its Vertex, and V G the Axis. Then let G be a radiant Point taken any where in the Axis, from whence a Ray of Light G C proceeds to any Point C very near to the Vertex V, and draw E C, the Perpendicular to the Point C, and make the Angle E C f = G C E, and C f will be the reflected Ray, meeting the Axis in f, which is called the *proper Focus*, or that which respects the Distance G V only. (See 1283.)

1302. All parallel Rays, D C falling on the Part C V, extremely near the Vertex V, will be reflected to a Point F, so that D C E the Angle of Incidence be equal to E C F the Angle of Reflection; which Point, or Focus F is the middle Point between E and V, or V F = $\frac{1}{2}$ Radius V E (1291.)

1303. And since this is the Case, with regard to the Sunbeams, which, by such a Mirror are all reflected or converged to the Point F, that Point is called the *Solar Focus*, or *Focus of parallel Rays*; and is relative to such Objects only as are at a very great or infinite Distance.

1304. Again, as the Point F is that where all the Rays of the Sun, falling on the Speculum, are collected into a very small Space, they will be there greatly condensed, and their Action on Bodies, with respect to Light and Heat, so very much encreased, as to produce *accension*, or burning of any combustible Body placed in that Point F, whence it is called

called the FOCUS, or *Burning Place*, and all such *Speculums* are called BURNING GLASSES.

1305. What we have said of a single Ray holds good for any Number, or Quantity of Rays issuing from a given Point; and hence it will follow, *that all the Rays which flow from any particular Point of an Object on a reflecting Speculum*, will all be *converged to one Point nearly*, or made to *diverge from one Point*; and that Point, therefore, will be a Representation of the said Point in the Object, and consequently, since every Point in the Object may easily be conceived to be thus form'd in the *Focus* of the Speculum, the whole Object will be there represented, formed, or depicted in *Imagery*; or there will be an IMAGE formed in the Focus of every distant OBJECT to which it is exposed.

1306. To explain this Matter more particularly, let O B be any Object placed before any Speculum C V D, (Fig. 13.) at the Distance A V in the Axis; let E be the Center of the Mirror, through which, from each extreme Part of the Object O and B, draw the Lines, or Rays O E D and B E C to the Mirror, and as they pass through the Center E, they will be perpendicular to the Surface in the Points D and C. Also from each Point O and B draw the Rays O V, B V, to the Vertex of the Mirror V. Lastly, join O C and B D.

1307. Now it is evident, that since the incident Ray O V on the Vertex on one Side the Axis A V, makes the same Angle O V A, as the reflected Ray V M does on the other, therefore the respective Focus M will be on the contrary Side of the Axis from the radiant Point O. And the same is to be observed with regard to the other extreme Point B, and its Focus I. *Therefore the Position of the Image* I M *before the Concave Mirror is inverted with respect to that of the Object* O B.

1308. The Points O, A, B, in the Object being represented by M, *a*, I, in the Image, it will be found by Computation, *that the Form of the Image is curvelineal*, when the Speculum CD is large, tho' very little so, when it is small in Diameter. The Rule for finding the focal Distances M D, *a* V, I C, for the respective Distances of the Radiant O D, A V, B C is this. *Multiply the Distance of the Radiant by the Radius of the Speculum, and divide that Product by the Difference between twice*

the

the ſaid Diſtance and the Radius; the Quotient will will be the Focal Diſtance required.

1309. *The Object and Image ſubtend the ſame Angle at the Vertex* V *and Centre* E *of the Speculum.* For at the Vertex they are both ſeen under the ſame Angle O V B; and at the Centre of the Angle which the Object ſubtends OEB = IEM the Angle ſubtended by the Image.

1310. *The Lineal Dimenſions, or Magnitude of the Object and Image, are as their Diſtances from the Speculum.* For OB : IM :: AV : *a*V.

1311. Therefore when the Diſtance of the Object is equal to the Radius, (*viz.* when it is placed in the Centre E) then the Image there meets it, *and is equal to it.*

1312. When the Diſtance of the Object exceeds the Radius EV, then will that of the Image be leſs; *and the Image in all ſuch Caſes will be leſs than the Object.*

1313. On the contrary, when the Diſtance of the Object is leſs than the Radius, that of the Image will be greater; *and the Image will be in Proportion larger than the Object.*

1314. Thus ſuppoſe IM a ſmall Object placed at (*a*) between the *Center and Solar Focus* (1291) then will OB be *its enlarged or magnified Image*; and this is the Caſe and Structure of what is properly called a REFLECTING MICROSCOPE, by a ſmall Speculum CVD.

1315. If CVD (Fig. 14.) repreſent the ſame Concave Speculum and E its Center, then if IM be any Object placed nearer to it than the Solar Focus (or half VE,) then by the ſame Reaſoning we ſhall have OB to repreſent *its enlarged and magnified Image on the other Side of the Speculum* (1291); which Image is in this Caſe *erect*, or in the ſame Poſition with the Object. And thus it is, *that all large Concaves* become MAGNIFYING MIRRORS.

1316. Thus it appears that a *Concave Speculum* has a *poſitive and a negative Focus*, and will *magnify or enlarge the Appearance of an Object* in either. And alſo, that it will *diminiſh Objects* in the poſitive Focus only.

1317. If CVD be a *Convex Speculum* (Fig. 14) then any Object OB placed before it will have a *Virtual Focus* only, or

 the

the Rays will be so reflected from it as if they came diverging from a Point behind it (1291) thus the Ray OV will be so reflected from V to B as if it came from the Point M, and the Ray BV will be reflected diverging from the Point I, and the same may be said of all other Rays from the Points O and B; therefore MI will be the Image of the Object OB.

1318. With respect to this Image we observe (1.) That it is always on the contrary Side of the Glass from the Object. (2.) That it is always erect. (3.) That it is ever less than the Object, the Proportion being that of their Distances IV to VO from the Vertex, as before. Hence a *Convex Mirror*, when large, will exhibit a *delightful Landscape of distant Objects*, which is its principal Use.

1319. As to a *Plain Mirror or Common* LOOKING-GLASS, it appears from the Theory (1286). (1.) That the *Focus is always Negative*, or behind the Glass. (2.) That the Distance of the Image behind is equal to the Distance of the Object before the Glass. (3.) That it is erect, and similarly situated with the Object. (4.) That it is of *equal Magnitude* with the Object. (5.) Therefore at the Distance of the Object, its Image on the Surface of the Glass will appear but of half that Length, and consequently a Person of six Feet Height will require a Glass 3 Feet long to view himself compleatly.

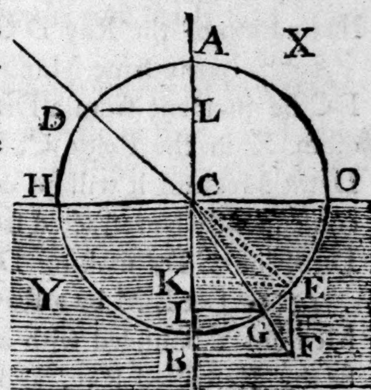
CHAP. III.

The Physico-Mathematical ELEMENTS *of* DIOPTRICS, *or* VISION *by* LIGHT *refracted thro' different Mediums, particularly adapted to* LENSES *for* OPTICAL USES.

1320. WE are taught to understand, by Sir J. NEWTON, that there is a *reflecting and refracting Power* which acts near the Surface of every Medium, or Body, in such Manner as to reflect the Rays of Light at one Instant, and at another to transmit and refract them thro' the Substance of the Medium.

And

And tho' the *Modus Agendi*, or particular Action of the Power be not so accurately ascertained, yet that Author has made it most certain by Experiments, that such a Mode of Action there is, and that its Effects on the Particles of Light are the *Inflection*, *Reflection*, and *Transmission* thereof in and thro' different *Media*. And the particular Modifications of Light from thence arising, he calls *Fits of easy Reflection*, and *Fits of easy Transmission*.



1321. Then admit D C were a Ray of Light incident in one Medium X, upon another Y of a different Density and refractive Power, in the Point C; and suppose it there in a *Fit of easy Transmission*, then if Y be the Denser Medium, or has the greatest Refractive Power, the Ray, by the Action of this Power, will be bent or refracted from its first Direction DCE into another CF, so as to make the refracted Angle F C B of a less Quantity than the Angle of Incidence D C A. The Line AB being supposed perpendicular to the Surface H O of the Medium Y in the Point C.

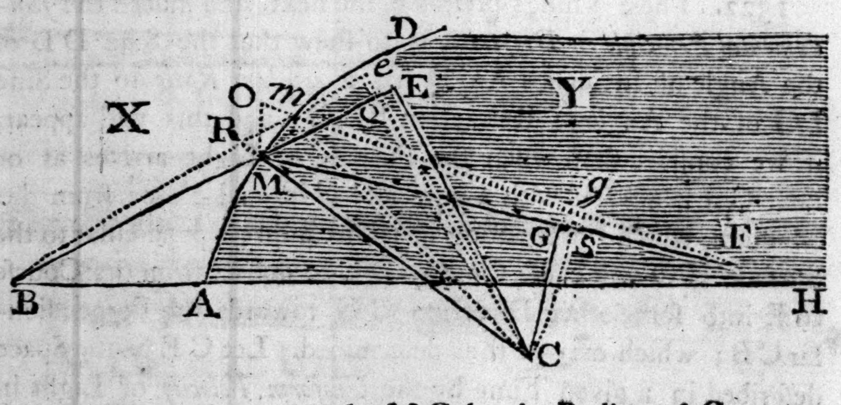
1322. These Things premised, the next, and indeed the *fundamental Principle in Dioptrics*, is to shew that the Sine D L of the Angle of Incidence ACD has a *constant Ratio* to the Sine I G of the Angle of Refraction BCG; and this will appear, if we consider that when the Particle of Light arrives at or near C it is affected with a new and additional Force from the Medium, which Force acting in a Direction perpendicular to the Surface H O will cause the Ray D C to deflect from its Course to E into some other Direction C G towards the Perpendicular C B; which may be thus determined. Let C E be the Space described in a given Time by the *Uniform Velocity* of Light in the Medium X; then will the said Line C E be as the Force which produces that Velocity, (998). From the Point E draw E F parallel to C B, and let E F represent the new acquired Force of the Medium upon the Particle in the Point C, and join C F, then will that be the *new Direction of the Ray*

 and

and the Space described in the same Time thro' the Medium Y, resulting from the two Forces C E and E F (1028). But C F is to C G (or C E) as B F (or K E) is to I G. That is, *the Sine of Incidence* K E *or* D L *is to the Sine of Refraction* I G *as the Velocity* C F, *in the Medium* Y, *to the Velocity of the Ray* C E *or* D C, *in the Medium* X. But these Velocities of Light in different Media are as constant as the Powers of Nature which produce them; therefore so is the Ratio of the Sines D L to I G in every Inclination of the Ray D C.

1323. In the same Manner it is shewn, that if a Ray of Light F C be incident from a Dense Medium Y upon a Rarer Medium X in the Point C, then by the superior Force of the Dense Medium it will be deflected towards its Surface H C, and consequently be refracted from the Perpendicular A C into the Direction C D, making D L to I G as F C to C D, as before.

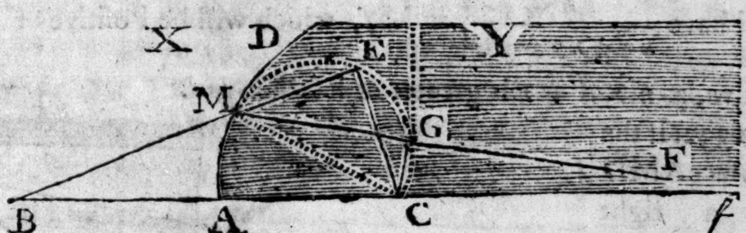
1324. Let B M be a Ray of Light incident on a Refracting Medium Y bounded with a curved Surface A M D, and let M F be the Refracted Ray, and F the Focus to which all the Rays falling on or near the Point M will be refracted. It is required to find the Point F by having given the Nature of the Curve A M D, the Radiant Point B, the Ray B M, and the Refraction of the Mediums X and Y.



1325. In order to this, let M C be the Radius of Curvature to the Point M, and draw B *m* infinitely near B M, and join *m* F and *m* C; from the point C let fall the Perperdiculars C E, C *e*, on the Incident Rays continued; and C G, C *g*, on the Refracted Rays; and on the Centers B and F describe the small Arches M R, M O. And put B M $= d$, M E $= a$, M G $= b$;

the

the little Arch MR $= \dot{x}$, and MO $= \dot{y}$, and the Sine of Incidence CE to the Sine of Refraction CG as m to n (the Radius being MC $= r$) then the right angled Triangles MEC, MRm; MGC, MOm; BMR, BQE, are Similar; becauſe, if from the right Angles RME, CMm, we ſubtract the Angle EMm, there will remain the Angle RMm = EMC; and if from the right Angles GMO, CMm we ſubduct the Angle GMm, there will remain OMm = GMC. Therefore we have the following Analogies, ME : MC :: MC : Mm; That is, $a : r :: \dot{x} : \frac{r\dot{x}}{a} =$ Mm. Again, MG : MC :: MO : Mm; that is, $b : r :: \dot{y} : \frac{r\dot{y}}{b} = \frac{r\dot{x}}{a}$; therefore $a\dot{y} = b\dot{x}$; and $a : b :: \dot{x} : \dot{y} =$ MO $= \frac{b\dot{x}}{a}$. Then BM : BQ (= BE) :: MR : Qe; that is, $d : d + a :: \dot{x} : \frac{d\dot{x} + a\dot{x}}{d} =$ Qe. But (1322) Ce : Cg :: CE : CG :: $m : n$ :: Ce — CE : Cg — CG :: Qe : Sg :: $\frac{a\dot{x} + d\dot{x}}{d} : \frac{na\dot{x} + nd\dot{x}}{md} =$ Sg. Laſtly, the Similar Triangles FMO and FSg, give MO : Sg :: MF : SF or GF; therefore MO — Sg : MO :: MF — SG, or MG : MF; that is $\frac{bmd\dot{x} - aan\dot{x} - and\dot{x}}{amd} : \frac{b\dot{x}}{a} :: b : \frac{bbmd}{bmd - aan - and} =$ MF, the Focal Diſtance required.



1326. If the Curve AMD be a Circle, then CM $= r$ is conſtant, and the Points E and G will be in a Semicircle deſcribed on the Diameter MC; and when the Point M is very near A the Vertex, then ME and MG are both equal to each other, and to MC; that is, $a = b = r$; and the Theorem will become

become $\frac{rmd}{md-nd-rn}$ Mf, in which Case the Focus f is in the Axis of the Sphere.

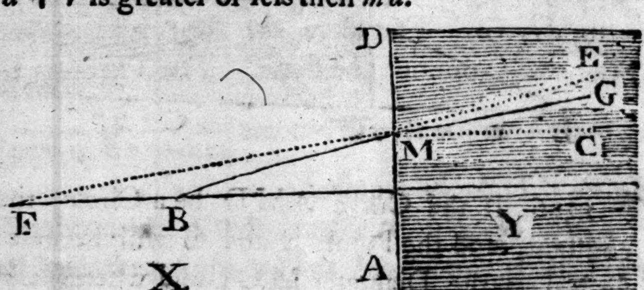
1327. If the Distance of the Radiant B (d) be infinite, or the Rays parallel, then the Theorem will be $\frac{mr}{m-n}=$ M$f=$ Focus of the *parallel Rays*.

1328. If the Rays fall diverging on the Convex Surface A M D; then d being Negative, or $-d$, the Theorem will become $\frac{-bbmd}{-bmd-aan+and}=\frac{bbmd}{bmd-and+aan}=$ M F (Fig. to 1324) and when A M D is a Circle, and M infinitely near to A, then will the Theorem be $\frac{mrd}{md+dn+rn}=$ Mf.

1329. If the Curve A M D were spherically Concave towards the Radiant B, then will the Radius be Negative or $-r$, and the Theorem is $\frac{-rmd}{md-nd+rn}=$ M f the focal Distance; which, because m is greater then n, will be Negative, or on the same Side with the Raidant B, or the Rays after Refraction will *diverge*.

1330. In the Case of *Parallel Rays* falling on the *Spherical Concave*, we have $\frac{-rm}{m-n}=$ Mf, Negative also, because d is infinite.

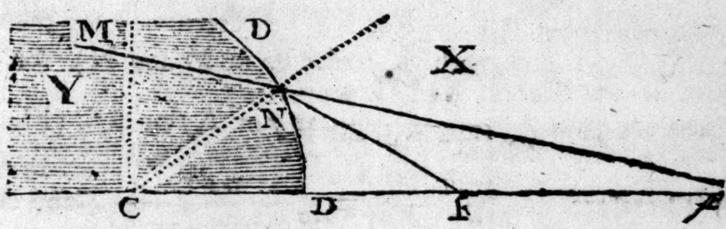
1331. But *Converging Rays* incident on the said Concave will have $\frac{rmd}{nd+nr-md}=$ Mf, which will be Positive or Negative, as $n\times\overline{d+r}$ is greater or less then md.



1332. If the Surface A M D be a right Line, or the Radius M C $=r$ infinite, then the Theorem (1325) will be $\frac{mbbd}{-naa}=$ M F; for in this Case a and b are also infinite; and

and consequently since $aa = bb$, we have $\frac{md}{-n} = f$, the focal Distance; and therefore $md = -nf$, and so $n : m :: d : -f$; whence it appears the Ray will be refracted towards the Perpendicular M C, having a negative Focus f on the same Side with the Radiant B (1321).

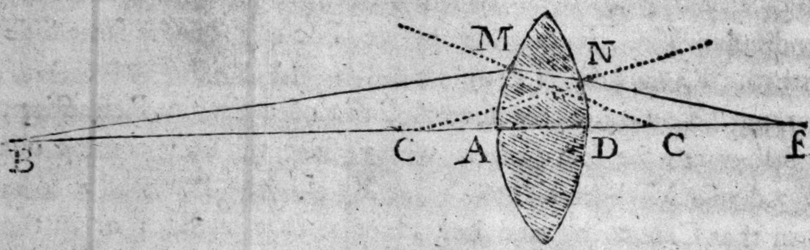
1333. These are the several Cases of a simple Refraction, and they are the same when we consider the Ray coming out of a denser Medium Y into a rarer X, only the Ratio of n to m is in that Case to be used instead of m to n in this; or in the foregoing Theorems, putting n for m, and m for n, interchangeably, and other Things altered as required.



1334. Thus for Example; *Suppose it required to find the Focus of Rays passing out of a denser Medium* Y *into a rarer* X *in a converging State, and refracted at a concave Surface of the Medium* X. Then it is plain the Theorem in (1326) will equally serve here with the following Alterations, *viz.* (1.) Because the Refraction is into a *rarer Medium*, we must write n for m, and m for n. (2.) Because the refracting Surface is *Concave*, the Radius will be Negative, and the Sign where (r) is found must be changed. (3.) Because the Rays are *converging*, the Distance is *negative*, and the Sign where (d) is found must be changed. (4.) Therefore the Sign where d and r are both found, will continue the same. The Theorem therefore, with all these Alterations, will become $\frac{rnd}{md - nd + rm} = \mathrm{f} = \mathrm{D\,f}$, the focal Distance required.

1335. Now it is evident, that if the incident Ray M N in this Case be considered as Part of the refracted Ray M f in the former (1326), then will the Focus f in that Case be the radiant Point in this; and therefore if we substitute f for d, and put

put r for the Radius of Concavity CD, the last Theorem will be $\frac{n\mathrm{r}f}{mf - nf + m\mathrm{r}} = D\mathrm{f} = \mathrm{f}$.



1336. If now we suppose the two refracting Surfaces AM and ND to be very near together, so that the Distance AD be inconsiderable, then may the Focus f be determined for the incident Ray BM after both Refractions at M and N; for the last Theorem gives $\frac{m\mathrm{fr}}{n\mathrm{r} + n\mathrm{f} - m\mathrm{f}} = f = Af$ or Df. (Fig. to 1326) Therefore $\frac{m\mathrm{fr}}{n\mathrm{r} + n\mathrm{f} - m\mathrm{f}} = \frac{mdr}{md - nd - nr}$; and consequently we have $\frac{ndr\mathrm{r}}{mrd - nrd + md\mathrm{r} - nd\mathrm{r} - nr\mathrm{r}} = \mathrm{f} = D\mathrm{f}$, as required.

1337. This Theorem may be abbreviated by putting $\frac{m-n}{n} = a$, (or if $n = 1$, and $m - 1 = a$) it will become $\frac{dr\mathrm{r}}{ard + a\mathrm{r}d - r\mathrm{r}} = \mathrm{f}$; and is thus accommodated to all Optical Purposes, and for a Lens AMND of any Sort, on which Rays diverging, parallel, or converging can fall.

1338. Thus if *parallel* Rays fall on the Lens, then $AB = d$, being infinite, we have $\frac{\mathrm{r}r}{ar + a\mathrm{r}} = \mathrm{f}$.

1339. If the Rays BM are *converging*, then d being negative, we have $\frac{-dr\mathrm{r}}{-ard - a\mathrm{r}d - r\mathrm{r}} = \frac{dr\mathrm{r}}{ard + a\mathrm{r}d + r\mathrm{r}} = \mathrm{f}$.

1340. If the Lens be *equally Convex* on both Sides, or $\mathrm{r} = r$; then the Theorem for *diverging Rays* is $\frac{ar}{2ad - \mathrm{r}} = \mathrm{f}$; for

parallel

parallel Rays where d is infinite, $\frac{r}{2a} = f$. And for *converging Rays*, where d is negative, $\frac{-dr}{-2ad-r} = \frac{dr}{2ad+r} = f$, always positive.

1341. If one Side N D be plane, or the Radius r infinite, then the Theorem for such a *Plano-Convex* Lens, for *diverging Rays*, is $\frac{dr}{ad-r} = f$; for *parallel Rays* $\frac{r}{a} = f$; and for *converging Rays*, $\frac{-dr}{-ad-r} = \frac{dr}{ad+r} = f$.

1342. If one Radius r be Negative, or the Side N D be Convex towards A M, then the Lens is a *Convexo-Concave*, or *Meniscus*; and the Theorem for *diverging Rays*, is $\frac{-dr\mathrm{r}}{adr-ad\mathrm{r}+r\mathrm{r}} = f$; for *parallel Rays*, $\frac{-r\mathrm{r}}{ar-a\mathrm{r}} = f$; and for converging Rays it is $\frac{dr\mathrm{r}}{ad\mathrm{r}-adr+r\mathrm{r}} = f$.

1343. If $r = \mathrm{r}$, then for diverging Rays, $-d = f$; for parallel Rays, $\frac{-r\mathrm{r}}{0} = f$; and for converging Rays, $d = f$.

1344. If both the Radii r and r be Negative the Lens becomes a *double Concave*, and the Theorem for diverging Rays, is $\frac{dr\mathrm{r}}{-adr-ad\mathrm{r}-r\mathrm{r}} = f$, always negative. For parallel Rays $\frac{r\mathrm{r}}{-ar-a\mathrm{r}} = f$, ever negative. And for converging Rays $\frac{-dr\mathrm{r}}{adr+ad\mathrm{r}-r\mathrm{r}} = f$.

1345. If one Radius $-$r be infinite, then the Lens is a *Plano-Concave*; and the Theorems become $\frac{-dr}{ad+r} = f$, for diverging Rays; $\frac{-r}{a} = f$, for parallel; and $\frac{-dr}{ad-r} = f$ for converging Rays.

1346. If the negative *Radii* are both equal, it makes an *equally Concave* Lens; where the Focus of diverging Rays is

found by this Theorem $\frac{-dr}{2ad+r} = f$, and of parallel Rays $\frac{-r}{2a} = f$, in each Case always negative. Converging Rays have $\frac{dr}{-2ad+r} = f$.

1347. These are all the Cases that can happen in the Theory of common Dioptrics, and may all of them be very easily applied in Practice by substituting such Numbers for the Ratio of m to n as we find by Experiments agreeing to Mediums we use.

Thus in	Water,	$m : n :: 4 : 3$,	whence $a = 0,3333$.
	Glass,	$m : n :: 31 : 20$	—— $a = 0,573$
	Diamond,	$m : n :: 5 : 2$	—— $a = 1,5$.

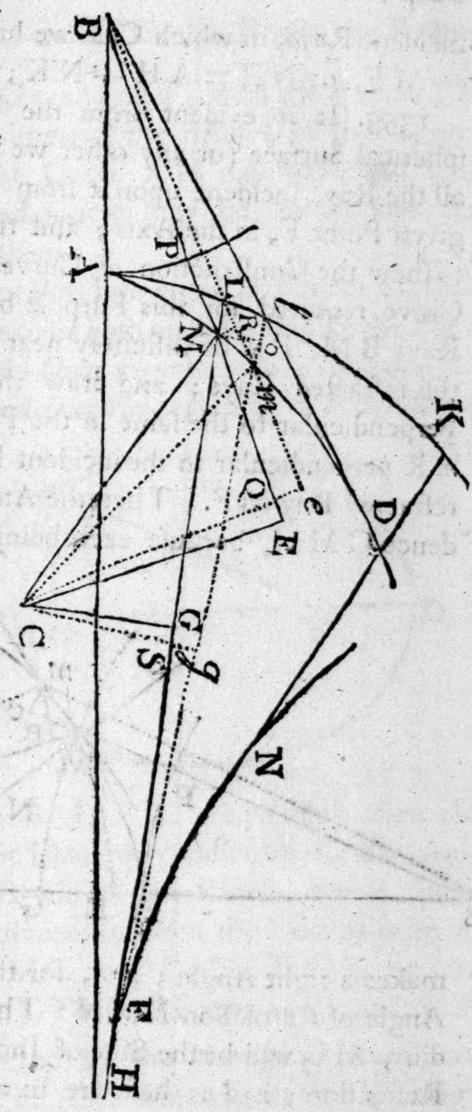
But for a larger View of this Subject see my *New* PRINCIPLES of OPTICS, lately published.

CHAP. IV.

The THEORY *of* DIOPTRICS *continued; the Nature of the* Diacaustic CURVE *explained; and the Method of finding a* GEOMETRICAL FOCUS *for Rays issuing from a given Point in the Axis of a* LENS *of a* Mechanical Figure.

1348. IT has been shewn (1325) how the Focus H, F, N, *&c.* of any Rays B A, B M, B N, *&c.* falling on the Curve A M D may be found, after Refraction, from proper *Data*; and the Curve N F H, which is the *Locus* of all those Points, is called the DIACAUSTIC, or *Caustic by Refraction.*

1349. If



1349. If the Caustic HFN be involved (see 914.) beginning at the Point A, the said Point A will describe the Curve ALK, the *Involute* of the said caustic Curve; whence the Tangent LF + the Part of the Caustic HF, will always be equal to the right Line AH; that is, LF + HF (or KN + HN) = AH.

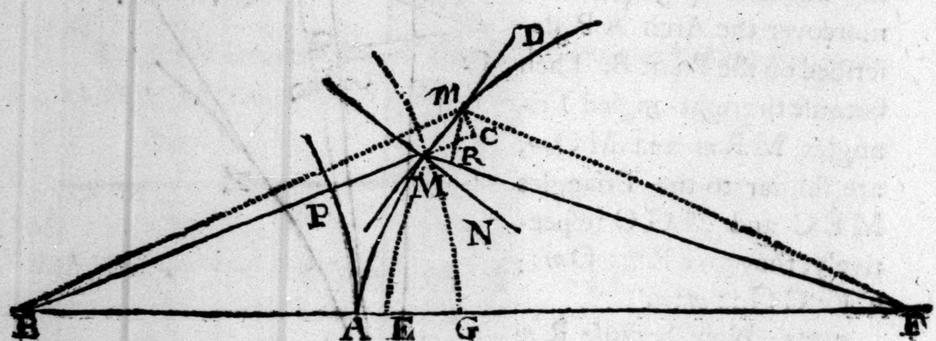
1350. Suppose the other Lines of the Figure drawn as directed (1325,) and moreover the Arch AP described on the Point B. Then because the right-angled Triangles MRm and MOm, are similar to the Triangles MEC and MGC respectively, they give Rm : Om : : CE : CG : : m : n.

1351. Now because Rm = d, the Fluxion of BM = d, and Om = Fluxion of LM, and because BP = BA is a constant Part of BM, the contemporary or proper Fluents of the foreging Fluxions are PM and LM; and since in this Case, the Fluxions and Fluents in the same Ratio (788) therefore PM : LM : : m : n : : BM — BA : AH — MF — FH; and consequently $n \times \overline{BM - BA} = m \times \overline{AH - MF - FH}$; from which Equation, we get $FH = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$, which is the *Rectification of the diacaustic Curve.*

1352. When the Radiant B is at an infinite Distance, then BA = BM; and AP is a right Line perpendicular to those

incident Rays, in which Case we have the Caustic $FH = AH - MF$, or $NH = AH - NK$; &c.

1353. It is evident from the foregoing Theory, that no spherical Surface (or any other we have treated of) can refract all the Rays incident upon it from a given Point B, to another given Point F, in the Axis; and therefore it becomes necessary to shew the Construction of Curves that will do this. Let the Curve required for this Purpose be A M D, and the incident Rays B M, B *m* be infinitely near each other, and M F, *m* F, the refracted Rays; and draw the Tangents D *m* and M N perpendicular to the same in the Point M. Lastly, draw *m* C, *m* R perpendicular to the incident Ray B M, continued out, and refracted Ray M F. Then the Angle M *m* C = Angle of Incidence C M N, because each being added to the Angle *m* M C



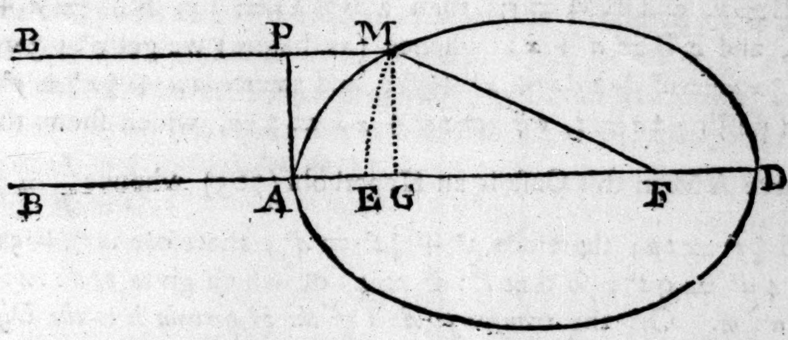
makes a right Angle; and, for the same Reason, M *m* R = the Angle of Refraction F M N: Therefore if we make M *m* Radius, M C will be the Sine of Incidence, and M R the Sine of Refraction; and as these are in a given Ratio *m* to *n* (1322) we have $MC : MR :: m : n$. But M C is the Fluxion of the incident Ray, and M R the negative Fluxion of the refracted Ray; the contemporary Fluents therefore of these Fluxions will be in the same Ratio of *m* to *n*. On the Point B describe the Arches P A and M G, and on the Point F describe the Arch M E, then P M and A E are the Fluents mentioned; and consequently we have PM or $AG : AE :: m : n$. Which is the Property of the Curve A M D.

1354. Therefore when the respective *Foci* B and F, and A, the Vertex of the Curve required, are given, the Curve may be thus constructed. Take A G at Pleasure, and say $m : n ::$

AG

$AG : \frac{n}{m} AG = AE$; then on the Center B with the Radius AG describe the Arch MG; and on F, with the Radius FE describe the Arch ME to intersect the Arch GM in M, and the Point M will be in the Curve AM required; and thus all other Points of the said Curve may be found, and the Curve AMD drawn thro' them.

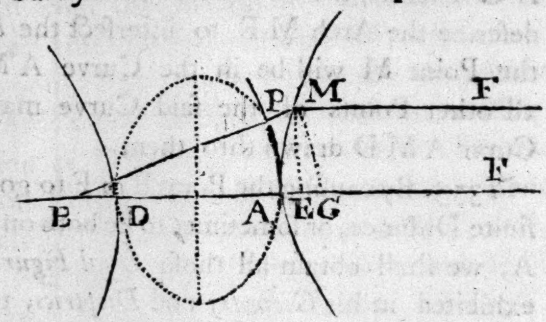
1355. By causing the Point B or F to go off sometimes to an infinite Distance, or sometimes to lie both on the same Side the Point A, we shall obtain all those *Oval Figures* which *Cartesius* has exhibited in his *Geometry* and *Dioptrics*, relating to Refractions.



1356. Thus if the Rays BM, BA, are parallel, then the Arch GM becomes a right Line, perpendicular to the Axis BF, and the Curve AMD will be an *Ellipsis*, whose transverse Axis AD is to the Distance between the Foci as m to n. For put $AF = a$, $AP = MG = y$, $PM = AG = x$; and suppose $m : n :: 3 : 2$ (1347) Then $GF = a - x$, and MF $\sqrt{aa - 2ax + xx + yy}$; and by the Nature of the Curve (1354) it is $\frac{2}{3}PM + MF = AF$, which gives this Equation $\frac{2}{3}x + \sqrt{a^2 - 2ax + x^2 + y^2} = a$. And by Transposition and Involution (212) we have $a^2 - 2ax + x^2 + y^2 = a^2 - \frac{4}{3}ax + \frac{4}{9}x^2$; and consequently $y^2 = \frac{2}{3}ax - \frac{5}{9}xx$; and so $\frac{9}{5}yy = \frac{6}{5}ax - x^2$. Put $\frac{6}{5}a = t = AD$, then $tx - xx = \frac{9}{5}yy$; therefore the Curve is an Ellipsis (764.) And $\frac{t}{p} = \frac{9}{5}$, consequently $t : p :: 9 : 5$. Let $d =$ Distance between the Foci, then $t^2 : tp :: 9 : 5$, and $t^2 - tp = d^2$ (768) therefore $t^2 - \frac{5t^2}{9} = d^2$, or $9t^2 - 5t^2 = 9d^2$, therefore $t^2 : 9 :: d^2 : 9 - 5 = 4$, that is

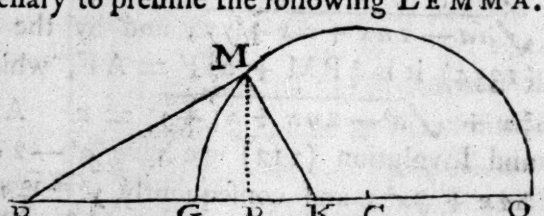
$t : d$

$t : d :: 3 : 2 :: m : n$, or, the *transverse Axis is to the Distance between the Foci as the Sine of the Angle of Incidence to the Sine of the Angle of Refraction*, as *Cartesius* has shewn in his *Dioptrics*.



1357. On the other hand, if the Point F be supposed at an infinite Distance, or the incident Rays F M, F A, parallel, then M E becomes a right Line; then $AB = a$, $AE = x$, and $EM = y$; then also $AG = \frac{3}{2}x$, $BM = a + \frac{3}{2}x$, and $EB = a + x$; whence (as before) we get $a^2 + 3ax + \frac{9}{4}xx = a^2 + 2ax + x^2 + y^2$; and thence $ax + \frac{5}{4}x^2 = y^2$; and putting $\frac{4}{5}a = t$, we get $tx + xx = \frac{4}{5}yy$, which shews the Curve A M in this Case is an Hyperbola (765) whence $\frac{t}{p} = \frac{4}{5}$, and $\frac{5}{4}t^2 = tp$; therefore $t^2 + \frac{5}{4}t^2 = d^2$; therefore $4t^2 + 5t^2 = 4d^2 = 9t^2$; so that $t^2 : d^2 :: 4 : 9$, which gives $t : d :: 2 : 3 :: n : m$. Or, the *transverse Axis of the Hyperbola is to the Distance between the Foci of the opposite Sections, as the Sine of Incidence to the Sine of Refraction.*

1358. Thus we have shewn what Curves will refract all Rays issuing from any one Point to any other given in a different Medium, or by a *single Refraction*. It remains now to shew how the same Thing may be effected by *two Refractions*; in order to which it will be necessary to premise the following LEMMA.



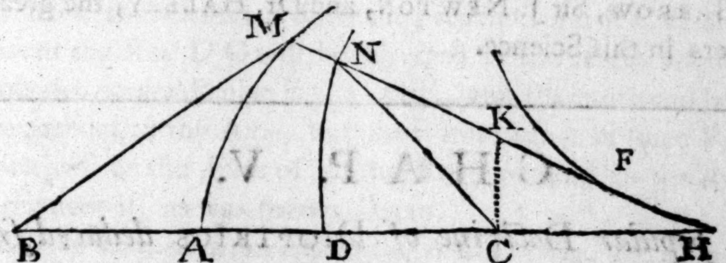
1359. *To describe the Curve* G M, *such, that from any Point* M, *two right Lines* B M, K M *may be drawn to two given Points* B *and* K, *which shall be to each other in the given Ratio of m to n, or that* $BM : KM :: m : n$.

Draw M R perpendicular to B K, and put $BK = a$, $BR = x$, and $MR = y$; then because of the right-angled Triangles B R M, K R M, we have $BM = \sqrt{xx + yy}$, and $KM = \sqrt{a^2 -}$

$\sqrt{a^2 - 2ax + x^2 + y^2}$; and therefore it must be $\sqrt{x^2 + y^2}$: $\sqrt{a^2 - 2ax + x^2 + y^2} :: m : n$; which gives $yy = \frac{2ammx}{mm - nn} - xx - \frac{aamm}{mm - nn}$; and since the Co-efficient of the Square of the Ordinate (yy) is Unity, (764) it is evident the Curve G M is a Circle.

1360. When $yy = 0$, then $\frac{2amnx}{mm - nn} - xx = \frac{aamm}{mm - nn} = \frac{am}{m + n} \times \frac{am}{m - n} = BG \times BQ$ (330.) Therefore GQ (= BQ — BG) is the Diameter of the Circle which is the *Locus* of the Point M.

1361. *The Curve* A M, *the radiant Point* B, *with the Ratio of m to n being given, it is required to describe the Curve* N D, *that shall refract the first refracted Rays* M N, *and make them converge to any given Point* C.



Suppose F H to be the Caustic by Refraction to A M, the radiant Point being in B; it is evident that the same Curve F H will also be the Caustic by Refraction to the Curve N D (1352) and therefore $FH = DH - NF - \frac{n}{m} DC + \frac{n}{m} NC = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$; which Equation will give $\frac{n}{m} AB - \frac{n}{m} BM + \frac{n}{m} DC + AD = MN + \frac{n}{m} NC$.

1362. Therefore to construct the Curve N D, in any refracted Ray A H, take at Pleasure the Point D, for one of the Points in the Curve required. And in any other Ray as M F take $MK = \frac{n}{m} BA - \frac{n}{m} BM + \frac{n}{m} DC + AD$; and

find

find the Point N (by 1360) so that N K : N C : : $n : m$, or N K $= \frac{n}{m}$ N C, and the Point N will be in the Curve required; and thus a sufficient Number of Points may be found thro' which to draw the Curve D N.

1363. From the foreging Theory it appears, that it is possible to determine the Figure for any *Mechanical* Lenses, *Convex* or *Concave*, by which either singly or conjointly, Rays of Light proceeding from any one given Point B in the Axis of the Lens may be accurately refracted to any other Point or Focus f, which in this Case may be called the *Geometrical Focus*: But certain it is, that no Figure for Glasses can be found for such a Focus of Rays procceding from a Point out of the Axis; and therefore it is naturally impossible that the *Defects of Dioptric Vision arising from the Figure of Glasses*, should ever be rectified by Art. We have here given the Substance of all that has been published by KEPLER, DESCARTES, HUGENIUS, Dr. BARROW, Sir J. NEWTON, and Dr. HALLEY, the greatest Masters in this Science.

CHAP. V.

The popular Doctrine of DIOPTRICS *deduced from the* THEORY; *with the Rules for finding the* FOCAL DISTANCES *of all Sorts of* LENSES, *also the* Proportion, Magnitude, Position *&c. of* IMAGES *formed thereby.*

(*Plate* I. *of* OPTICS *and* PERSPECTIVE.)

1364. HAVING thus premised the Theory, we now proceed to make such practical Deductions from it as will be sufficient to acquaint our Readers (not versed in Mathematics) with all the useful Part of *Dioptrics*, and its Application to *Optical Instruments* in every Branch of the *Visual Science*.

1365. Therefore let A B (Fig. 6.) be the Surface of any refracting *Medium* BK denser than Air, as *Water*, *Glass*, *&c.* and let D C be a Ray of Light incident thereon in the Point C; thro'

thro' the Point draw E H perpendicular to the Surface A B, and the Angle E C D is the *Angle of Incidence.* When the Ray arrives at C it is refracted or bent out of its first right-lined Direction D C K into another C F towards the perpendicular C H, and the Angle F C H is the *Angle of Refraction*, less than the Angle of Incidence K C H by the small Angle F C K, which is called the *refracted Angle.*

1366. On the other hand, if a Ray of Light C F in a dense Medium be incident at C upon any rarer Medium, it will be refracted out of its first Direction F C into another C D which will be farther from the perpendicular C E. And the greater the *different refractive* Powers of the Mediums, the greater will be the Difference of the Angles E C D and F C H. All which is evident from the Theory (1322.)

1367. If the refracting Surface A C B be not plain, but spherical; then let E be the Center of the Sphere, and V E F, the Axis thereof. Let C D be a Ray of Light falling on the *convex Surface* (Fig. 7) in the Point C and parallel to the Axis V F; from the Center E draw the perpendicular E C H and it is evident the Ray D C will be refracted in the denser Medium towards the perpendicular E C (1365), and therefore can be no longer parallel to the Axis, but must interfect it at some Point F which will be the *Focus* of all the Rays parallel to the Axis, and very near it, as was shewn (1327.)

1368. In Case of a *concave Surface* A C B (Fig. 8) a Ray of Light D C parallel to the Axis E V, will, at its Entrance into the denser Medium, be refracted alfo towards the perpendicular C H, in such a Manner to F, that were F C to be produced, it would cut the Axis produced in the rarer Medium (suppose *Air*) beyond F. (1329) The Distance of the Piont F from the Vertex V in Water, is *four Times the Radius of the Sphere*, *viz.* $VF = 4VE$; and in *Glass* it is $VF = 3VE$ (1347) or the Focus is distant *three Semi-diameter of the Sphere.*

1369. If a Ray of Light be twice refracted, first into a dense Medium and then into a rare one, its Course after the second Refraction will be variable, according to the Figure of the Surfaces which bound on each Side the denser Medium, and may easily be determined by the Theory (1336.) This Case in *practical Dioptrics* brings us to the Consideration of *Lenses*,

which are the essential Parts of *Telescopes*, *Microscopes*, and all other Instruments of the Dioptric Kind.

1370. Let A V B (Fig. 9) be a *Plano-convex Lens*, then Rays of Light D C, D C, which fall upon it parallel to the Axis V E will, after Refraction thro' it, be converged to a Focus in the Axis at the Point F, which is *nearly equal to the Distance of the Diameter of the Sphere* V F (from the Vertex V) *of which the Lens is a Segment* (1341.)

1371. On the contrary, it follows that if Rays of Light diverge from a Point F, at the Distance of twice the Radius V E of Convexity in any Plano-convex Lens A V B, *they will, after Refraction thro' it, proceed parallel to the Axis* V E.

1372. In like Manner, if A V B (Fig. 10) be a double and equally convex Lens. Then parallel Rays D C falling upon it *will be refracted to a Focus* F *very near the Point* E *or Center of Convexity, as is demonstrated* (1340.)

1373. Therefore on the contrary, when Rays of Light F A, F B, diverge from a Point in any Object in the Focus F of an equally convex Lens A V B, *they will, after Refraction thro' it, proceed parallel to the Axis of the Lens* V E.

1374. Let parallel Rays D C, D C, fall upon a *Plano-concave Lens* A C B (Fig. 11) then will they be so refracted thro' it to f, f, *as if they came diverging from a Point or Focus* F *at the Distance of the Diameter of the Sphere of Concavity of the Lens* (1345) and the Point F is in this Case called the *virtual Focus.*

1375. If parallel Rays D C fall upon a *double and equally concave Lens* (Fig. 12.) *then they will be refracted to* f *and* f *diverging from a Point* E *which is the Center of Concavity* (1346) and the *virtual Focus* of the Lens.

1376. In each of these two last Cases, if Rays f C, f C, converging to a Focus F or E, are intercepted by a single or double concave Lens A B placed at the Distance of the Diameter or Radius of Concavity from the said focal Point, *then the Rays, after Refraction, will proceed parallel among themselves, and to the Axis of the Lens.*

1377. If the Lens be not *equally convex* on both Sides, then the Focus of *parallel Rays* will be found by the following

RULE.

RULE.

Divide the Product of the Radii, by half their Sum, and the Quotient will be the focal Distance required. (1338) *

Example. Suppose one Radius 15 Inches, the other 9; their Product is 135, which divided by half the Sum 12, gives 11 $\frac{2}{10}$ Inches for the focal Distance of such a Lens.

1378. If the incident Rays are not *parallel*, but come *diverging* from a distant radiant Point in the Axis of a Lens of equal Convexity, then the focal Distance is found by this

RULE.

Multiply the Distance of the Radiant by the Radius of the Lens; and divide that Product by the Difference between the said Distance and Radius; and the Quotient will be the focal Distance required. (1340.)

Example. Let the Radius of the Convexity be 15 Inches, and the Distance of the Radiant 60; then their Product is 900, and their Difference 45; therefore 45)900(20 = the focal Distance in Inches, for that Distance of the Radiant.

1379. When the Radii of Convexity are unequal, the Focus of diverging Rays is found by the following

RULE.

Multiply twice the Product of the Radii by the Distance of the Radiant; and then divide by the Difference between the Sum of the Radii multiplied by the Distance, and twice the Product of the Radii; and the Quotient will be the focal Distance required. (1337)

Example. Let one Radius be 15 Inches, the other 9; and the Distance of the Radiant 60; then twice the Product of the *Radii* is 270, which multiplied by 60 makes 16200; the Sum of the Radii 24 multiplied by 60, is 1440, from which take 270, there will remain 1170; then 1170)16200(13 $\frac{85}{100}$ Inches, the focal Distance required.

1380. If the Rays fall *converging* on the Lens, then if we divide by the *Sum* instead of the *Difference*, the Rule will in all other Respects be the same, in each of the two last Cases, for finding the focal Distance for any given Distance to which the Rays tend (1339.)

* This Rule is in general exact enough for Use, since in common Glass the Value of (a) in (1338) is very little more than ($\frac{1}{2}$).

1381. In all that has been hitherto ſaid, it is ſuppoſed that the Lenſes are of Glaſs, and that the Sine of Incidence is to that of Refraction in Glaſs as 3 to 2; agreeable to (1347.) But as there is a conſiderable Difference in the refractive Powers of different Kinds of Glaſs, if we have Regard to that (as in ſome Caſes will be neceſſary) then the Rules above will be ſomewhat more complicated, and we muſt operate according to the Theorems referred to, and take in the Value of (a) as it is found by Experiment, in each particular Sort of Glaſs. †

1382. Let O B be any Object placed at a Diſtance from the convex Lens C D (Fig. 15.) Then it is evident that a Pencil of Rays C A D which flow from the Point A in the Axis will all be converged to another Point (a) in the Axis (if the Diameter of the Lens C D be but ſmall (by 1336) whoſe Diſtance V a may be found (by 1377 or 1378) *and this Point* (*a*) *will be the Repreſentation or Image of the Point* A *in the Object* (1305.)

1383. Let O be a Point in the extreme Part of the Object which ſends a Pencil of Rays D O C to the Lens C D; among theſe one Ray O V will be refracted thro' the Center of the Lens, and therefore the Poſition of the refracted Part V M will be ſimilar or parallel to the incident Ray O V (as will appear from 1332) and when the Thickneſs of the Lens C D is inconſiderable (as in moſt Optical Caſes it is) then O V and V M may without ſenſible Error be eſteemed *one right Line*; and therefore the Axis of that Pencil of Rays, in which at the Point M they will all be united after Refraction, *wherefore* M *will be the Image of the* Point O in the Object.

1384. And in like Manner I will be the Image of the Point B in the Object, and ſo the whole *Object* O B *will be repreſented in its Image* I M. And by calculating by the above Rules it will be found that the Image I a M will be *curvelineal*, more or leſs, as the *Aperture* C D of the Lens is greater or ſmaller.

1385. The *Poſition of the Image*, with reſpect to the ſimilar Parts of the Object, is *inverted*. The Reaſon of this is evident by Inſpection of the Figure, ſince the Axis of the Pencils from each extreme Part of the Object croſs each other in the Center of the Lens.

1386. *The*

† See my NEW PRINCIPLES of OPTICS, lately publiſhed.

1386. *The Object and Image subtend equal Angles at the Center of the Lens.* For since O M and B I may be considered as strait Lines, the Angle B V O, under which the Object appears, will be equal to the Angle I V M, under which the Image is seen from the Center or Vertex V of the Lens.

1387. *The lineal Dimensions of the Object and Image are as their Distances from the Lens respectively*; for since the Triangles O V B and I V M are similar, we have O B, the Length of the Object, to I M the Length of the Image, in the same Proportion as O V or A V the Distance of the Object, to I V or *a* V the Distance of the Image. Hence their *Surperficies* will be as the *Squares*, and their *Solidities* as the *Cubes* of the Distances from the Lens.

1388. *The Object and Image are reciprocal*; for if I M be considered as an Object, then will O B be the Image thereof; and A, *a*, are called the *conjugate or respective Focusses*, for those Distances of the Object and Image.

1389. Hence it appears, that the nearer the Object is to the Lens, the farther off the Image will be formed; and when the Object comes to the Focus of the Lens, the Image will then be at an infinite Distance, since the Rays from every Point will after Refraction be parallel among themselves (1373.) Lastly, if the Object be nearer the Lens than the Focus, the Rays after Refraction will diverge, and in this Case no real Image can be formed of that Object at all by the single Lens.

1390. There is another Form of a Lens called a *Meniscus*, which is concave on one Side, and convex on the other (Fig. 16.) But tho' it was in great Use and Esteem some Years ago, it is now entirely useless, as the *Plano-convex Lens* is known now much to exceed it in those Properties for which it was then so much valued.

SCHOLIUM.

1391. The Lenses hitherto considered are of a spherical Form, and will not admit of a *Geometrical Focus*, not even of Rays flowing from a Point in the Axis (1348, *&c.*) and therefore it is impossible that any such Lens should form a perfect Image of any Object, not so much as in a single Point thereof. We have shewn indeed from the Theory, (1361) that a Lens

may

may be formed that shall have a Geometrical Focus for any radiant Point of its Axis, but not for any other; and therefore it is plain, *no Image of an Object can be formed in any Degree perfect by any Lens whatsoever*; and consequently, *that refracting Telescopes will ever be, in their own Nature, imperfect, if constructed with a single convex object Lens.* But this Imperfection from the Figure will admit of a little Correction from the Addition of *another convex Lens*, but will be increased by joining therewith a *concave one*, as I have largely shewn in my NEW PRINCIPLES of OPTICS. *

CHAP. VI.

Of the NATURE *and* STRUCTURE *of the* EYE; *and the* ELEMENTS *of* VISION *explained from the foregoing* THEORY.

1392. FROM the Elements delivered in the preceding Chapter we are enabled to explain the *true* THEORY OF VISION, as it is performed by the most exquisite of all Dioptric Instruments, the EYE. For which Purpose it will be necessary

* Before we leave this Subject it may be necessary to observe, that the Hypothesis of the Passage of a Ray of Light by Reflection, from A to C and D (see 1282) being a *Minimum*, or the least Possible, is not an arbitrary Position; for if we consider the Reflection of Light as an *Operation of Nature*, and that perfect Wisdom has established the Oeconomy of Nature's Laws, it is necessary to conclude, that every Thing is done in the most direct and simple Manner, and therefore that the Passage of the Ray A C + C D is the least Possible from A to D, by Reflection from the given Plane B E; for this cannot be denied, without asserting, *that the Author of Nature has not taken the most direct and ready Way of doing Things*; which is inconsistent with our natural Notion of Deity, and consequently is irreligious, as well as absurd. Therefore it ought to be esteemed an *Axiom*, or a primary Postulate in Optics, on which the whole Science depends. And notwithstanding it is a self-evident Principle, it is at the same Time the Result of the strictest *Mathematical Theory*; it is also demonstrable on the *Principles of Mechanics*; and *all Experiments testify* that the Angles of Incidence and Reflection are equal, and therefore that the Passage of the Rays of Light from one given Point to another, either by *Reflection* or *Refraction*, must be the least possible.

neceſſary to deſcribe the ſeveral Parts that are immediately concerned in producing this wonderful Effect.

1393. And here it muſt be obſerved, that Viſion is effected by a Refraction of Light thro' the Humours of the Eye to the Bottom or *Fundus*, where the Images of external Objects are formed on a fine Expanſion of the Optic Nerve called the *Retina*, and therefore the anterior Part of the Eye muſt neceſſarily be of a *convex Figure*, and of ſuch a preciſe Degree of Convexity as the particular refractive Power of the ſeveral Humours require for forming the Image of an Object at a given focal Diſtance, *viz.* the *Diameter of the Eye*.

1394. Hence we find, *Firſt*; The external Part of the Eyeball C D (Fig. 17.) is a Pellucid, properly convex, and ſtrong Subſtance, which, when dried, has ſome Reſemblance to a Piece of tranſparent *Horn*, and is therefore called the *Cornea*, or *horney Coat* of the Eye.

1395. *Secondly*; Immediately behind this Coat there is a fine clear Humour which from its Likeneſs to *Water* (in a general View) is called the *Aqueous* or *watery Humour*, and is contained in the Space between C D and G F E.

1396. *Thirdly*; In this Space there is a Membrane or Diaphragm, called the *Uvea*, with a Perforation or Hole in the Middle as at F, of a muſcular Contexture for altering the Dimenſions of that Hole (or *Pupil*) for the adjuſting a due Quantity of Light.

1397. *Fourthly*; Juſt behind this Diaphragm is placed a lenticular Subſtance G E, called from its Tranſparency, the *Cryſtaline Humour*, tho' it be not a fluid Body, but of a conſiderable Conſiſtence. It is contained in a fine Tunic called the *Arachnoides*, and is ſuſpended in the Middle of the Eye by an *Annulus* of muſcular Fibres called the *Ligamentum Ciliare*, as at G and E. By this Means it is capable of being *moved a little nearer to, or farther from the Bottom of the Eye*.

1398. *Fifthly*; All the remaining interior Part of the Eye is made up of a large Quantity of a jelly-like Subſtance called the *Vitreous* or *glaſſy Humour*, tho' there is not the leaſt Likeneſs to Glaſs in it, except its Tranſparency; it being moſt like the *White of an Egg* of any Thing.

1399. *Sixthly*; On one Side of the hinder Part of the Eye as at K, the Optic Nerve enters it from the Brain, and is ex-

panded

panded over all the interior Part of the Eye to G and E all around. This delicate Part is by Nature appointed the *immediate Organ of* SIGHT. On this wonderful Membrane the Image I M of every external Object O B is formed according to the *Optic Laws of Nature*, in the following Manner.

1400. Let O B be any Object, placed at a great Distance A L, from the Eye. Then a Pencil of Rays proceeding from any Point L will fall on the *Cornea* D C, and be refracted by the *Aqueous Humour* under it to a Point in the Axis of that Pencil continued out. Then supposing the *Radius of Convexity* of the Cornea to be 3,3 Tenths of an Inch; and the Sine of Incidence in Air to that of Refraction in the Aqueous Humour to be as 4 to 3 (as it is nearly) then if the Object be infinitely distant, or the Rays parallel, we shall find (per Theorem 1327) that the *focal Distance* after the first Refraction will be 13,3 Tenths of an Inch from the *Cornea*.

1401. The Rays thus refracted by the *Cornea*, fall converging on the *Crystaline Humour*, and tend to a Point 12,28 Tenths of an Inch behind it; also the *Radii* of Convexity in the said Humour are 3,3 and 2,5 Tenths respectively; and the Sine of Incidence is to that Refraction of the *Aqueous* into the *Crystaline Humour* as 13 to 12. Therefore (per Theorem 1328) the *focal Distance* after Refraction in the *Crystaline* will be 10,6 Tenths of an Inch from the fore Part thereof.‡

1402. The Rays now pass from the *Crystaline* to the *Vitreous Humour* still in a *converging State*, and the Sines of Incidence and Refraction being here as 12 to 13 (as found by Experiment); and since the Surface of the vitreous Humour is *Concave* which receives the Rays, and is the same with the *Convexity* of the posterior Surface of the *Crystaline*, the Radius will be the same, *viz.* 2,5 Tenths of an Inch. Then the focal Distance after this third Refraction will be found (by 1331) to be 6,1 Tenths from the hinder Surface of the *Cornea*.

1403. Now the Distance from the hinder Part of the Crystaline to the Bottom of the Eye, or *Retina*, is nearly equal to that focal Distance; and *therefore all Objects at a great Distance have their Images formed on the Retina in the Fund of the Eye,* and

‡ In 1328 for diverging, read converging.

and thereby *distinct Vision* is produced by this *Organ of Optic Sensation.*

1404. When the Distance of Objects is not very great, the focal Distance, after the last Refraction in the *Vitreous Humour*, will be a little increased, and to do this we can move the Crystalline a little nearer the *Cornea* by Means of the *Ligamentum Ciliare* (1397) and thus on all Occasions it may be adjusted for a due focal Distance for every Distance of Objects, excepting that which is less than six or seven Inches, in good Eyes. Many, I know, are of Opinion, that this is effected by a Power in the Eye to *alter the Convexity* of the Crystalline Humour as Occasion requires, but this does not very easily appear.

1405. By what has been said, it appears that Rays of Light flowing from every Part of an Object O B, placed at a proper Distance from the Eye, will have an Image I M formed thereby on the *Retina* in the Bottom of the Eye; and since the Rays O M, B I, which come from the extreme Parts of the Object, cross each other in the Middle of the Pupil, the Position of the Image I M will be contrary to that of the Object, or *inverted*, as in the Case of a Lens (1385.)

1406. The *apparent Place* of any Part of an Object is in the Axis and conjugate Focus of that Pencil of Rays by which that Part or Point is formed the Image. Thus O M is the Axis, and O the Focus proper to the Rays by which the Point M in the Image is made; therefore the *Sensation of the Place* of that Part will be conceived in the Mind to be at O; in like Manner the *Idea of Place* belonging to the Point I, will be referred, in the Axis I B, to the proper Focus B, therefore the *apparent Place* of the whole Image I M will be conceived in the Mind to occupy all the Space between O, B, and at the Distance A L from the Eye.

1407. Hence likewise appears the Reason *why we see an Object upright by Means of an inverted Image*; for since the apparent Place of every Point M will be in the Axis M O at O; and this Axis crossing the Axis of the Eye H L in the Pupil, it follows, that the sensible Place O of that Point will lie, without the Eye, on the contrary Side of the Axis of the Eye to that of the Point in the Eye; and since this is true of all other Parts or Points in the Image, 'tis evident the Position of every Part of the Object

will be on the contrary Side of the Axis to every correſponding Part in the Image, and therefore the whole Object O B will have a *contrary Poſition to that of the Image* I M, *or appear upright.*

1408. The *Dimenſions, or Magnitude*, of an Object O B, we judge of by the Quantity of the Angle O A B which it ſubtends at the Eye. For if the ſame Object be placed at two different Diſtances L and N, the Angles O A B and *o* A *b*, which in theſe two Places it ſubtends at the Eye, will be of different Magnitude; and the *lineal Dimenſions* (*viz. Length* and *Breadth*) will be at N and at L as the Angle *o* A *b* is to the Angle O A B. And the *Surfaces* and *Solidities* of the Objects will be as the *Squares* and *Cubes* of thoſe Angles (670, 675, 1387).

1409. It is found by Experience, that two Points O, L, in any Object will not be diſtinctly ſeen by the Eye till they are near enough to ſubtend an Angle O A L of *one Minute*. Hence when Objects, however large in themſelves, are ſo remote as not to be ſeen under an Angle of *one Minute*, they cannot properly be ſaid to have any apparent Dimenſions or Magnitude at all; ſuch as is the Caſe of the large Bodies of the Planets, Comets, and fixed Stars. But the *Optic Science* has ſupplied Means of enlarging this natural ſmall Angle under which moſt diſtant Objects appear, and thereby encreaſing their apparent Magnitudes to a very ſurpriſing and delightful Degree in that noble Inſtrument we call a TELESCOPE, as we ſhall elſewhere explain.*

1410. On the other hand, we find in the Creation an Infinity of Objects, whoſe Bulks are ſo ſmall, that they will not ſubtend the requiſite Angle (1409) if brought to the neareſt Limits of *diſtinct Viſion*, *viz.* 6, 7, or 8 Inches from the Eye, as found by Experience; and therefore in order to render them viſible at a very near Diſtance, we have a Variety of Glaſſes, and Inſtruments of different Conſtructions, which we uſually call MICROSCOPES, by which thoſe minute Objects appear many Thouſands, yea Millions of Times larger than to the naked Eye; and thereby enrich the Mind with Diſcoveries of the ſublimeſt Nature, in regard to *creating Power, Wiſdom, and Oeconomy.*

1411. If

* The practical Part of Optics containing the Deſcription and Uſe of TELESCOPES, MICROSCOPES, and other Optical Inſtruments, the Reader will hereafter find in the Young *Gentleman* and *Lady*'s *Philoſophy*.

1411. If the Convexity of the Cornea C D happens not exactly to correspond to the Diameter of the Eye, considered as the natural focal Distance, then the Image will not be formed on the *Retina*, and consequently *no distinct Vision can be effected in such an Eye.*

1412. If the *Cornea* be too *convex*, the focal Distance in the Eye will be less than its Diameter, and the Image will be formed short of the *Retina*. Hence the Reason why People having such Eyes are obliged to hold Things *very near* to them, to lengthen the focal Distances (1340) and also why they use *concave Glasses* to counter-act or remedy the Excess of Convexity, *in order to view distant Objects distinctly.*

1413. If the Eye has *less than a just Degree of Convexity*, or is *too flat*, as is generally the Case with old Eyes, by a natural Deficiency of the Aqueous Humour, then the Rays tend to a Point or Focus beyond the *Retina* or Bottom of the Eye; and to supply this Want of Convexity in the Cornea, we use convex Lenses in those Frames we call *Spectacles*, or VISUAL GLASSES.†

1414. Since the Rays of Light O A, B A, which constitute the visual Angle O A B, will, when they are intercepted by a Lens, be refracted sooner to the Axis, (1380) the said Angle will thereby be enlarged, and the Object of Course become magnified; which is the Reason why those Lenses are called *Magnifiers*, or READING GLASSES.

† These *Visual Glasses* are very different from Spectacles in two Particulars; for (1.) They are made with proper Apertures to admit of no more Light than what is requisite. (2.) They are bent to an Angle, that the Rays may fall directly and not obliquely on the Eye; both which Precautions are necessary for easy and distinct Vision.

CHAP. VII.

Of the different Refrangibility *of* Light; *and the* Doctrine *of* Colours *from thence explained by the* Prism, *and applied to* refracting Telescopes.

1415. THE Science of Colours is one of the most delightful Parts of *Physics*; is but of modern Invention; depends entirely on *Optical Principles*; and is the Life of all *painting* and *picturesque Arts*, both Natural and Artificial. Therefore it will be necessary here to lay down the Elements which explain it.

1416. We have already shewn the general Nature of the Refraction of Light in different *Mediums*, (1312) from which it appears that when the two Surfaces of a refracting Medium are parallel, the Ray after Emergence will proceed in a Direction parallel to that which it had at its Incidence; ‡ and farther, we have taken for granted, that all Rays of Light are equally refrangible, or uniformly refracted by any Medium. But when we consider that the Rays are really very differently refrangible in the same Medium, and that the Direction of the Incident and emergent Rays will be different also, when the Surfaces of the refracting Medium are not parallel, but inclined to each other, I say, when these Things are considered, we shall find Matter for new Speculation, and soon unfold the *Doctrine of Colours*.

1417. The Form of a Prism, it is presumed, is well known, consisting of three plain Sides inclined to each other in certain Angles; and therefore if a Beam of Light, as D C (Fig. 18) fall on one Side A B of the Prism in the Point C, it will leave its first Direction, and be refracted to the other Side at E (1312) At its Emergence into the Air at E it will be refracted from the Perpendicular P E to the Side of the Prism, and therefore make still a greater Angle with the Direction of the incident Ray D C. (1323.)

1418. And

‡ This is evident from the Theorem in (1337) by making *r*, and r, infinite, in the usual Method.

1418. And becauſe it is found by Experience, that the Rays of common ſolar Light are *differently refrangible*, that is, ſome Rays are *more* and others *leſs refrangible* by the *ſame Medium*, it will follow, that at the firſt Refraction at C, and at the ſecond at E, the Beam of Light will be dilated, and rendered of a different Form from that of the incident Beam D C. Thus for Inſtance, at E the Part of the Beam which is leaſt refrangible, will be refracted to (*r*) making the leaſt Angle *r* E P with the Perpendicular E P; and thoſe Rays in the Beam which are moſt refrangible will be refracted to (*v*) making the Angle *v* E P the greateſt of all; ſo that the Beam by Refraction at E, is dilated or diſſipated into the Form *v* F *r*, very different from that of the incident Beam.

1419. And moreover, we obſerve the different Rays of the Beam at *r*, *o*, *y*, *&c.* appear of a different Colour; thus the Rays at *r* are *Red*; thoſe at *o* are *Orange*; thoſe at *y*, *Yellow*; at *g*, *Green*; at *b*, *Blue*; at *i*, *Indico*; and at *v*, *Violet-colour'd.* Now its evident, that the different Colours of the Rays muſt be owing to ſome *peculiar Mode of Action* in them on the *Optic Nerve.*

1420. That the *Senſation of Colour* is the Effect of *Light alone*, is manifeſt from hence, that no Sort of Object on which the refracted Beam falls, nor any Difference in the *Mediums* by, and in which it is refracted does ever *change the Colours* peculiar to the ſeveral Parts of the refracted Beam. They are more or leſs intenſe, according to the greater or leſſer refractive Powers of the Mediums, but ſtill the Colours of the ſame Parts of the refracted Beam are always the ſame.

1421. Hence then it follows, that the *Senſation* or *Idea of Colours* is excited in the Mind by the *Action of Light*, as the *efficient Cauſe*; and that the different *Phænomena* of *Colours* are the Effects of different Rays of Light, varying in ſome Property or Quality, which perhaps we do not certainly, if at all, comprehend. Sir *Iſaac Newton* ſuppoſes, with great Reaſon, that this *colorific Quality of the Rays* depends on the *different Sizes or Magnitudes of the Particles of Light which compoſe them*; but to this Hypotheſis there are ſome Objections; and we know of none without any.

1422. It

1422. It must suffice therefore to know, *that Light is the Cause of Colour*; and that where there is *no Light* there can be *no Colour*; and as *Darkness*, or *total Shadow*, is nothing more than the *Absence*, or *Privation of Light*, so BLACKNESS is no other Thing *in itself* than a *Want* of the *natural Operation of Light*; and with respect to *us*, it is the *Want of all Colour* in Bodies. Hence *Black* is, properly speaking, *no Colour at all*.

1423. Since all the various *colour-making Rays r* E, *o* E, *y* E, *&c.* before they are separated by the Prism A B, compound one common Beam of Light, whose Colour is *White*; we may easily thence infer that WHITENESS is not a simple Colour, but only the Result or Compound of all the simple Colours before mentioned (1419) blended together.

1424. Therefore such Bodies which imbibe all the Light incident upon them, or reflect none, will appear absolutely *black*, or colourless. And those which reflect all the Light which falls on them, will appear *White*; and the same in regard to Refraction.

1425. But such Bodies as reflect or refract one simple Sort of Light only, will appear of the Colour peculiar to that homogeneous Ray; thus if any Object reflects or refracts only the Rays E *r*, its Colour will be *Red*; if the Rays E *g*, it will be *Green*; and the Ray E *v* will, when reflected or refracted alone, shew the Object of a *Violet-colour* (1419) and so of the rest.

1426. Again, if Bodies reflect one simple Colour, and refract another, they will appear of one Colour by Reflection, and another by Refraction; as is the Case of *Leaf-gold*, Decoction of *Lignum Nephriticum*, *&c.*

1427. Rays of Light are also *differently reflexible*, and those which are *most or least refrangible*, are also *most or least reflexible*; therefore Bodies will appear of *different Colours in the same Part*, if made to receive the Beam of Light under such Angles of Incidence as are proper to each respective Sort of Rays, for a given Position of the Object and the Eye.

1428. Those Objects which reflect or refract two or more of the homogeneal Rays will appear of a Colour compounded of them; and it is observable, that of three different Rays (next to each other) the two extreme ones produce nearly the Colour of the middle One; thus *Red* and *Yellow* made an *Orange*; *Yellow* and *Blue* make a *Green*; *Blue* and *Purple* make an *Indigo*-colour;

colour; and from hence all the *Phænomena* of *Colours* in natural Bodies arise, and are easily confirmed by Experiments.

1429. The Image of any Object, formed by reflected or refracted Light, must be of the same Colour in every Part with the Object; for the Rays of Light which proceed from the several Parts of the Object are not altered or changed in their Nature by Reflection or Refraction; and therefore whatever Colour they excite in the Object, the same must they shew in the corresponding Part of the Image; the whole Image will therefore be variegated and painted with the same Colours in every Respect as we view in the Object itself.

1430. I think, then, it deserves to be considered, that every PICTURE formed by an *Optic Glass*, ought to be looked upon as the *Portrait of Nature itself*, and consequently deserves a much greater Regard than we usually pay to it. ——— The Performance of *Titian*'s Pencil being as much inferior to the PAINTINGS of NATURE, as a *created Being* is below the CREATOR.

1431. For the same Reason that a Prism separates the Beam of Light into its original or simple Rays, so likewise does a *Lens*, *viz.* because its Sides are inclined to each other (1369) and therefore the Image in the Focus of a single Lens must be as compounded as Light itself, and consequently in some Measure confused; for each particular Species of Rays does in Reality form a distinct Image in its own peculiar Focus. And of Course, when this compound Image is viewed with a deep Magnifier, it will appear both coloured and confused, as we find by Experience in all our Microscopes, Telescopes, *&c.* of the *refracting Sort*.

1432. But Images formed by *reflected Light* are not subject to either of those Imperfections, because there is no different Reflection of Light while the Angles of Incidence are the same; and therefore only one simple Image is formed in the Focus of a *Speculum*; and so perfect, that it will bear to be magnified a *second Time* with sufficient Distinctness; and consequently a *double Power of magnifying* in a REFLECTING TELESCOPE will have the same Effect in a small Length as we have in a very great Length by Refraction; and this is the Reason of that noble Invention.

1433. There

1433. There have been Methods invented to remedy, or rather to palliate, the Imperfections of a refracting Telescope, which I have considered at large in my *New Elements of Optics*, to which I refer the *Reader*, as being a Subject too prolix for a *System of elementary Principles* only. In that Treatise I presume, it is demonstrated, that as there are two Defects in refracting Telescopes, *viz.* one from the different Refrangibility of Rays, and the other from the Figure of Glasses, so the very Means of correcting the former will inevitably augment the latter; ‖ and in those very Telescopes where this Correction has been applied, by *joining a Concave with a convex Object Glass*, the Rays are afterwards made to pass thro' a single convex Lens before the Image is formed, and therefore if the Colours were taken away by the two Glasses, they must be again produced by the first of the five next the Eye. And therefore we have not yet any such Thing as an *achromatic* Refractor, or one that will shew Objects *entirely free from Colours*.

1434. But as in the abovementioned Treatise I had omitted one or two Particulars, relative to this new Refractor, and also for the Sake of the Inquisitive, I shall here give the following Diffection of the compound Object Glass as I found it in one of those Telescopes I purchased for $3\frac{1}{2}$ Guineas, and was three Feet long.

1435. The *convex Lens* was of *Crown-glass*, doubly and equally convex on both Sides; and its solar focal Distance was precisely $9\frac{1}{2}$ Inches. The *concave Lens* was of *White-flint*, or Crystal; it was a Plano-concave, and its focal Distance by Reflection was four Inches from its Surface. The focal Distance of both these combined together, was just $29\frac{1}{2}$ Inches.

1436. The Radius of the convex Lens was 10,1 Inches, as will appear from (1340,) for in Crown-glass, $a = 0{,}532$, and $2a = 1{,}064$; then $1 : 2a :: f : r$, or $1 : 1{,}064 :: 9{,}5 : 10{,}1$. But with regard to the Plano-concave of White-flint, since the Radius is double the solar Focus by Reflection, (1291) it is in that 8 Inches. The two Radii, therefore, in these two Object Glasses, are as 10,1 to 8, or as 160 to 127 nearly: That is

‖ This is to be understood of a convex and a concave Lens of the *same Sort* of Glass; and how little the Case will be altered, by having one of *Crystal* and the other of *Crown-glass*, will appear bye and bye.

is (if R be put for the Radius of the Plano-concave) $r:R::160:127$.

1437. Now it is known by Experience, and is by all confess'd, that two Lenses, one a Plano-convex of *Crown-glass*, and the other a Plano-concave of *Flint*, must have their Radii of Sphericity very nearly as 2 to 3, in order to prevent the Error of Refraction arising from the different Refrangibility of the same Beam of Rays, and that in such a Case, we have $R:r::3:2::8:5{,}35$; therefore in a double and equally convex Lens of Crown-glass to produce the same Effect, the Radius must be $r=10{,}7$; but that in the Telescope is only 10,1 and therefore too convex to prevent a *coloured Image*, when compounded with a Plano-concave of Flint whose Radius is 8 Inches.

1438. Then if the Sine of Incidence be to the Sine of Refraction (of the same Ray) out of Crown-glass into Air as n to m, and out of White-flint into Air as n to M; it is demonstrated by Sir *Isaac Newton* * the Error of Refraction arising from the Figure of the Lens is $\frac{m^2 y^3}{4n^2 r^2}$ in Crown-glass, or $\frac{M^2 y^3}{4n^2 R^2}$ in White-flint, in a Lens of a *Plano-convex* Form. In these Expressions, (y) is the Semi-aperture of the Lens.

1439. Because there is the same Refraction, and Error from thence arising, in an equally Plano-concave Lens, and being made the contrary Way, therefore when the Errors are equal in a Plano-convex of Crown-glass, and a Plano-concave of White-flint, they will destroy each other; or when two such Lenses are combined together they will correct each other, and prevent any Aberration of Rays from the Figure.

1440. Therefore let $\frac{m^2 y^3}{4n^2 r^2} = \frac{M^2 y^3}{4n^2 R^2}$, from whence we have $\frac{m^2}{r^2} = \frac{M^2}{R}$, or this Analogy $M:R::m:r$; but by Experiments it appears, that $M:m::160:153$ in Crown and Crystal. And therefore supposing, the Radius R of a Plano-concave to be 8 Inches, (as in the abovementioned Telescope) then $160:153::8:7\frac{3}{4}=r$, the Radius of a Plano-convex Crown-lens that combined with the other, shall prevent any Error from the spherical Figure.

* See also *Philosophia Britannica*, 2d Edit.

1441. Now ſince the Effect or Refraction is the ſame in a double and equally convex Lens, of a double Radius, *viz.* $r = 15\frac{1}{2}$ Inches; therefore ſuch a Lens only, combined with a Plano-concave of White-flint, whoſe Radius is 8 Inches, can cauſe the Aberration of Rays from the Figure of the Glaſſes to vaniſh. But that in the Teleſcope has a Radius of $10\frac{1}{10}$ Inches only; whoſe Sphericity is therefore much too great to anſwer this Purpoſe.

1442. Therefore ſince in a Plano-convex of Crown-glaſs, and a Plano-concave of White-flint, the *Radii* for preventing the Error from the different Refrangiblity of Rays muſt have the Ratio following, *viz.* R : *r* : : 3 : 2 : : 160 : 107; and for annihilating the Error from the Figure, the Ratio muſt be R : *r* : : 160 : 153; it appears to be impoſſible, that the ſame two Glaſſes, *viz.* Crown and Cryſtal, which correct one Error, ſhould at the ſame Time correct the other; and farther it appears, that the Ratio of the Radii of the Object Glaſſes in the Refractor under Conſideration, being in a Ratio different from either of theſe (1436), can correct neither of the Errors.

1443. But we are told, "*That the Surfaces of ſpherical Glaſſes admit of great Variations tho' their focal Diſtances be limited.* With regard to parallel Rays on a Plano-convex, or Plano-concave, we have ſhewn (1341, 1345.) the focal Diſtance is $f = \frac{r}{a}$; and ſo $r = fa$; but (f) is limited or given by Suppoſition, and (a) is the Refraction of the particular Species of Glaſs (1347) and therefore is given of Courſe; how then does it appear, that the Radius (r) or ſpherical Surface of the Glaſs can admit of ſuch *great* (or indeed any) Variation at all? 'Tis true, the ſame Degree of Sphericity may be divided and variouſly proportioned between the two Surfaces, but while the focal Diſtance is limited, the Refraction, and the Error occaſioned thereby, will be ſtill the ſame.

1444. Therefore the "*Poſſibility of making the Aberrations of any two Glaſſes equal,* is a Thing that does not appear;" no more than how, a "*Perfect Theory for making Object Glaſſes can be obtained,* from any Principles of Optics hitherto publiſhed." A Demonſtration of theſe great Poſitions is a Satisfaction we have

yet to come, and which the Public has long, and with great Impatience, waited for.

1445. In the mean Time it may not be unacceptable to many Persons to be informed and directed how to make this compound Object Lens for their own Use; and so procure at an easy Rate a Telescope, which they have been told is of *infinite Service to Mankind.* For this Purpose, *take a double Convex* and a *Plano-convex* of Crown-glass, and a *double Concave of White-flint*, all ground upon the same Tool, and let the Concave be put between the two Convexes, and place them in the End of the Telescope with the Plano-convex outwards, and they make the *triple compound Lens for taking away Colours* in Refractors of a small Length.

1446. But if the Telescope is to exceed the Length of 18 or 24 Inches, then two Glasses will do, *viz.* one Convex of Crown-glass, and the other Concave of Flint, whose solar focal Distances are to each other as 3 to 4 † very nicely; and they will form an Image without Colours, for the Telescope proposed. If these two Lenses are held together in the Sun-beams, they will converge them to a Focus, and thereby shew the focal Distance of the compound Object Glass, and consequently of the Telescope itself.

1447. But as to the Error from the Figure of the Glasses, I confess it is not in my Power to give any Directions for preventing or extenuating the same in any great Degree. If any Person can find among the different Sorts of Glass, or any transparent Mediums, any two, which being formed into Prisms, *shall have their refracting Angles, which take away Colours, reciprocally proportioned to their Sines of Refraction into Air, respectively*; then he may be assured of a *perfect Theory of making Object Glasses.* —— And he that shall do this, *erit mihi* plusquam *Magnus Apollo.*

† For let $\frac{r}{a} = f$ in a Plano-convex of Crown-glass, and $\frac{R}{b} = F$, in the Plano-concave of White-flint. Then $f : F :: \frac{r}{a} : \frac{R}{b} :: \frac{2}{0,53} : \frac{3}{0,6} :: 0,376 : 0,5 :: 3 : 4$, as in the Prescript above.

INSTITUTIONS OF PERSPECTIVE:

CONTAINING

The *Mathematical* THEORY thereof deduced from OPTICAL PRINCIPLES, and applied to a *General* PRAXIS.

CHAP. I.

The Optical ELEMENTS *of* Lineal PERSPECTIVE.

1448. WE are now prepared to treat of the *Elements*, or *first Principles* of PERSPECTIVE, the most delightful and necessary of all the Mathematical Sciences. These Elements are immediately derived from *Optics*; for *the Art of delineating Objects as they appear on a given transparent Plane, or Superficies, to an Eye at a given Height and Distance*, is the true DEFINITION of PERSPECTIVE.

1449. The Want of these preliminary Principles has rendered the Treatises on this Subject defective in the most essential Part, and the Art of Perspective itself very difficult to be understood: In short, it would be absurd to suppose any Man can understand Perspective, without being acquainted at least with as much of the *Optic Theory* as we have premised, and shall here superadd in this Chapter.

1450. It appears by the above Definition, that in order to delineate the true Appearance of an Object on a given Plane, it will be first necessary to know the Law according to which the apparent linear Dimensions of Object increase or decrease; and here we must observe (1.) That the visual Angle, or the apparent Magnitude of a Line will be less at a *greater Distance*; and *vice versa*. (1408) (2.) That it will be less as the said Line is viewed

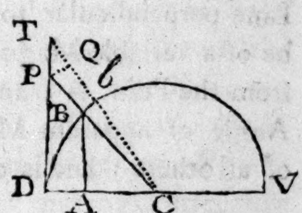
viewed *more obliquely.* (3.) Therefore the Law of Diminution will be nearly *in Proportion to the Distance and Obliquity of the View conjointly.*

1451. It is true, the Diminution of any Line viewed directly increases with the Distance, but not exactly in Proportion thereto, unless the Distances be very great. Let O B (Fig. 19) be an Object viewed directly by an Eye at C. On the Center C, with the Radius C B describe the Semi-circle A D B, and join C O; then will O C B be the Angle under which the Object O B appears at C. Again, suppose the Eye at A, and draw A O; then the Angle O A B is the visual Angle under which it appears at A. Now A B is double the Distance B C, but the Angle O A B is *more than half* the Angle O C B; for the Angle O A B or E A B is just half the Angle O C B or E C B (642.) But at great Distances, the differential Angle E A O will become insensible; and therefore in such Cases, *the Diminution will be directly as the Distance of the Object.*

1452. When any Object as B D (Fig. 23) is viewed obliquely (that is, when the Angles A D B and A B D are not equal) then a Diminution of its apparent Magnitude will ensue; for in all such Cases, the Length B D is reduced to B O, which subtends the visual Angle D A B at the Distance A B. Therefore the Proposition is evident. (1408)

1453. It is further evident, that if the same Object B D were removed to twice the Distance of A B, then the Angle of apparent Magnitude O A B would be but one half so large (642); and therefore the apparent Magnitude of any Object on Account of its Obliquity, is also diminished in Proportion to its Distance; and therefore the whole Diminution of Magnitude on Account of its Distance and oblique Position jointly, is proportional to the *Square of the Distance.*

1454. But we have a more direct Demonstration of this fundamental Proposition in Perspective, in the Diagram of (823) which we shall here insert.



If T P be any Object viewed obliquely by an Eye at C, at the Height of C D above it, then the visual Angle is P C T, and its Measure the Arch B b to the Radius C B, or P Q to the Radius C P.

And

And because $TP : QP :: CP : CD$; and $PQ : Bb :: CP : CB = CD$; therefore (*ex Equo* 652.) it is $TP : Bb :: CP^2 : CD^2$. But when TP is a given Quantity, then (since Radius $CD = 1$) we have $Bb : \frac{1}{CP^2}$ or *the visual Angle* B *b of apparent Magnitude is ever in the inverse Proportion of the Square of the Distance* C P.

1455. Hence it appears, that in any Line A F (Fig. 20.) equal Parts A B, B C, C D, D E, E F, are seen by an Eye at C under such Angles, whose Measures *g h*, *h i*, *i k*, *k l*, *l m*, are reciprocally as the Squares of the Distances A C, B C, C C, C D, C E. Therefore the Circle I K may be called *the Line of Measures in Optical Perspective*, or such *Delineations as are made on spherical and cylindrical Surfaces.*

1456. If the right Line G H be parallel to the Line A F, then will the visual Rays A C, B C, &c. divide the Line G H in a similar Manner to that in which they divide the Line A F; that is, if the Divisions in A F are equal, they will also be equal in G H; but if they are unequal in A F, they will have the very same Ratio of Inequality in G H. For the Triangles F C E and *f* C *e*, also E C D and *e* C *d*, are similar; which give these Analogies $ef : EF :: eC : EC$; and $ed : ED :: eC : EC$; consequently $ef : ed :: EF : ED$; and so of the Rest.

1457. If any Objects A B, C D, (Fig. 21.) are seen by the Eye at C under equal Angles *a b*, and *c d*, *then will they be as the Squares of their Distances from the Eye directly.* For in the foregoing Figure to (1454) we had $TP : Bb :: CP^2 : CD^2$; and therefore since in this Case B *b* and C D are constant Quantities, T P will be directly as CP^2; that is (in Fig. 21.) $AB : CD :: BC^2 : DC^2$.

1458. The Object B D (Fig. 22.) of a given Length, placed at a given Height A B will be seen by the Eye at C in a right Line perpendicular to A B, under an Angle B C D, which will be of a variable Magnitude as the Eye approaches to, or recedes from the Point A; and there is one Distance A C where that Angle of apparent Magnitude will be a *Maximum*, or greatest of all others; and is determined in the following Manner.

1459. Every

1459. Every Angle P C Q (see Fig. to 1454) is as the Arch P Q directly, and the Radius P C inversely or as $\frac{PQ}{PC}$. But $TC : CD :: TP : PQ = \frac{TP \times CD}{TC}$; also we have $PC = \sqrt{CD^2 + DP^2}$. Therefore the visual Angle P C Q is as $\frac{TP \times CD}{TC \times \sqrt{CD^2 + DP^2}}$. That is, if (in Fig. 22) we put $Ab = a$, $BD = b$, $AD = a + b = c$, and $AC = x$; then will the Angle B C D be as $\frac{bx}{\sqrt{a^2 + x^2} \times \sqrt{c^2 + x^2}}$; then by taking the Fluxion thereof (800) and making it equal to *Nothing* (818) we shall get $x^2 = ac$, whence $a : x :: x : c$, or the Distance A C is a *geometrical Mean* between A B and A C, when the visual Angle B C D is the greatest possible.

1460. If any Line (Fig. 25) O B appears to an Eye at A, under the same Angle with another Line C D, then at any other Point E, the two Lines will have the same apparent Magnitude (however the visual Angle C A D may vary) if the Distances of the Eye from each Line preserve the same Ratio. For let $o\,b$ be equal and parallel to O B, and draw C E and D E; then by similar Triangles, we have $AH : AK :: CH : OK$; and $EH : EL :: CH : oL = OK$; therefore $AH : AK :: EH : EL$. Also a right Line passing from the Eye to those Lines divides them in the same Ratio; for it is $GH : HD :: OK : KB :: oL : Lb$.

1461. The Appearance of any distant Line or Object O B upon another Line or Plane C E given in Position, will increase or decrease with the Distance from the said Line or Plane, tho' not in the same Ratio; for at the greater Distance A it will occupy the Length C E, but in a less or nearer Distance at D, it occupies only the Space $c\,e$, much less than before, as is evident by Inspection.

1462. Having thus premised such Optical Principles of *lineal Perspective*, or the *Appearance* of LINES among themselves with regard to their Positions, and Distance of the Eye; we shall now proceed to *a generally* THEORY of *universal* PERSPECTIVE, and demonstrate the same from its genuine Principles. And tho' the common Methods of making Perspective

Draughts

Draughts are very easy, and in every ones Hands, yet as the judicious *Architect*, *Designer*, *Painter*, *&c.* will find it no easy Matter to come at a concise and plain *Theory*, or easy *Rationale* of this necessary or fundamental Part of his Art, in any of the Books hitherto published; it is presumed, the following new Method of demonstrating the Reason of such useful and common Rules will not be unacceptable to the ingenious Artists of all such Professions, as require the Assistance of this excellent Science.

CHAP. II.

The THEORY *of* PERSPECTIVE *demonstrated from the* PRINCIPLES *of* OPTICS *and* GEOMETRY.

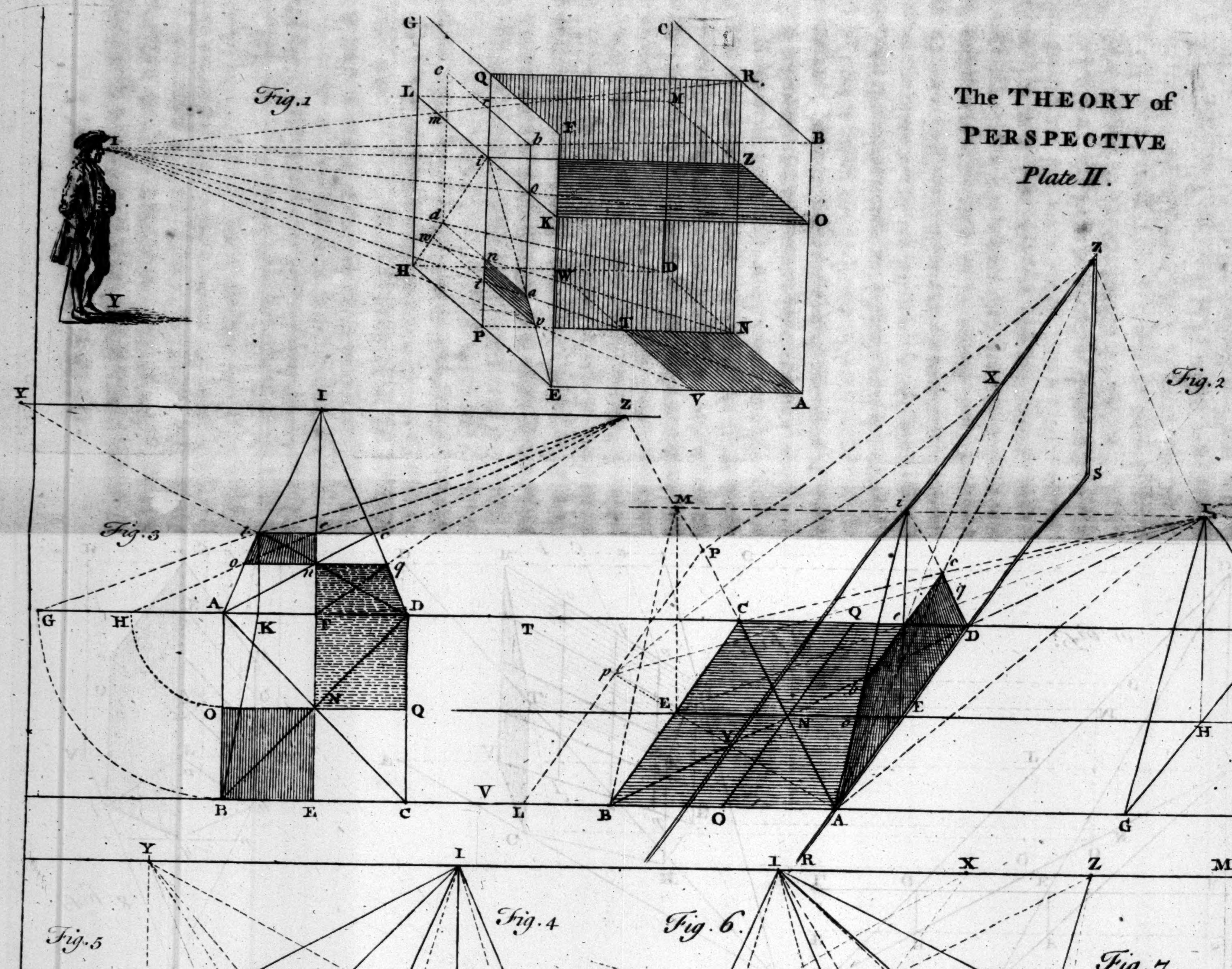
[PLATE II. *of* PERSPECTIVE.]

1463. FROM the Definition of Perspective, (1448) it appears, that in order to give a *Rationale* of the common Practice, we must premise a *Theory of the Vision of Objects* on a transparent Plane at a given Distance and Height of the Eye. Thus suppose the Eye of the Spectator at I, and E F G H the Plane on which it observes the Appearance of Objects, (Fig. 1.) This is called the *Perspective Plane.*

1464. As the Appearance of Objects will be variable according to the Situation of the Planes in which they are posited in regard to the *Perspective Plane*; these particular Situations of Object-planes are principally to be considered. And, first, it is evident, that any Plane passing thro' the Eye cannot be seen as a *Plane*, but as a *Line* only on the perspective Plane; for the Eye having no Elevation above such a Plane, can see no Part of its Surface; the Edge or bounding Line of such a Plane being all the Appearance it can have to the Eye.

1465. Of these *Ocular Planes*, there are two of principal Note, *viz.* the *horizontal Plane* O K L M parallel to the Horizon; and the *vertical Plane* N R Q P, which is perpendicular thereto. The first intersects the perspective Plane in the Line

The THEORY of PERSPECTIVE Plate II.



LK, which is therefore called the *horizontal Line*; and the other, in the Line QP, which is called the *vertical Line*, on the perspective Plane.

1466. Secondly; those Planes which do not pass thro' the Eye, will have a *direct or an oblique Situation* with respect thereto. If it be a direct Situation, it will be parallel to the perspective Plane which is supposed to be placed directly before the Eye. Thus the Plane A B C D is a direct one, and parallel to the perspective Plane H F. And among all Planes situated obliquely to the Eye, that which is in the Plane of the Horizon, as A E H D, is the most considerable; and is called the *Ground Plane*. This is perpendicular to the perspective Plane H F.

1467. From what has been said, it appears, that Objects in the Surfaces of the *horizontal and vertical* Planes cannot be seen at all by the Eye at I; and therefore they are not to be regarded in Perspective. We must therefore consider of Objects in and upon the direct and oblique Planes, and the Appearance they make on the perspective Plane.

1468. Thus let O B be an Object in the direct Plane; and from the extreme Points O and B draw the visual Rays O I, B I, to the Eye at I. They will pass thro' the perspective Plane in the Points *o* and *b*; and by joining those Points with the right Line *o b*, that Line will be the Picture of the Line or Object O B, upon the perspective Plane. Thus also the Perspective of the Line O A is *oa*; and the Proportion of Objects and their perspective Appearances is the same in this direct View, *viz.* that of their Distances from the Eye; for O B : *ob* : : O A : *oa* : : O I : *o*I, by similar Triangles.

1469. Thus (*b r*) and (*a n*) will be the Perspectives of the Lines B R and A N in the object Plane; and *r b a n* will be the Perspect of R B A N of half the Plane A C, and *a b c d* the Perspective of the whole Plane A B C D. And all Lines parallel to A B or C B in the object Plane, will have their perspective Lines parallel to *a b* and *c d* in the Picture on the perspective Plane. And in what Manner soever the object Plane A C is divided by Lines drawn upon it, their Representatives will divide the Picture (*ac*) in a similar Manner. (1456.)

1470. Any Point B in a direct Plane has the same Ratio of Distance from the *horizontal and vertical Planes*, as its perspec-

tive (*b*) has from the *horizontal and vertical Lines*, **viz.** that of the Distances of the Planes from the Eye. For B R : *b r* : : B O : *b o* : : I O : I *o* : : I *z* : I *i*, from the Nature of similar Triangles; hence it is easy to delineate the Appearance of any Objects, or form their Pictures on the perspective Plane when they are presented in a direct View; as in the Front of Buildings, *&c.* as will be exemplified hereafter.

1471. The last Sort of Plane, in and upon which we are supposed to view Objects, is that of the *Horizon* itself, as A D H E, above which the Eye has an Elevation, more or less, as *i* P = I Y. This is therefore called the *ground Plane*, and its Intersection H E with the perspective Plane, is called, the *ground Line*. And this is the most considerable of all others, as being the common Table or Plan of all perspective Views, Landscapes, and picturesque Draughts of every Kind.

1472. With regard to this horizontal Plane, it has been shewn (1469) that the two remote Angles thereof A and D are represented by *a* and *d*, in the perspective Plane; and the other two Angles E and H, are in the said Plane also, as being common to both; therefore by drawing the Lines *a* E and *d* H, there will be formed the Figure E and H on the perspective Plane which will be the true perspective Appearance of the *ground Plane* A D H E.

1473. Hence it follows, that *a* E is the Perspective of A E, *n* P of N P, and *d* H of D H. Whence it appears that Lines which are parallel in the ground Plane, and perpendicular to the perspective Plane, are not so in their perspective Picture, but they all converge to a Point *i*, which is called by the *Point of Sight* in the perspective Plane as being exactly opposite to the Eye, or that Point in which a Perpendicular from the Eye falls on the Plane.

1474. In the ground Plane draw V W parallel to A D; its perspective *v w* will be parallel to *a d*, in the Picture; and *a d w v* will be the Perspective of the Part A D W V in the original Plane. But for a Demonstration of what relates to forming the Picture or perspective Appearance of the ground Plane and Objects upon it, and consequently, of the common Practice of Perspective, we must have Recourse to the following Method of Representation, which is the most natural, concise, and perspicuous, of any I have been able to think of.

1475. Let A B C D be a right-lined Figure in the ground Plane V G K T, contiguous to and at right Angles with the perſpective Plane Y Z S R; F H the Diſtance of the Plane; and H I the Height of the Eye at I. H E is parallel to G B or C K and biſects A D and B C in the Points F and E. On the Point E raiſe the perpendicular E M = H I, and draw the Lines BM, C M, G I, K I.

1476. Draw the viſual Lines I A, I B, and I M, which is called the *principal Ray*, and is perpendicular to the perſpective Plane in the Point *i*. Then it is evident the Plane I G B M interſects the perſpective Plane in the Line A *i*, and the Ray B I being in the ſaid Plane G M, muſt inerſect the Line A *i* in ſome Point (*b*) which is therefore the Perſpective of the Point B; and of Courſe A *b* is the Perſpective of the Line A B.

1477. In like Manner it is ſhewn, that as the Plane I K C M interſects the perſpective Plane R Z in the Line D I; and the Ray I C being in that Plane and interſecting the Line I D in the Point (*c*), that Point (*c*) will be the Perſpective of the Point C; and D *c* the Perſpective of the Line D C. And joining the Points *b*, *c*, the Line *b c* will be the Perſpective of the Line B C in the ground Plane.

1478. Let A B = D C, then B C will be parallel to A D, and ſince in this Caſe A *b* = D *c*, therefore *b c* will be parallel to A D alſo. And hence it appears, *that all right Lines, as* B C, *in the ground Plane which are parallel to the ground Line* A D, *will alſo be parallel to the ſame in their Picture on the perſpective Plane.*

1479. Hence alſo it is evident, *that the Perſpectives* A *b*, F *e*, D *c*, *of all Lines* A B, F E, D C, *which are perpendicular to the ground Line* A D, *do converge or tend to the Point of Sight* (*i*) *in the perſpective Plane.*

1480. If the Line A B be continued out *in Infinitum*, towards V, then ſuppoſing the Point B to move along that Line continually, the viſual Ray B I, will keep riſing on the Plane I G B M towards I M making the Angle B I M ſtill leſs and leſs, till the Point B arrives to an infinite Diſtance, and then the Ray I B will coincide with I M; and conſequently the Line A *i* will be the Perſpective of A B continued to an infinite Length. Thus alſo D *i* will be the Perſpective of the Line D C infinitely continued towards T. *And therefore the Triangle* A *i* D *will*

on the perspective Plane be the Picture, or true Perspective, of the Plane A B C D *continued out upon the Plane of the Horizon to an infinite Length.*

1481. *Hence the Line* Y *i* Z, *is the Perspective of the Horizon, or Boundary of the Sight at an infinite Distance*; and therefore all Objects on the Plane of the Horizon, will, in their perspective Appearance, or *Landscape*, keep rising from the ground Line, or Base A D, towards the Point of Sight (*i*) ; and lessen in their Dimensions as they are more remote, till at last they vanish in the *horizontal Line* Y Z.

1482. We have seen how Lines parallel or perpendicular to the ground Line A D are to be delineated or drawn in Perspective; and we are next to shew how those right Lines appear, or are to be drawn upon the perspective Plane, which lie oblique to, or make any Angle with the ground Line A D, or any other parallel to it. In order to this, make A L = A G = I *i*, and draw A *p* to make any Angle *p* A R or *p* A D with the Base A D, acute or obtuse. Then in the horizontal Line Y Z, take *i* X = L *p*; and draw *p* X and I X; and the Plane I X *p* A will intersect the perspective Plane in the Line A X. Draw the visual Ray I *p* which as it is in the Plane I X *p* A must go thro' the perspective Plane somewhere in the Line A X, which suppose at (*r*), then is the Point (*r*) the Perspective of (*p*) ; and since while the Point *p* is supposed to pass from A to *p* in describing the Line A *p*, its Perspective (*r*) will move in the Plane A Z from A to *r*, and describe the Line A *r*; which therefore will be the Perspective of the Line A *p*.

1483. If the Line A *p* were continued out to an infinite Length, and the Point (*p*) supposed to move constantly therein, its perspective (*r*) will appear to move towards X, till at length the Point (*p*) being at an infinite Distance, the Point (*r*) arrives at, and coincides with X, in the horizontal Line; *the Line* A X *is therefore the Perspective of the Line* A *p infinitely continued*; and X is called the *accidental Point, to which the Perspectives of all Lines parallel to* A *p tend.*

1484. Let L P be taken equal to A L; and *i* Z equal to *i* I, and then joining A, P, and I, Z; we have the Triangles A P L, and *i* Z I, equal to each other; then will the Plain I A P Z intersect the prespective Plane in the Line A Z, which will

will be the Perſpective of the Line A P continued out to an infinite Diſtance.

1485. But ſince A L = L P, and L P is parallel to A D; therefore A P is the *Diagonal of a Square*, and contains an Angle D A P of 45 Degrees with the ground Line A D; *therefore the Point of Diſtance* Z *is that to which all Rays parallel to* A P *tend in the perſpective Plane.*

1486. Let A B = A D, then is A B C D a geometrical Square, and its Diagonal A C, of which the Perſpective is A *c*; and the Point (*c*) is therefore that in which the perſpective diagonal A Z interſects the Ray or radial Line *i* D. Make *i* Y = *i* Z, = *i* I; and join D Y, then will that Line D Y be the perſpective Diagonal of D B, (the other Diagonal of the Square A C) infinitely continued, and D *b* the Perſpective of the Diagonal D B determined by the Interſection of the Lines D Y and A *i*, as before.

1487. Thus it is demonſtrated that A *b c* D on the perſpective Plane A S Z Y is the true *Picture or perſpective Delineation* of the *original Square* A B C D, on the ground Plane, as required. And from hence is deduced the *Rationale* of the common Method of drawing the Perſpective of a given Square or any Figure inſcribed therein; as alſo of a ſtill more univerſal Method of aſſigning the Perſpective of any Point, Line, Superficies, or Solid, on the Plane or perſpective Table, which we ſhall next proceed to explain.

CHAP. III.

The RATIONALE *of the* common METHODS *of drawing the* PERSPECTIVES *of* OBJECTS, *explained from the preceding Theory.*

[PLATE II. *of* PERSPECTIVE.]

1488. THE common Methods of *Drawing in* PERSPECTIVE are much better known than the Reaſon of them; we ſhall here give both together, and then our Endeavour will at leaſt have the Face of Novelty. Let T G (Fig. 3.) be

be the *Base* or *ground Line*, in any *perspective Plane* or *Table* T G Y Z. The Method of drawing the *perspective Appearance* of any Object on this Table is as follows.

1489. *Let it be required to draw the Perspective of the given* SQUARE A B C D *and all its Parts, in a front View.* The Square is supposed to be contiguous to the Perspective Table, and therefore its Side A D will be in the ground Line. As the View is direct, or in Front, the Eye must be supposed directly against the Middle of the Side A D; and therefore in the Perpendicular E F continued out, take F I equal to the given Height of the Eye, and (I) will be the *Point of Sight* (1476) set off the given Distance of the Eye from the Table each Way from I, to Y and Z; and these will be the *Points of Distance* (1486) in the horizontal Line Y Z.

1490. From the Points A and D draw the *Radials* I A, I D; and the *Diagonals* A Z, and D Y, intersecting the *Radials* in the Points *b*, and *c*; then draw *b c*; and the Figure A *b c* D is the Perspective of the given Square A B C D as required. For A *c* is the Perspective of the Diagonal A C, and D *b* that of D *b* (by 1486); therefore A *b* and D *c* are the Perspectives of the Sides A B and D C (1479) and *b c* of the remote Side B C (1477.) Also the Point *n* is the Perspective of N the Intersection of the Diagonals in the Square; and *o q*, F *e* are the Perspectives of O Q and F E which bisect the Square orthogonally, (1478, 1479.)

1491. It is not necessary to make use of more than one Point of Distance, Z; because one Diagonal A Z, gives the Point (*c*) in the Radial I D which determines the Perspective of the Diagonal A C; and then by drawing thro' the Point (*c*) the Line *b c* parallel to the ground Line A D, the Perspective of the Square A B C D is compleated as before (1477.)

1492. From hence we may derive a Demonstration of an *universal Rule* for putting all Objects into Perspective from given Point therein. The Rule is this.

From the given Point in the ground Plane, let fall a Perpendicular on the ground Line, from whence draw a Radial to the Point of Sight; then from the Radial set off that perpendicular Distance in the ground Line, and from thence draw a Diagonal to the Point of Distance, the Intersection of the Radial and Diagonal will give the Seat

or

or Place of the Perſpective of the given Point. And thus the Perſpectives of any Number of Points may be found, which being connected will form the Perſpective of the given Figure which they terminate.

1493. To exemplify this Rule; let C be any given Point in the ground Plane, from which let fall the Perpendicular C D to the ground Line T G; and from the Point D draw the Radial D I to the given Point of Sight I. Then in the ground Line make D A = D C, and from the Point A draw the Diagonal A Z to the given Point of Diſtance Z, which will croſs the Radial in the Point (*c*) the Perſpective of the Point C as required. This is evident from the common Method beforegoing (1476), becauſe A C is the Diagonal of a Square.

1494. Suppoſe B any other given Point; let fall the Perdicular B A, and draw the Radial A I; then make A G = A B, and drawn the Diagonal G Z, it will alſo interſect the Radial in the Point (*b*) the Perſpective of B as required. To prove which, let us ſuppoſe B C = A D = D C = A B. Then joining (*b c*) that muſt be the Perſpective of B C, the ſame as before determined by the commen Method; and ſo it will prove. For in the ſimilar Triangles A I D, *b* I *c*, we have A D : *b c* : : I A : I *b* (656,) according to the common Method; alſo from the ſimilar Triangles Z G A, Z *b c*, we have Z G : Z *b* : : A G : *b c*, according to the Rule; but from the ſimilar Triangles *b* G A and *b* Z I, we have I A : I *b* : : Z G : Z *b*; and therefore it is A D : *b c* : : A G : *b c*; but A D = A G, by ſuppoſition; therefore the Line (*b c*) is the ſame in both Caſes.

1495. Hence we have eaſy Methods of drawing the Perſpectives of any ſuperficial Figures whatſoever; for if they are circumſcribed with a Square, and Lines drawn thro' their terminating Points and Angles to the Sides of the Squares, then by taking the Perſpective of the Square you will at the ſame Time have the Seat of every Point or Line therein which bounds the Figure propoſed, and theſe being connected, the Perſpective of the Figure is formed as required.

1496. Thus ſuppoſe any Line B K were to be put into Perſpective, then one Extreme being placed on the ground Line at K, let the Angle B of the Square be placed on the other; then the Points K and (*b*) being connected in the Perſpective, will

give

give K b, for the Perspective of the Line K B required. Or if from the Point B you let fall the Perpendicular B A, and make A G = A B; then drawing the Radial A I, and Diagonal G Z, they will intersect each other in the Point (b), which therefore is the Seat of the Point B, and so K b is of Course the Seat of the Line K B, according to the Rule (1492.)

1497. Since any right Line is so easily put into Perspective, there can be no Difficulty in finding the perspective Appearance of any given right-lined Figure. Thus the Perspective of the Parallelogram C E F D is $c\,e$ F D; the Perspective of the Square D F N Q is D F $n\,q$, and the Perspective of the remote Square C E N Q is $c\,e\,n\,q$, all determined by inscribing them in the geometrical Square A B C D.

1498. If the Figure be not right-lined, but bounded by a Circle or some other Curve, yet if they are properly circumscribed by a Square, the Intersections of the Diagonals and other Lines with one another in the several Parts of the Perimeter of the Curve, will give a sufficient Number of Points in the Perspective Table, thro' which to draw the Perspective of the Circle or Curve proposed. Thus for Example (Fig. 4), if in the Square A B C D you draw the Circles O F Q E, and T V W S; then by drawing the Diagonals of the Square and other proper Lines G H, I K, L M, R P, O Q, E F, &c. their Intersections in the Periphery of the Circle, being all found in the Perspective Table, and a curve Line drawn thro' them; such curve Lines are the Perspectives of the Circles, as required.

1499. The Square O F Q E with an Angle F in Front, has its Perspective F $o\,e\,q$ at the same Time determined. Also the Triangle X V Z, is projected by the visual Rays into its Perspective $x\,v\,z$ on the Table; and the Perspectives of its several Parts (which are six *Triangles* and a *Trapezium*) are formed by the Lines which divide it; all which is evident by Inspection of of the Figure.

1500. If it be required to delineate the Perspective of a Square A B C D viewed obliquely (as in Fig. 5.) then the given Obliquity of the View will determine the Position of the Point of Sight I in the horizontal Line, and the given Distance being set off from I to Y; then if the Radials I A, I D, are drawn, and one Diagonal D Y to intersect the Radial I A in (b)

that

that will be the Perſpective of the Point B; *b c* drawn parallel to A D, will be the Perſpective of B C; and A *b c* D will be the Perſpective of the Square A B C D as required. The Theory (1475) being in this Caſe the ſame as in a direct View, the Practice muſt be conducted by the ſame Rules of Courſe.

1501. We muſt not quit this Subject without a Remark on the Abſurdity of ſeveral Writers on Perſpective, *viz. that tho' they give the practical Rule of drawing the Perſpective of any Superficies right, yet they place the Perſpective itſelf wrong.* Or they place that Part of it neareſt the ground Line, which is in Reality moſt remote from it; for Example (in Fig. 4.) the Perſpective *v z x* of the Triangle V Z X, they place contrary to its Poſition here, *viz.* with its Baſe *z x* towards the ground Line A D, and its Vertex *v* towards the Point of Sight I. In ſhort, they give the ſame Poſition to all the Parts of the Perſpective, which the Originals themſelves have in the ground Plane, contrary to what really appears to the Eye viewing thoſe Objects thro' a tranſparent Plane, as we have ſhewn. This erroneous Repreſentation of Gardens, Fortifications, *&c.* gives ſuch an unnatural Idea thereof as muſt be very diſagreeable to a nice Genius. Many Inſtances of this you will find in the JESUITS PERSPECTIVE. And even POZZO himſelf has puzzled his Readers with this Piece of Nonſence.

CHAP. IV.

The THEORY *of the* PERSPECTIVE *of* CIRCLES, *and* circular AREAS, *demonſtrated*; *with the Solution of ſome very uſeful* PROBLEMS *relative thereto.*

(PLATE I. *of* OPTICS *and* PERSPECTIVE.)

1502. AS there is ſomething very curious in the Speculation of the *Perſpective of a Circle*, and very little known, as being rarely found in Treatiſes on this Subject; I preſume it will be agreeable to the ingenious Reader, if I employ a Page or two, in order to ſet that Matter in a clear Light.

1503. In the firſt Place, then, it is to be conſidered, that the Rays which proceed from a Circle to the Eye (in any Elevation above the Plane in which it lies) do form a *Cone*; whoſe Baſe is the Circle, and whoſe Vertex is in the Eye. (625.)

1504. If this Circle be viewed thro' a tranſparent Plane parallel to the ground Plane of the Circle, then its Perſpective will be a CIRCLE alſo; becauſe in this Caſe, the viſual Cone is cut by a Plane parallel to its Baſe, and therefore the Figure of the Section will be the ſame as that of the Baſe, in any Poſition of the Eye above the perſpective Plane.

1505. If the perſpective Plane paſs thro' any Part of the Circle or Baſe of the Cone, and the Poſition be parallel to either Side of the Cone, then that Part of the Circle's Periphery which is ſeen through, or under the Plane, is projected on the ſaid Plane in the Figure of a PARABOLA, (by what has been ſaid of the Geneſis of that Curve in (740); for the Eye at F will view the Point *e* at R on the Plane, and the circular Periphery B *e* B will by the viſual Rays of the Cone be depicted in the Curve of the *Parabola* B R B, which will therefore be the true Perſpective of that Part of the Circle in this Caſe. *

1506. If the perſpective Plane paſſes thro' the circular Baſe of the viſual Cone, in a Poſition not parallel to either Side, and yet ſo as not to paſs thro' both in the ſame Cone if they were continued out beyond the Circle or Baſe, then ſo much of the Circle as could be ſeen thro' the Plane would be projected thereon in the Form of an HYPERBOLA (by 765); for the Eye at C will project the Point *e* on the Plane at V, and the circular Periphery B *e* B into the *hyperbolic Curve* B V B on the Plane, which therefore will be its perſpective Appearance.

1507. In all other Poſitions of the perſpective Plane with regard to the Sides of the Cone of Rays, the Perſpective of the Circle will be an *Ellipſis* therein (by 763.) For ſuppoſe the Plane to cut the Cone in the Line T V, then to an Eye at the Vertex L, the Point F will appear at T, and the Point P at V on the ſaid Plane; and of Courſe the whole circular Baſe will be projected in the Ellipſis T B V B, which is therefore its perſpective Appearance.

1508. If a Circle E D be placed on an horizontal Plane A E, and viewed thro' a perpendicular tranſparent Plane D G (Fig. 28.) by

* See the Figures of the conic Sections here referred to.

by an Eye at F at the Distance D B, and Height B F; then its Perspective D C will be an *Ellipsis* also. But because the visual Cone E F D is in this Case in an oblique Position (and what they call a *scalenous Cone)*, and as the Obliquity of the Cone, or its Inclination to the Horizon, will vary with the Distance and Height of the Eye, so it will be cut in a various Manner by the Plane D G, producing as many different Ellipses by those Sections in some of which the Diameter E D of the Circle will be the longer Diameter of the Ellipse, in others it will be the shortest; consequently between both there will be one Case were the *Ellipse* will become a *Circle*, or have all its Diameters equal on the perspective Plane.

1509. Now this will happen when the Cone E F D is cut in a similar Manner by both the Planes, *viz.* the ground Plane A E, and the perspective Plane D G, or when the Angle G D F is equal to the Angle D E F; for if we suppose the Eye to move thro' the Arch of a Circle G F A from the vertical to the horizontal Plane, it is evident, when the Eye is at G the Angle D E C will be of some Quantity less than a right Angle; and the Angle F D C will be Nothing. But as the Eye departs from G, the said Angle F D C begins, and encreases to a right Angle when the Eye arrives at A; on the other hand, during all that Time the Angle D E C decreases, and at last vanishes; and therefore at some Moment of that Interval the two Angles must be equal, which let us suppose to happen when the Eye is in the Position F.

1510. Then are the Triangles E F D and D C F similar; for the Angle C D F = D E C, and the Angle C F D is common to both; therefore the Angle E D F = D C F, and consequently the visual Rays of the Cone make *subcontrarily* the same Angles with each Plane A E and D G; therefore the Section of the Cone made by each Plane must be of the same Figure, and consequently since E D is a *Circle* its Perspective D C will be a Circle also.

1511. From hence some curious Problems will arise of the last Importance in Perspective. The first is, *To determine the Position of the Point* F, *or Point of Sight*, *from the given Diameter of the original Circle* D E, *and Distance from the Table* D B. In order to this, let D E = D, D B = s; also put the Height

of

of the Eye $BF = h$; the Diameter of the perspective Circle $DC = d$; and let $EF = e$, and $DF = f$. Then $D : e :: d : f$ (1510) let $D + s = r$; then $e = \sqrt{r^2 + h^2}$ and $f = \sqrt{h^2 + s^2}$ (636) also $D : d :: r : h$, therefore $d = \frac{Dh}{r}$. Whence $D : \frac{Dh}{r} :: 1 : \frac{h}{r} :: \sqrt{r^2 + h^2} : \sqrt{h^2 + s^2}$; whence we get $h^2 = rs = \overline{D + s} \times s$; and therefore $h = \sqrt{\overline{D + s} \times s} = BF$, the Height of the Eye at F required.

1512. The above Equation gives this Analogy, $D + s : h :: h : s$, whence it appears (645, 660) that the *Locus* of the Point F is the Circle E F A. Hence $AB = BD = s$; $AF = DF$; the Diameter $AE = D + 2s$; and the Radius $AN = \frac{D + 2s}{2} = \frac{DE + 2DB}{2}$; from whence the Construction of the Problem is evident.

1513. The *second Problem* is, the *Diameter of the Circle* D E, *and Height of the Eye* B F *being given, to find the Distance* D B *from the perspective Table.* The Solution is thus; $h^2 = Ds + ss$, (1511); let the Square be compleated (338) then $h^2 + \frac{1}{4}D = ss + Ds + \frac{1}{4}D$, and therefore $s + \frac{1}{2}D = \sqrt{h^2 + \frac{1}{4}D}$; consequently $s = \sqrt{h^2 + \frac{1}{4}D} - \frac{1}{2}D = DB$ the Distance required.

1514. A *third Problem*, is *to determine the Distance* D B, *at which the Perspective* D C *of a given Circle* D E *shall be a Circle, and its Area be to the Area of the given Circle in any given Proportion of* z *to* y. To solve this Problem, it must be considered that since $DB = BA$, (1512) we have $DF = FA$; and since the Areas of Circles are as the Squares of their Diameters (840) we have $DE^2 : DC^2 :: EF^2 : DF^2 = FA^2$ (1512) but $EF^2 : FA^2 :: EB : AB$, (660) therefore $DE^2 : DC^2 :: (y : z ::)$ $EB : AB\ (:: D + s : s)$ whence we have $ys = zD + zs$; and $ys - zs = zD$, therefore $y - z : z :: D : s = \frac{zD}{y - z} = DB$, the Distance required.

1515. The Center of the *original Circle* will not be projected into the Center of the *perspective Circle*; neither will the Diameters

Diameters of the latter be the Perſpectives of the Diameters of the former. Nor will any right-lined Figure (as a *Square*, *Triangle*, *Polygon*, *&c.*) inſcribed or circumſcribed about the original Circle have its Perſpective in, or about the perſpective Circle of the ſame Kind or ſimilar to it; for that of a *Square* will be a *Trapezium*; that of a *Triangle* a *diſſimilar Triangle*, that of a *Pentagon* will be an *irregular Polygon*; and ſo of others. All which the Reader will be eaſily aſſured of by drawing this Caſe in the uſual Manner and Form as directed (1492.)

CHAP. V.

The Theory of CATOPTRIC PERSPECTIVE; *or the perſpective Appearance of Objects on a* REFLECTING PLANE.

[PLATE I. *of* OPTICS *and* PERSPECTIVE.]

1516. WE have conſidered the *Dioptric* THEORY of PERSPECTIVE ſo far as it relates to *Lines* and *Superficies*; and before we go farther it will be proper to conſider the perſpective Appearance of the ſame Objects by *Light reflected from a poliſhed Plane*; in as much as we ſhall find the *Perſpective* of any *Line* or *Superficies* is identically the ſame on the *Catoptric* or reflecting Plane as it is on the *Dioptric* or refracting one.

1517. Preparatory to a Demonſtration of this Poſition, let HD (Fig. 26) be a Line or Section of the Speculum, perpendicular to the ground Line AF, and B the Place of the Eye at the given Diſtance AD, and Height AB; from any Point E in the Line AD, let a Ray of Light proceed to the Speculum in the Point C, and be reflected from it to the Eye at B; then continuing the reflected Ray out beyond the Speculum, it will interſect the Line AF in the Point F, making the Line DF = DE; and conſequently, the *Point* E *by reflected Light will appear upon the Speculum in the ſame Point* C *with another Point* F *viewed*

viewed by the Eye thro' a transparent Plane H D, *and placed at the same Distance behind it.*

1518. The Demonstration is easy; let C G be perpendicular to the Plane H in the Point C, it therefore will also be parallel to the Line A F; then because the Angle G C B = G C E (1282) and G C B = C F D, and G C E = C E D (631) therefore we have the Angle C E D = C F D; and therefore the Line C E = C F, and consequently the Line D E = D F. Wherefore *the apparent Place of the Point* F, *by Reflection, is the Point* F *at an equal Distance beyond the Speculum.*

1519. Proceeding in the same Steps of Demonstration here as in Dioptric Perspective (1475), we shall find the same Appearance also of Lines and Superficies on a reflecting Plane as we did there on the refracting one. Let A Y Z D (Fig. 27.) be the Speculum, A *B C* D be a Square placed before, and contiguous to it; extend the Sides A *B*, and D *C* indefinitely on each Side, and draw E H between and equidistant from both. Let I be the Place of the Eye at the Distance F H, and Height H I; take F E = F *E*; and thro' E and H, draw B C and G K; also on the Point E erect the Perpendicular E M = H I, and draw I M passing thro' the Speculum in the Point *i*; and join I G, I K; M B, M C.

1520. Then because the Point *B* of the Square is in the Plane I G B M which intersects the reflecting Plane A Z in the Line A *i*; and because the incident and reflected Rays are in the same Plane (1282) therefore the Ray of Light which proceeds from the Point *B* and is by the Speculum reflected to the Eye at I, must imping on the Speculum in some Point (*b*) in the Line A *i*; and the reflected Ray will be *b* I. If this Ray be continued out, it will meet the Side of the Square A *B* produced in B, making A B = A *B*. (1518)

1521. Now because the same Thing is demonstrable of every other Point in the Side of the Rectangle A *B*, *therefore the Perspective of the Line* A B *by Reflection is the Line* A *b*, *or that very Part of the Radial* A *i* *which is the Perspective of the equal Line* A B *by Refraction thro' a transparent Plane* (1476).

1522. In like Manner it is demonstrated, that the Perspective of the Side D *C* by Reflection is D *c*, the same Part of the Radial as is that of the equal Line D C by transmitted Rays.

By

By joining *b c*, the Line (*b c*) is the Perſpective equally of *B C* by reflected, and B C by refracted Light. And conſequently, A *b c* D *is the perſpective Appearance of the Rectangle* A B C D *by Reflection, and every Way the very ſame as it was ſhewn to be of an equal Rectangle* A B C D *beyond the pellucid Plane* (1487).

1523. Therefore whatever was demonſtrated relative to the Perſpective of Lines and Superficies any how poſited or formed with reſpect to a common tranſparent Plane in the Sequel of the Dioptric Theory, will hold equally true here in Catoptric Perſpective. And tho' this Part of the Science has been the leaſt of all conſidered by Writers on this Subject, yet when compared with common Perſpective it will be found not only more delicate in its Nature, but alſo more adapted to Uſe. This we ſhall illuſtrate more particularly when we come to treat of *Picturesque Perſpective*, or that Part of the Science which treats of the PRINCIPLES of PICTURES or LANDSCAPES formed by OPTICAL GLASSES both of the *reflecting*, and *refracting* Sort.

CHAP. VI.

Of INVERSE PERSPECTIVE; *where the Nature and Principles* of the ANAMORPHOSIS *or* Deformation *of* Figures, *and their Rectification, are explained.*

1524. THE *inverſe Method of Perſpective* is not leſs pleaſant than uſeful, nor is it in the leaſt Degree difficult, as it is only going backwards in the ſame Steps we took in the direct Method; and it muſt be obſerved, that with regard to any Science, unleſs the Student can go backwards as well as forwards, he makes but an imperfect Proficiency.

1525. *Inverſe Perſpective* ſhews how from a given Piece of Perſpective to determine the *Original* or *Prototype* thereof under the Circumſtances of *Size*, *Situation*, *Diſtance*, &c. And here, as in the direct Method, we muſt begin with finding the Original of a *Point*, then of a *Line*, and laſtly of a *Superficies*; and

and being able to determine theſe, there will remain no Difficulty with reſpect to *Solids*, becauſe the Perſpective of a Solid is only the complex Perſpective of its ſuperficial Parts; and for this Reaſon, we think this the moſt proper Place to treat of the Inverſion of a perſpective Piece, and the Nature of *Anamorphoſis* or the *Deformation of Pictures*, which naturally reſults from thence.

1526. For by the direct Method it appears, that the Perſpective is very diſſimilar to its Original, and therefore if the latter be formoſe and regular, the former will be diform and irregular, or in the uſual Phraſe, *deformed* and *diſtorted*. So on the other Hand, if any natural and well-formed *Picture* be conſidered as a *Piece of Perſpective*, formed on a perſpective Plane, then when it is reſolved into its *Prototype*, that muſt be a very unnatural, miſhapen, and ill-proportioned Figure; all which will be evident from the following Proceſs.

1527. Let T V W X be a Perſpective Table (Fig. 29.) ſtanding on the ground Line S V. Alſo let I be the Point of Sight, and Y the Diagonal Point of Diſtance; and then ſuppoſe it required to find in the ground Plane the Seat of the Original of a given Point P in the Perſpective Table. You proceed thus; from the Point of Sight I, draw thro' P the Radial I K, and on K in the ground Line erect the Perpendicular K G; then from Y draw thro' the given Point P, the Diagonal Y L to meet the ground Line in L. Laſtly, on the Point K, with the Diſtance K L, deſcribe the Arch I H to interſect K G in H; and the Point H is the Original of the Point P, as required (1476, 1493.)

1528. In the ſame Manner any other Point Q in the Table will have its Prototype determined in C; and the Diſtances of any of thoſe original Points H, and C, from the ground Line S V will always be equal to the Diſtances K L and L S contained in the ground Line between the Interſections of the reſpective Radials and Diagonals.

1529. To find the Original of a given Line P Q in the Perſpective Table; nothing is required but to determine the Original Seats H and C of its two extreme Points P and Q (1494) then by connecting the Points H, C, the Line H C is formed on the ground Plane and is the Original of the Perſpective

tive PQ, as required. The Diſtance of this Original at each Extreme from the ground Line, as alſo it's Magnitude or Length, is meaſured on the ſaid Line S V, in *Inches*, *Feet*, &c. according to the Meaſure by which that Line is divided.

1530. In the ſame Manner any other Perſpective Line O P is found to have it Prototype B C, on the ground Plane; and thus any *Perſpective Angle* O P Q on the Table V X, will have its *original Angle* B C H determined on the ground Plane; whoſe Quantity is there alſo meaſured in the uſual Manner.

1531. Therefore, *to find the Original or Prototype of any given Perſpective Figure or Superficies*, O P Q R *on the Table*, nothing more is neceſſary than to find the original Seats E, B, C, H, of it's angular Points O, P, Q, R, (1493) for theſe being connected, from the Figure E B C H, which is the true Original of the Perſpective given; whoſe Diſtance and Dimenſions will then be eaſily known by common Geometry.

1532. As in the *direct Method*, the Perſpective of every regular Figure is irregular and diſſimilar, (1487) ſo in the *Inverſe Method*, the *Original* of every regular Piece of Perſpective, muſt itſelf be irregular and of a different Form; and therefore, if the Perſpective Figure O P Q R be a *Square*, the Original E B C H is a *Trapezium*, (1531) and if that Square be divided into equal Parts, by Lines drawn thro' the equidiſtant Points *f*, *e*, *d*, the Original will be divided very unequally by correſpondent original Lines drawn thro' the original Points *a*, *b*, *c*, at unequal Diſtances in the Line H C, the Original of P O.

1533. Therefore it will follow, that if any *Portrait*, or other *Picture* be drawn on the Perſpective Square, it will, when projected by the viſual Rays in the original Trapezium, be there deformed, or appear of a monſtrous Shape. For the natural Form and juſt Proportion of Parts in the Portrait, which depend on equal and ſimilar Spaces which they fill in the Square, will be all deſtroyed, when projected on the Trapezium into unequal and very diſſimilar Spaces. And this Diſtortion of the Figure is called, an *Anamorphoſis*, *Deformation*, or *monſtrous Projection*.

1534. The *Anamorphoſis* is eaſily effected in any Degree by the Rules of Art. For if the Perſpective Square P R be divided

into a Number of small Squares, or other equal or similar Spaces; then since by the Rules above the Original of the Square and all its Parts are to be found, and drawn on the ground Plane, it remains only to draw the same Parts of the Picture in each irregular Space of the Trapezium, as you see in the corresponding Part of the Perspective Square; and the Deformation will be compleated, as required.

1535. The Distortion or Deformation will be in Proportion to the Height and Distance of the Eye from a given Perspective Figure O P Q R. For if the Distance of the Eye I Y continue the same, the Breadth of the Projection at each End, *viz.* E H and B C will lessen as the Height of the Eye W V increases, and consequently, the Deformity of the Picture will be the greater, as its Length continues the same nearly.

1536. Also, if the Height of the Eye be the same, but the Distance greater, then will the Deformation or Trapezium vastly encrease, both in Length and Breadth, in all the remote Parts towards B C, in Comparison of those towards H E; and therefore the Deformation becomes greatly augmented in this respect also; and, indeed, much more than by altering the Height of the Eye at the same Distance.

1537. Hence it appears, that there is a certain Position of the Eye, in which it will view any given artificial Deformation or monstrous Picture E B C H, so that it shall appear perfectly natural, or in a just Proportion and Symetry of all the Parts of the Object it is intended to represent. For it is evident, from the foregoing Theory, if the Deformation E B C D be placed on the Horizon, at the Distance M E, from the Perspective Table V X, then an Eye placed at I, at the Distance I Y, and Height T X, will *view that monstrous Projection on the ground Plane, as a regular and perfect Square on the Table; and the deformed Image contained in it; as a well-proportioned and natural Portrait or Picture.*

1538. Now, because the Image in the Eye is every Way similar to the Object it views on a Plane placed before it, and parallel to the *Fundus* or Bottom of the Eye; therefore, since this Image on the Retina in the Eye is the same as would be formed of a Picture well drawn or designed on the Plane or Table V X, and since it does no ways depend on the

said

ſaid Table, but on the peculiar Pencil of Rays paſſing through it only, which is ſtill the ſame when the Table V X is removed; it follows, that the Eye in the Poſition aſſigned at I, will view, by itſelf alone, the Deformation contained in B E H C, as a juſt and well drawn Picture.

1539. This Invention has given Riſe to thoſe common *Anamorphoſes of King* Charles's *Head*, *St.* George *and the Dragon*, &c. on long Slips of Paper (like Fig. 30.) ſold at the Shops, which being held in a Poſition parallel to the Horizon, and viewed at the proper Diſtance E D, and ſmall Height of the Eye at I, do very agreeable ſurprize the Spectator (inconſcious of the Deſign) with a regular and beautiful View of thoſe Objects, of which in the Deformation they ſaw ſcarce any Appearance.

1540. From what we delivered in the Theory of Catoptric Perſpective, it is evident, that if T V W X be conſidered as a *reflecting Speculum*, and E B C H a Deformation placed before it, then its Appearance O P Q R, in the *Speculum*, will be a regular Picture; for that will be the true Perſpective of the Figure on the ground Plane, in the ſame Manner by a reflected, as by a tranſmitted Pencil of Rays (1522.)

1541. Since the Law of Reflection is the ſame in Surfaces of every Figure (1300) it will follow, that if we conceive the *Speculum* T V W X to be pliable and formed into a Cylindrical Surface, as A B C D (Fig. 31.) There will ſtill be a regular Perſpective Picture formed by Reflection from a proper Deformation E F G H; the greater Divergency of Rays in this Caſe, cauſing only a greater Diſtortion in the *Anamorphoſis*, and a Diminution of the Picture in the Speculum.

1542. Hence ariſe all thoſe Experiments of *poliſhed Cylinders*, *Cones*, and other figured *Speculums*, which rectify the Appearance of thoſe ſeemingly unmeaning and ludicrous Deformations painted round about them, on the Planes on which they are placed; any of which are eaſily drawn by any Perſon ſkilled in the Theory of Catoptrics; but thoſe who are not, may very ſucceſsfully uſe the following practical Method which is univerſal for all *Speculums*.

1543. Let a ſquare Hole be cut in a Piece of Paper, Paſtboard, Vellum, &c. and fine Threads paſted on the Paper over the Hole, horizontally and perpendicularly, ſo as to divide

the ſquare Hole into a proper Number of leſſer Squares; then paſt this Paper with the *Lattice-Square* on the Surface of your poliſh Cone, Cylinder, Looking-Glaſs, *&c.* and place a Candle at a proper Diſtance and Height (ſuch as you intend the Eye ſhall have to view the Picture) then will that polliſhed Lattice be reflected on the Table in the Form of the required Anamorphoſis, with the Shadow of the Threads, dividing it in a proper Manner for drawing the Deformation of any propoſed Object, whoſe Picture is drawn in a Square, of the ſame Size with that on the *Speculum*.

CHAP. VII.

Of SCENOGRAPHIC PERSPECTIVE; *or the* METHOD *of drawing* FIGURES *of* THREE DIMENSIONS *in* PERSPECTIVE.

1544. THE Superficies which terminate or bound a Solid, conſtitute its Form, or external Appearance; and therefore the PERSPECTIVE *of a* SOLID, is nothing more than the *complex Perſpective of all its ſuperficial Parts*; ſo that, properly ſpeaking, *there is nothing beyond ſuperficial Perſpective in Nature*, for that rightly applied, gives the Perſpective of every Figure or Form of Solids, or rather Figures of three Dimenſions *; as will appear in the following Articles.

1546. We ſhall begin with the moſt regular and ſimple Form of a Solid, *viz.* that of a CUBE, whoſe Sides are ſix in Number, and all equal; but if a Cube be placed direct before the Eye, only one or two of the ſix Sides can then appear. For if Fig. 1. (Plate II.) be conſidered as a Cube, then, becauſe its Height

* We have thought proper here to make a Diſtinction between a *Solid*, and a Figure of three Dimenſions; for though every Solid has always three Dimenſions, yet a Figure of three Dimenſions may not be a *Solid*, but a hollow or concave Body; thus there may be a *ſuperficial Cubic Inch*, as well as a *Solid one*, but the Perſpective of both is the ſame; for this Art is not concerned with the Solidity or internal Parts of Bodies that do not appear to the Eye.

Height QP exceeds the Height of the Eye IY, it will be impossible for the Spectator to see more than the Side in Front, EG. For Rays of Light proceeding from any Point in the two side Planes EB, HC, and that on the Top GB, can never come to the Eye at I; and the Bottom ED, and farthest Side DB, are precluded from the Sight by the Solidity of the Cube, so that one Side only can in this Case appear.

1546. But if the Height of the Eye IE (Fig. 4.) exceeds the Height of the Cube AB, then not only the Front-side ABCD, but likewise the Top or uppermost Side will appear to the Eye. And only these two can appear for the Reasons before-mentioned (1545.) The Perspective of the Side ABCD of the Cube in Front, will be similar to the Side of the original Cube, and therefore a *Square*, as is evident from (1468, 1469, 1470.)

1547. But since the Top, or upper Side of the Cube is below the Eye, and parallel to the Horizon, it may be conceived as placed on a ground Plane at the Height of the Cube above it; and its Perspective on the Table, will therefore be the same as of any Square ABCD for the Height IF, and Distance IY of the Eye, (1487) that is, A*bc*D will be the Perspective of the upper Side of the Cube; and consequently, the Perspective of the whole Cube, viewed in Front, will consist only of those two Sides, as in the Figure.

1548. A Cube of a less Height than that of the Eye, placed with one Side parallel to the perspective Table, but removed from the direct Ray IF towards either Side, to more than half it's Width AF or FD, will discover a third Side to the View; and in a solid Cube, three Sides are the most that ever can be seen; this is called a View in *Profile*.

1549. Therefore to give the most general and easy Idea of the Perspective of a Cube (or any solid Body) it will be best to divest it (as it were) of its Solidity, and represent it as hollow, or consisting only in Form, or the external Sides or Superficies.

1550. Wherefore (Fig. 7.) let ABCD be the Side of a hollow Cube, placed contiguous the perspective Table, and viewed in Profile. Then will the Perspective of the Base be B*bc*C (by 1487.) This is called ICHNOGRAPHY or *perspective Plan* of the Cube. The Perspective A*ad*D of the Top of the Cube

Cube is determined in the ſame Manner, by ſuppoſing A D the ground Line in regard thereto. The Perſpective of the Front A B C D, as it coincides with the Table, is the very Side itſelf. And the Perſpective of the Oppoſite or remote Side, as it is parallel to that Front and to the Table, will be of the ſame Figure, *viz.* a Square *a b c d* (1468, 1469.)

1551. Therefore the ſame Part A *a* of the Radial A I which bounds the Perſpective of the Top of the Cube, terminates alſo the upper Part of that Side of the Cube which is in View, as being the common Interſection of both thoſe Sides or Planes. For the ſame Reaſon B *b* terminates the lower Part of the viſible Side of the Cube on the Table; therefore the Perſpective of that Side of the Cube is A *a b* B. And ſince the Cube is hollow, and the Sides open, the other Side will appear internally; and its perſpective Profile will be D *d c* C.

1552. All we have ſaid is evident from the Conſideration of the mutual Interſection of Planes, as indeed from the Figure itſelf; for if we turn the Figure Side-ways, and look upon C D continued out to M as a ground Line, then will the two Sides, which are now perpendicular to the Horizon, be parallel to it, or they will be the Bottom and Top of the Cube, and the Height of the Eye above thoſe Sides or Planes will be I M, as it is now C M. And C *c d* D, A *a b* B, will be their Perſpectives, drawn by the common Rules (1489, 1492.) as B *b c* C, A *a d* D, are for the lower and upper Sides in the preſent Poſition of the Cube.

1553. If the Baſe of the Cube be divided by Lines parallel to the ground Line, like the Square in Fig. 5. then by ſetting off thoſe equal Diviſions in the ground Line from B to 4, 5, 6; and from them drawing Diagonals to the Point of Diſtance Z, they will cut the Side B *b* in the Points I, O, G, thro' which the Lines I K, O Q, G H, are to be drawn for their Perſpectives in the ichnographic Plan B *b c* C (1478, 1479.) and they will be all parallel to the ground Line B C, as in the original Baſe.

1554. In the ſame Manner, if C D be conſidered as a ground Line, and the equal Diviſions in the Side C D, be ſet off from D towards M, then Diagonals drawn from them to the proper Point of Diſtance in this Caſe, will interſect the Side D *d* in the

Points

Points F, *q*, *k*, through which the Lines FK, *q*Q, H*k*, being drawn parallel to DC (1552) will be true Perſpectives of the original Lines in the Side of the Cube.

1555. Hence we ſee that the perſpective Parallels IK, OQ, GH, on one Side, contain right Angles with the perſpective Parallels KF, Q*q*, HK, on the other adjacent Side, and are therefore perpendicular to each other, as in the original Cube. And conſequently, the perſpective Sections of the Cube through thoſe Lines, *viz.* EIKF, *o*OQ*q*, *l*GH*k*, will be all of them *Squares*, and parallel to the Side ABCD, as they are in the Cube itſelf.

1556. If each Side CB, CD, be divided by perpendicular Lines in the Points 1, 2, 3; then if from thoſe Points, Radials are drawn to the Point of Sight I, they will in the Plains B*bc*C, C*cd*D, be the Perſpectives of the original Lines in the Sides of the Cube reſpectively (1552). And thus it appears, that to put a Cube into a Perſpective, is nothing more than to find the Perſpective of a Square for its ſeveral Sides ſeparately, and that thoſe ſix perſpective Plans together compleat the *Perſpective of the Cube*, which is uſually called the SCENOGRAPHY of it.

1557. The Cube having an Angle placed in Front, or in ſuch Manner that a Diagonal of its Baſe may be perpendicular to the ground Line, is put into Perſpective by the ſame Rules, and with the ſame Eaſe, as the other; in this Caſe we ſee the upper Side of the Cube, and have an equal Profile of two of its Sides. Thus Fig. 8. is the *Scenography* of a Cube, whoſe Side is equal to the Square EOFQ in Fig. 4. Then *abhg* is the Perſpective of the upper Surface (the ſame as F*oeq* in Fig. 4.) and EOFQ is the *Ichnography* of its Baſe, found as directed (1485) for the Height EI, and Diſtance IZ, of the Eye. Having theſe two Plans, the Perſpective of the Cube is compleated by drawing the Lines *a*E, *b*O, *h*F, *g*Q.

1558. Becauſe the Sides of the Cube, or parallelopiped in this Caſe, make an Angle of 45 Degrees with the ground Line, therefore their Perſpectives *ag*, *eq*, will converge to the *Point of Diſtance* Z, (as in the other Caſe they converged, to the *Point of Sight* I.) As will all other Lines drawn parallel to them on the Sides of the original Solid. (1485)

1559. If on any one Part, the Solidity of the Cube be ſuppoſed to be uniformly continued for any given Length E *e*, then the Cube becomes a *Parallelopiped*, the Perſpective of which is ſtill determined by the ſame Rules, as is evident from the Figure.

1560. And if any Side of the Parallelopiped be divided by parallel Lines perpendicular to the Baſe; then by ſetting off the Diſtance of thoſe Lines from *e* to C in the ground Line to the Points 3, 2, 1, reſpectively; if a Ruler be laid from each of thoſe Points, to the Point of Sight I, it will give the Points in the Line *eq* of the perſpective Baſe, through which, if Lines are drawn parallel to *ae*, the Side of the Perſpective of the Cube or Parallelopiped, will be divided in the ſame Manner as the Prototype. All which is evident from the Figure, and what we have before taught (1552). For the *Radials* divide the Baſe Lines in this Caſe, as *Diagonals* did in Fig. 7. where the Cube was viewed by the Side. But this Diviſion is to be made equally by Radials or Diagonals at Pleaſure, as the Reader will eaſily underſtand from the Theory.

CHAP. VIII.

The RATIONALE *of the practical* METHODS *of* SCENOGRAPHIC PERSPECTIVE, *with the* THEORY *of Perſpective* MEASURES *for that Purpoſe.*

1561. AS in ſuperficial Perſpective there are two practical Methods of putting any plane Figure in Perſpective, *viz.* one by inſcribing them in a Rectangle, (1488) the other by given Points (1492). So in like Manner we here proceed to draw the Perſpective of any Upright, Solid, or ſuperficial Figure of three Dimenſions, two different Ways, *viz.* The *firſt* by inſcribing the given Figure, if regular, in a Cube, or proper Parallelopiped; the *Second*, is by finding the Perſpective of the ſeveral Lines which are formed by the Interſection of Planes, compoſing

posing the Superficies of the given Figure; and these Lines connected, from the *Scenography* thereof. This second Method is general for all figured Bodies, regular, or irregular.

1562. As a Specimen of the first Method, let it be required to put a PRISM into Perspective; to do this, we may conceive it circumscribed by a hollow Parallelopiped, whose Base and Height are equal to those of the given Prism; then draw the Scenography or Perspective of this Parallelopiped, which let be A B *b a d c* C D, the Base of which B *b c* C is also the Perspective of the Base of the *Prism*; and since the Edge of the Prism touches the Surface of the Parallelopiped, let the Distance thereof be measured, and set off on the ground Line from B to 4; then a Ruler laid from Z to the Point 4, will cut the Side B *b* in I: Draw I E parallel to A B, and it will be the Perspective of the Perpendicular let fall from the Edge or Angle of the Prism on the Base. Therefore drawing E F parallel to A D, it shall be the Perspective of the Edge of the Prism. Then drawing the Lines B E, *b* E, and C F, *c* F, they compleat the Perspective of the Prism in the Figure B E *b c* F C, which is shaded for the Sake of Distinctness.

1563. After the same Manner, a *Pyramid*, *Cone*, *Globe*, &c. may be drawn in Perspective, which cannot be difficult to those who understand the foregoing Principles. What I here speak of relates to Matters *merely Perspective*; but to give a natural *Relievo* to *conical* and *spherical Figures*, whether Convex or Concave, requires more a *skilful Management and Disposition of Lights and Shadows*, as we shall hereafter shew.

1564. The *second Method* for determining the *Scenography* of Solids, is by finding the Perspective of Lines which are formed by their intersecting Planes. To understand the Reason of this Method, the following *Theorem* must be premised. Let I D *be a given Line*, *placed on the ground Line* C D (*Fig.* 10.) *then from any two Points* A, B, *in the horizontal Line*, *draw the Radials* AD, AI; BD, BI; *and parallel to* CD *draw any Line* LK, *cutting the Radials* AD, BD, *in* G *and* E; *from those Points draw* GH, EF, *cutting the Radials* AI, BI, *in* H *and* F. *This done*, *I say*, *the Line* GH *is equal to the Line* EF. The Demonstration is from similar Triangles (656.)

For $AG : AD :: GH : DI$.
And $BE : BD :: EF : DI$.
Alſo $AD : BD :: AG : BE$.
Therefore $AD \times BE = BD \times AG$.
Therefore alſo $DI \times EF = DI \times GH$. (652.)
Conſequently $EF = GH$. *Q. E. D.*

1565. If F D (in Fig. 9.) be made equal and parallel to I D (in Fig. 10.) and from the Point of Sight I, the Radials I F, I D, be drawn; then if any Line L M be drawn parallel to the common ground Line D C D, interſecting the Radials I D, A D, in the Points L and M; and L G and M O be drawn parallel to F D or I D, they will be equal to each other alſo, as is demonſtrable in the ſame Manner as the above Theorem.

1566. Therefore let B L K I be the Perſpective of the Baſe of the Solid (Fig. 9.) drawn for an oblique View; and neither Side or Angle in Front. Then when the oppoſite Sides of the Solid are equal and parallel, and one Angle placed in the ground Line at B; if A B be the Height of the Solid, then through L draw the Radial I L D, meeting the ground Line in D; on the Point D draw D F equal and parallel to A B, and join I F; on the Point L erect the Line L G parallel to D F interſecting the Radial I F in G; and join A G; then is A B L G the Perſpective of one Side of the Solid. In like Manner you find the perſpective Lines I E, K H; and drawing A E, E H, and G H, the Perſpective of the whole Solid is compleated.

1567. Or thus more univerſally, let I D be the Meaſure of the Solid's Height, and placed on the ground Line at D (Fig. 10.) and make any Triangle A I D at Pleaſure, whoſe Vertex A is in the horizontal Line; then from the Angles of the perſpective Baſe I, L, K, (Fig. 9.) draw Lines I C, L M, K P, interſecting the Radial A D in C, M, and P, on which Points erect the Lines C N, M O, P O, and they will be the perſpective Heights of the Solid at the Angles I, L, K; that is, $CN = IE$, $MO = LG$, and $PQ = KH$; (by 1564) which Heights are all determined by the parallel Lines E N, G O, and H Q, (in Fig. 9, 10.)

1568. Hence it appears, that if any Line be perpendicular to the ground Plane, and its Diſtance from the ground Line, and Height,

Height, be given; then its Perſpective may be eaſily drawn. For the perſpective Seat of any given Point on the ground Plane is found by (1492) and the Perſpective of any given Height or Line is found for that Seat or Point, by (1494.)

1569. If the given Line be perpendicular to the ground Plane, but placed above it, at a given Height; then that Height is to be added to the given Length of the Line, and both conſidered as one Line; whoſe Perſpective is to be found for the Whole firſt, and then for the lower Part or Height above the Plane; then this latter ſubducted from the Whole, leaves the Perſpective of the given Line.

1570. A Line of a given Length, Poſition, and Inclination to the ground Plane, if placed upon it, is thus put into Perſpective. Firſt let the Perſpective of its Seat on the ground Plane be found (1492) then from the elevated End let fall a Perpendicular to the ground Plane, and find the Perſpective of that Perpendicular (1494). Laſtly, draw a Line from the perſpective Seat to the Top of the perſpective Perpendicular, and that will be the Perſpective of the given Line.

1571. For Example; ſuppoſe (*c*) the Seat of the Line in the perſpective Plane (Fig. 7.) and K the Seat of the Perpendicular, then through K draw the Radial I C, cutting the ground Line in C; upon the Point C erect C D equal to the Perpendicular, and draw I D; and from the Point K draw K F parallel to C D, and it will be the Perſpective of the ſaid Perpendicular; then join *c* F, and it will be the Perſpective of the Oblique Line, as required.

1572. If the Line be wholly elevated above the ground Plane, then its Perſpective may be drawn two Ways; for, firſt, you may continue the given Line to the ground Plane; and then finding the Perſpective of the Whole, and its Parts, that of the given Line will be known of Courſe. Or, ſecondly; from each End of the elevated Line let fall a Perpendicular to the ground Plane, and then find the Perſpective of thoſe Perpendiculars, the Summits of which being joined, give the Perſpective of the Line, as required.

1573. Having thus ſhewn how all Lines may be drawn in Perſpective of any given Length, Poſition, or Elevation on, or above the ground Plane, it follows, that all Bodies of what Form

or Figure ſoever they may be, are drawn in Perſpective at the ſame Time, and by the ſame Rules. Since the Lines formed by their various interſecting Planes will be given in *Length*, *Poſition*, and *Inclination* from the Nature of the Solid; their Perſpectives therefore are eaſily drawn by the Rules above (1492 to 1495) and the Extremities of thoſe perſpective Lines being joined, give the Scenography or Perſpective of the Body required.

1574. It farther appears how, by the *menſural Triangle* AID (Fig. 10.) the perſpective Heights of Men, Houſes, Trees, Beaſts, Birds on the Wing, *&c.* may all be duely adjuſted and meaſured; and their proper Diminutions proportioned to their Diſtances from the ground Line in *Landſcapes*, *Prints*, and all *perſpective Views*. Thus ſuppoſe ID were a Tree on the firſt Ground of a Landſcape, then the Radials AI and AD determine the Diminution of Trees of the ſame Height in all remote Diſtances from the ground Line. So that at C the Height is CN; at P, it is PQ; at G, it is GH, and ſo on, till it vaniſhes in the Horizon at A.

1575. But by ſimilar Triangles we have CN : ID :: AC : AD; and PQ : ID :: AP : AD; and GH : ID :: AG : AD, *&c.* Therefore CN : GH :: AC : AG; or the *Linear Dimenſions* of Objects, in a Piece of Perſpective or Landſcape, are *directly as the Diſtances from the Point of Sight.*

1576. Again, the Diminutions of the Heights of Objects in a Landſcape, will always be as the Diſtances from the fore Ground, or ground Line; for AD — AC : ID — NC :: AD — AG : ID — HG (649.) That is, ID — NC : ID — GH :: CD : DG.

1577. Hence it appears, that while Objects are at the ſame perpendicular Diſtance from the perſpective Plane, their perſpective Appearance or Magnitude will always be the ſame; and therefore if a Bird, or any other Body aſcend or deſcend in a perpendicular Direction through any Height, it will ſtill have the ſame Dimenſions in every Part of the Aſcent or Deſcent on the perſpective Plane, and muſt be ſo repreſented in Drawing, as we obſerved before. And thus we have premiſed every Thing neceſſary in the THEORY and PRACTICE of *common* PERSPECTIVE: We now proceed to illuſtrate the ſame by a Variety

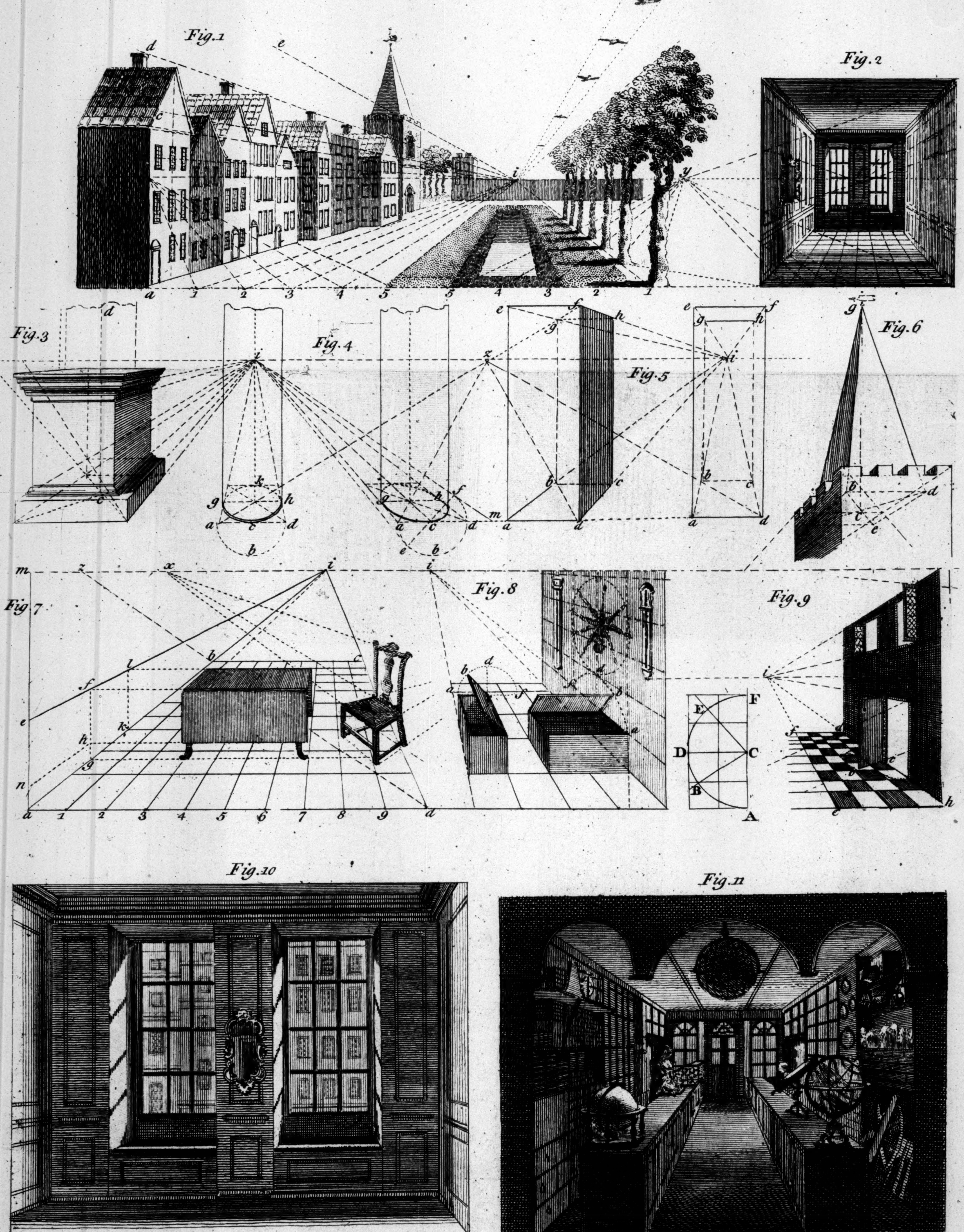


Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 10
Fig. 11

Variety of useful Examples; with Observations thereon, as Occasion requires.

CHAP. IX.

The RULES *of* PERSPECTIVE, *illustrated by a Variety of useful* EXAMPLES, *with proper Observations thereon relative to a* general PRAXIS.

[PLATE III. *Of* PERSPECTIVE.]

1578. AS we have given the fundamental Rules for drawing all Kinds of Superficies and Solids in Perspective; it remains now, that we illustrate the same by proper Examples, with such farther Observations and Instructions as may tend to facilitate the Practice of those Rules to the young Designer.

1579. In Fig. 1. we have Instances of a *long Parade*, *gravel Walks*, *grass Plats*, a *Canal*, &c. all put into Perspective according to the Rules laid down (Chap. III.) by drawing the proper Radials and Diagonals from the Point of Sight (*i*) and Distance (*z*); as also the *Scenography* of a whole Row or *Street of Buildings* on one Side, and of Trees on the other, by the Rules contained in Chap. VII. Here we may observe, that in order to preserve Uniformity, and Identity of Measures, in any Perspective or Landscape, there ought to be but *one Point of Sight, and Distance*; for if the Breadth or Distances of the Houses were determined from the Point of Distance (*z*) and the Distances of the Trees from another (*y*) there could be no Measure found, nor any Judgment formed, of any Distances at all in such a Piece.

1580. To the same Point of Sight (*i*) the same Rules of Perspective are applicable for Objects on all Planes in any Situation about the Eye; for the *Elevation of the Plane above the Eye*, and *the Elevation of the Eye above a Plane*, are the *same Thing* in Perspective; and consequently the Diminution of Objects in the *Air* follow that same Rules as those on the *ground Plane*. Thus

Birds

Birds (Fig. 1.) diminiſh as they recede from the fore Ground of the Sky (if we may uſe that Expreſſion) to remoter Diſtances, and at Length vaniſh at the Point of Sight in the horizontal Line.

1581. Thus alſo (in Fig. 2.) which is a direct perſpective View of the Inſide of a ROOM, the CIELING above, the *Floor* below, and each Side, are to be conſidered as ſo many Planes put into Perſpective from the given *perpendicular Diſtance* of the Eye from each (called the *Height* of the Eye) and the given Diſtance (*i y*) from the Front of the Room. For ſince the Interſections of the ſide Planes with the *Cieling* above, and the *Floor* below, are ſuppoſed perpendicular to the ground Line, or perſpective Plane, they will converge to the Point of Sight (*i*) according to (1479) and thereby terminate the Perſpective of the whole inſide View; which, indeed, cannot be difficult, as it contains the plaineſt Caſes of *direct*, *ſuperficial*, and *Scenographic Perſpective*.

1582. In Fig. 3. you have a ſide View of the PEDESTAL of a *Pillar*, whoſe *Front* being formed by the Meaſures proper to the Order, you proceed to form the Perſpective of its Baſe and Capital for the given Height and Diſtance of the Eye, as in the Figure. Then the Scenography of the Whole is finiſhed by the Rules in Chap. VII. and VIII.

1583. This Figure is added to ſhew how the Pillar is to be placed on its Pedeſtal in Perſpective; for it is eaſy to obſerve, that the Axis or central Line of the Pillar muſt be perpendicularly over the perſpective Center (*c*) of the Baſe, and conſequently the Perpendicular (*c d*) is that Axis or Line which determines the perſpective Poſition of the Pillar on its Pedeſtal.

1584. Fig. 4. contains the Perſpective of two PILLARS or COLUMNS in Front; one in a *Direct*, the other in a *Side View*; the *geometrical circular Baſe* (*a b d*) being given, the *perſpective elliptic Baſe* of each is determined by (1498) and then the Pillars themſelves are drawn or defined by the two parallel Lines which are perpendicular to the ground Line, and touch the Ellipſis on either Side.

1585. Hence it appears, that a *Colonade* of Pillars, viewed in Front, will not have all an equal perſpective Magnitude, or they

muſt

muſt not be drawn all of an equal Size; but that which is neareſt to, or directly before the Eye, has the leaſt perſpective Magnitude; and others appear larger on the perſpective Plane, as they are farther removed from the Eye or Line of direct View. However paradoxical this may ſeem, it is thus eaſily demonſtrated — The Ellipſis (*gchk*) of the ſame Circle on the ſame ground Line is greater or leſs in Proportion to the Obliquity or Diſtance of the View, as is evident from the Figure here, but more ſo in Fig. 4, 5. of Plate II. And ſince the Sides of the Pillars are Tangents to thoſe Ellipſes, and perpendicular to the ground Line, they muſt be at a greater Diſtance in an oblique than in a direct View, and therefore give a greater perſpective Magnitude of the Pillar.

1586. It may be proper alſo to give an optical Demonſtration of the Truth of this Poſition; let (*ad*) be the Diameter of the geometrical Baſe; this in the direct View ſubtends the greateſt Angle (*a i d*) under which the Baſe of the Pillar (*gchk*) can appear, and the perſpective Diameter is *g h*; but *a c* and *g h* are the ſame in both Views, and a Pillar on that Diameter would be the ſame in both Caſes. But the apparent Magnitude of a Pillar is in Proportion to the Angle which is ſubtended by that Diameter that is perpendicular to the viſual Ray (*i c*.) Suppoſe (*ef*) to be that Diameter, and draw the Rays (*i e*) and (*if*) which produce to the ground Line at (*m*) and let (*i e*) croſs it in (*l*) then it is evident the Diameter (*ef*) will eclipſe from the Eye a Part of the ground Line equal to (*lm*) in the oblique View; and therefore the perſpective Magnitude of the Pillar will be as much greater here than it is in the direct View, as (*lm*) is greater than (*a d*.)

1587. Hence it appears, that though it is always the Practice of *Architects* to repreſent all Pillars in a front View of an equal Size, yet the Abſurdity of ſo doing is as great as it would be to give the ſame Objects the ſame Heights at different Diſtances from the ground Line. And thoſe who vindicate ſuch a Proceedure, muſt be looked upon as either very ignorant, or elſe prevaricating with the Rules of Art in a moſt licentious Degree.

1588. The Axis of a Pillar in a direct View is equidiſtant for either Side; but it is not ſo in the oblique View; for as it

riſes

rises from the Center of the elliptic Base, it will, in that Case, be nearer to that Side next the Eye than to the other, as is evident by Inspection of the Figures. The same Thing is to be understood of any other regular Body in an oblique View.

1589. In Fig. 5. we have two PARALLELOPIPEDS, or *square Columns*, put into Perspective, on their proper Bases (*a b c d*, *e f g h*) which are supposed to be square; one in a direct View, in which only one Side can be seen; the other in an oblique View, shews the same equal Side in Front, and a Profile of the other Side (*d c f h*) next the Eye at (*i*) the Point of Distance being (*z*). These are Cases similar to those before observed of the Cube (1546 to 1549) in the same Points of View.

1590. The Manner of placing a SPIRE or STEEPLE on the Top of the *Tower* of a Church, &c. in Perspective, is shewn in Fig. 6. The Spire is supposed to be a *square Pyramid*, the Perspective of whose Base is (*a b d e*) (being a side View, like that in Fig. 1.) Here also you can see only two triangular Sides (*viz. b g d* and *a g b*) so far as they rise above the Top of the Tower. If from the Center (*c*) of the perspective Base a Perpendicular be raised, and its perspective Height (*c g*) proper to the perpendicular Height of the Pyramid, than (*g*) will be the Vertex of the Spire, which will be much nearer to that Side of the Tower next the Eye, than to the other, for the Reason before-mentioned.

1591. To place a TABLE, CHAIR, &c. on a Floor, as (*a b c d*,) in true Perspective, is no difficult Thing by the Rules prescribed. For Example, let *a d* be 10 Feet, and suppose the Table 4 Feet long, and to stand 3 Feet from the Side of the Floor (*a b*) and $1\frac{3}{4}$ Foot from the ground Line (*a d*) as in Fig. 7. is evident; let the Height of the Table be $2\frac{1}{2}$ Feet, and its Width 16 Inches, then on (*a*) erect the Perpendicular (*a m*) and from (*a*) to (*e*) set off $2\frac{1}{2}$, the Height of the Table, and draw (*e i*) to constitute the *mensural Angle* (*e i a*). Then draw Parallels to the ground Line at the Distance $1\frac{3}{4}$, and 3 Feet; they will cut the Radial (*i a*) in (*g* and *h*) then the Perpendiculars *g f*, *k l*, will be the perspective Heights of each Side of the Table; all which, with every other Circumstance, is directed by the Rules in Chap. VIII.

1592.

1592. As to the CHAIR, if the Perſpective of its Baſe be found, then the Height of the Seat and Back are in the ſame Manner determined by an Angle of Meaſures, or by the common Rules. The Bottom or Seat of the Chair not being a *Square* but a *Trapezium*, will have its Perſpective determined by the Point of Sight (*i*) and accidental Point *x* (1483) with regard to the two Sides which are *parallel*; but the other two which are inclined converge to a Point before they arrive at the horizontal Line, which is ſuppoſed at an infinite Diſtance from the Eye, and where only parallel Lines can meet, nor they neither but in Perſpective.

1593. If a *Box* is to be opened in Perſpective, then the Poſition with regard to the ground Line is to be conſidered. If the End be parallel thereto, then the Corner of the Lid, in opening, will deſcribe a Circle in Perſpective, as (*a d f*) Fig. 8. (1469, 1504) and therefore if a Circle be deſcribed on each End of the Box, with a Radius equal to the perſpective Width of the Box, and the given Angle to which the Lid is opened, be ſet off from (*a*) to (*b*); then you will have the four angular Points of the Lid by which it may be drawn as required.

1594. But if the Side of the Box be placed parallel to the ground Line, then the angular Point of the Lid, in opening, will deſcribe a Semi-ellipſis in Appearance on the perſpective Plane, as (*a b d f*), ſuppoſing the End of the Box contiguous to the Side of a Room. Here the elliptic Angle (*a c b*) is determined from the given circular Angle in which the Lid of the Box ſtands open, as directed (1498), or if the Angle (*b*) has its perpendicular Height meaſured, and its Perſpective found (1570) the Perſpective of the Lid or Cover is thereby determined.

1595. Again, if Doors, Windows, *&c.* are to be put into Perſpective as they ſtand open in any given Angle; the Rules for doing it are ſtill the ſame. Thus ſuppoſe the Door be 2 Feet wide = A C (Fig. 9.) then on the Center C deſcribe the Semicircle A D F, upon which the angular Point of the Door moves; ſuppoſe the given Angle to which it is opened be A C B; then if on the perſpective Floor (*b e f d*) you find the Perſpective of the Semicircle A D F; and therein make the elliptic Angle (*a c b*) correſpondent to the given Angle A C B in the Prototype; you will have the Poſition of the Point (*b*) on which you

raiſe the perpendicular Perſpective (*b o*) of the given Height of the Door (1568) and then, laſtly, by forming the Perſpective of the Door-place or Aperture (*a m n c*) in the Side of the Room, you will have the four angular Points of the Door determined, by which it may be readily drawn in the Poſition required.

1596. In the ſame Manner the Windows on the upper Part of the Room may be drawn. But it is to be obſerved, that ſince the Angle is given in which the Door or Window is opened, and alſo the Diſtance on, and Height, above the ground Plane, the Angles of the Door, or Caſement, may be moſt eaſily determined by the Method of Perpendiculars as directed in (1567.)

1597. If a Perſon in a Room, on one Side of a Street, views through the Windows the Buildings on the other Side, the Window then becomes the perſpective Plane or Table for the orthographic Delineation of whatever appears in Front on the other Side of the Way, (1649,) as is here repreſented in Fig. 10.

1598. The whole *Scenography* or perſpective View of the Inſide of a Shop is preſented in Figure 11. in order to compare it with the Picture made thereof by a Speculum or Lens which we ſhall hereafter have Occaſion to take Notice of.

CHAP. X.

The PRINCIPLES *of* Catoptric *and* Dioptric PERSPECTIVE *conſidered with regard to* VIEWS, PICTURES, LANDSCAPES, *&c. formed by* MIRRORS *and* LENSES; *with the* RULES *for exhibiting them* in any required Proportion *to the* OBJECTS.

1599. WE have already conſidered ſo much of *Catoptric Perſpective* as relates to *reflecting Planes*; but it was neceſſary to treat of *Scenographic Perſpective* before we could properly handle the Doctrine of *Optical Perſpective* univerſally as concern'd in the Formation of *Images*, *Pictures*, *Landſcapes*, and all Kind

Kind of *Views* by reflected and refracted Light from and through Convex and Concave Glasses of every Kind.

1600. The Subject we are now entering upon, though the most essential and exquisite Part of the Science, has not (that we know of) been touched upon by any Author on Perspective, and therefore we shall be the more particular and explicit on this Head. In order therefore to determine how far the *Picturesque* IMAGES of OPTICAL GLASSES agree and coincide with the *Drawings made* by the *Rules of common Perspective*, it must be remembered, that the Linear Dimensions of an Object and its Perspective are proportioned to their Distance from the Eye (1468) that is (in Fig. 1. Plate II.) $RN : rn :: IZ : Ii :: AB : ab :: AD : ad$.

1601. Now suppose A G were an open Cube or other rectangular Body whose opposite Sides are equal, and let the Side E G in Front be called the *Proscene*; the hinder Side A C the *Postscene*; the Sides E B, H C, the *Lateral Scenes*; the ground Plane A H, the *Primary* SCENE, and C B the *Ærial Scene*, if placed above the Eye; or *Secondary Scene*, if below it. Then conceive a perspective Plane placed contiguous to the Proscene, on which the whole *Scenography* of the Body will be viewed in Perspective by the Eye at I.

1602. Put the linear Dimension of the *Proscene* E H or E F $= P$, and the Perspective of the *Postscene* ad or $ab = p$; also let their Distances from the Eye be $Ii = D$, and $IZ = d$. Then we have $P : p :: d : D$. And this Analogy will hold for any Situation of the perspective Plane, between the Eye and the given Cube.

1603. If instead of the Eye, a *Convex* or *Concave* Speculum were placed at I, then an Image will be formed of the said Cube or Body, and the linear Dimensions of every Part will be proportioned to the Distance from the Glass, as we have shewn (1310) Then, if we call A B or A D = an Object, and put it equal to O, and its Image $= i$; we shall have $O : i :: d : f =$ focal Distance of the Image; and therefore $\frac{di}{O} = f$ in all Kinds of Speculums.

1604. In a CONVEX MIRROR, we had $\frac{dr}{2d+r}=f$ (1291) therefore $\frac{dr}{2d+r}=\frac{id}{O}$; from whence we get $Or = 2id + ir$, which gives this Analogy, $O : i :: 2d + r : r$. Or, *the Proportion of the Length or Breadth of the Postscene is to that of the Image thereof, as twice the Distance added to the Radius, is to the Radius of the Speculum.*

1605. The Proscene being equal to the Postscene (1601) its Length or Breadth EF or EH will still be denoted by O, but its Distance being less, let that be $= D$; and because the Image will be larger in Proportion, put it $= I$, and proceed as before; we have $Or = 2ID + Ir$, and the same Analogy $O : I :: 2D + r : r$.

1606. But because in both Cases Or is the same, therefore we have $2id + ir = 2ID + Ir$, and therefore $I : i :: 2d + r : 2D + r$. From hence we have the Ratio of the linear Dimensions of the Proscene to that of the Postscene; and consequently the *Optic Scenography* of the Solid is thereby determined, and its Difference from that in common Perspective.

1607. In the CONCAVE *Speculum*, because $\frac{dr}{2d-r}=f$, (1291) we have $Or = ir - 2di = Ir - 2DI$; whence $I : i :: r - 2d : r - 2D$. From whence it appears, that when the Radius of the Speculum exceeds twice the Distance of the Objects, then the Images will be *positive* or *behind* the Glass; but otherwise *negative* or before it, agreeable to (1315, 1316.)

1608. In Case of a LENS, it is every where $O : i :: d : f$, and $\frac{id}{O}=f$ (1340). Also for a *double and equally Convex Lens* we have $f=\frac{dr}{d-r}=\frac{id}{O}$ (1387) and therefore $Or = id - ir = ID - Ir$; whence we have $I : i :: d - r : D - r$; or in the *Scenography of the Image* of such a Lens, the *linear Dimensions of the Proscene and Postscene are as the Differences between the Distances and Radius of the Lens respectively.*

1609. After the same Manner the *Scenography of Images* for all different Forms of Lenses may be found, and compared with those

thoſe of common Perſpective. And ſince, in all, the Terms of Compariſon conſiſt of the Sums or Differences of the Radius and Equimultiples of the Diſtances; therefore *while the Radius of the Glaſs bears any conſiderable or ſenſible Proportion to the Diſtances of Object, the optical* SCENOGRAPHY *of their* IMAGES *will differ* more or leſs from the common Perſpective thereof.

1610. But it is evident, when the Diſtances are ſo great that the Radius of the Glaſs bears no ſenſible Proportion thereto, it will then vaniſh out of the Terms of the Compariſon, and then the Analogies

$$\text{become in the} \begin{cases} \text{Convex Mirror, } I : i :: \quad 2d : \quad 2D :: d : D. \\ \text{Concave Ditto, } I : i :: -2d : -2D :: d : D. \\ \text{Convex Lens, } I : i :: \quad d : D. \end{cases}$$

1611. Conſequently, in all ſuch Caſes the optical Scenography of the Image is the very ſame with that of common Perſpective on the tranſparent Plane; for by all the Glaſſes, we have $I : i :: d : D$, and on the perſpective Table, we have $P : p :: d : D$ (1468) therefore $I : i :: P : p$. Whence it appears, *that all the Parts in the optical and perſpective Scenography are perfectly ſimilar, and that, therefore, the* IMAGES *of* OPTICAL GLASSES in ſuch a Caſe, are PICTURES, PROSPECTS, or LANDSCAPES in *true Perſpective.*

1612. Hence the excellent Uſe of a *Convex Speculum*, in exhibiting to the Eye a *genuine perſpective View* of all diſtant Objects, as a perfect Copy for the Artiſt to draw by. ——— It preſents him with an inſtantaneous Conſtruction of the perſpective Scenography of the interior Parts of a Room, and all its Furniture, Tables, Chairs, Scrutores, Book-caſes, Pier-glaſſes, Pictures, &c. juſt as they ought to be drawn. —— An Inſtance of its Uſe in this reſpect, is Fig. 11. Plate III. which is a *perſpective View* of a *Mathematical* SHOP, with its *Counters*, *Spheres*, *Globes*, *Air-pumps*, *Teleſcopes*, *Glaſs-caſes*, &c. actually delineated from the *perſpective* PICTURE of a CONVEX MIRROR (1598.)

1613. Another great Uſe of ſuch a *Speculum*, is to exhibit the true Perſpective of an Object diminiſhed in any given Degree. For, becauſe $Or = 2id + ir$ (1607) it is $Or - ir =$ $2id$,

$2\,i\,d$, and therefore $d = \frac{Or - ir}{2i}$. For Example, suppose it required to draw a Microscope, Air-pump, &c. 4 Times less than the Life; then O : i :: 4 : 1; and let the Radius of the Speculum be $r = 12$ Inches; then $d = \frac{48 - 12}{2} = 18$ Inches, the *Distance from the Mirror to make the Image 4 Times less than the Object.*

1614. Since the Image is larger in Proportion as the Radius of Convexity is so, it appears how excellently well adapted such Mirrors are, when very large, for exhibiting the most perfect LANDSCAPES of distant SCENES, whether *Gardens*, *Fields*, *Lawns*, *Woods*, *Mountains*, *Vales*, *Rivers*, *Sea*, &c. with all the natural COLOURS, LIGHTS, and SHADES, MOTION, and every other Incident which can tend by this *perspective Miniature*, to improve and out-vie even NATURE itself.

1615. The same Things may be said of a *Concave Mirror*, with regard to the Form and Proportion of the several Parts of the Image, and the just Perspective of the Whole, but then the *inverted Position* of the Picture before the Glass, and the Inconvenience of observing it, renders it not so useful in the Arts of drawing and designing, as that of the Convex Form.

1616. But then it has this most entertaining Property of resolving the perspective Picture, or Landscape formed by the *Convex Mirror* into its *Original* or *Prototype*, and gives each Part Distances, Size, and Situation. In this Case it is supposed that the Radius in both Speculums is the same. Thus, for Illustration, suppose a *View of 7 Miles round St.* PAUL's were drawn from a Convex Mirror of 10 Feet Radius, that Drawing held at the same Distance before an *equal Concave Mirror* as it appeared to be behind the Convex One, the said Concave would revert the Landscape, and present the Eye with a *delightful View* of the large *City of London*, and County about it, as *large as the Life*, and at the *same Distance* in every Part, as appears to the naked Eye. The Reason of all which, is very evident from the Theory we have above premised. Hence *the great Use of Concave Mirrors in viewing perspective Prints*, in the *portable* CAMERA OBSCURA.

1617. From the ſame Theory (1608) it appears, that a *Convex Lens* does alſo preſent us with the ſame *perſpective Figure* or *Image* of diſtant Objects in the Focus, which as the Radius is longer, will be larger in Proportion; and therefore, in a real *Camera Obſcura*, when ſuch a Lens is applied in a *ſcioptric Ball and Socket*, you view upon a Screen, at a proper Diſtance, the *Scenography* of *Buildings*, and a LANDSCAPE of every rural Scene, ſo heightened by Colours, and animated by Motion, as juſtly excite our Admiration, and we readily pronounce the Pencil of Nature perfect, and all her Paintings inimitable.

1618. Upon the Whole we may conclude, that as all artificial Paintings are but Copies of Nature, the more they approach to, and are regulated by the Art of Perſpective, the more natural and valuable they will be, and Beauty and Harmony will ſo much the more evidently appear; for we have ſhewn in the Theory (1292) that in all Nature's Painting the ſtricteſt *harmonical Proportion* is obſerved; and conſequently NATURE is all PERSPECTIVE and HARMONY.

CHAP. XI.

Of the PERSPECTIVE APPEARANCE *of* OBJECTS *on* INCLINED PLANES.

[PLATE IV. *Of* PERSPECTIVE.]

1619. WE have hitherto treated of ſuch perſpective Views as are ſuppoſed to be taken on Planes perpendicular to the ground Plane; but as it will, on many Occaſions, be neceſſary to conſider them on Planes or perſpective Tables inclined in any Angle thereto, I ſhall here deliver the Theory and Rules for that Purpoſe.

1620. Let A D N R be a perſpective Table inclined to the ground Plane B C G K in a given Angle (Fig. 1.) A B C D is a Rectangle viewed upon the Table by an Eye at I, at the Height I H, and Diſtance from the ground Line H F. Biſect B C in

E,

E, and draw E H, and I i parallel thereto, then will (i) be the Point of Sight on the Table, and R Y, drawn through it parallel to A D, will be the horizontal Line, in the ſame Manner as in the upright Plane (1465).

1621. Produce C K and E F indefinitely, and make the Angle I M H = Angle of the Plane's Inclination; then will I M be parallel to i F. Draw the Radials i D, i A; make K L = H M, and join I L, which is then parallel to i D. From the Points B and C, draw the viſual Rays I B, I C; they will interſect the Radials i A, i D, in the Points S and P, then joining S P, the Area A S P D will be the Perſpective of the Rectangle A B C D; the Demonſtration is the ſame as was uſed for the upright Table (1475, &c.)

1622. But for a practical Method, the following Theorem in this Caſe may be preferable. In the ground Line A D produced, take D O = D C; and in the horizontal Line R Y, take i Y = i I, the Diſtance of the Eye from the Table; draw Y Q parallel to i D, and draw Y O, which will interſect the Radial i D in p, the ſame as the Point P found by the other Theory. For in the ſimilar Triangles L C I and D C P, we have I C : D C :: L I : D P; and in the ſimilar Triangles Y O Q and p O Q, we have Q O : D O :: Y Q : D p. But the three firſt Terms of each Analogy are ſeverally equal to each other; for L C = L D + D C = Q D + D O = Q O; and D C = D O, by Conſtruction; alſo L I = i D = Y Q. Therefore, alſo D P = D p, and conſequently the Points P and p coincide or are the ſame. *Q. E. D.*

1623. Whence the practical Rule for finding the Perſpective of a given Point C on the ground Plane, upon a given inclined Table is this. *From the given Point draw a perpendicular* C D *to the ground Line, and draw the Radial* i D; *in the ground Line* A D *produced, take* D O = D C, *and* i Y = i I, *and draw* Y O, *which will cut the Radial* i D *in the Point* P, *the Seat or Perſpective of the given Point* C, *as required.*

1624. The Diſtance of the Eye from the Plane is equal to its perpendicular Diſtance from the ground Line, together with the Cotangent of the Plane's Inclination; for i I = F M = F H + H M, but H M is the Tangent of the Angle H I M, the

the Complement of the Inclination H M I to the Radius I H the Height of the Eye.

1625. Hence it will be easy to draw the Perspective of any given Object upon an inclined Table B C T V (Fig. 2.) Let the Object be a hollow Cube whose Side is M N O P; suppose it to touch the Plane on its upper Part; and that the Inclination is equal to the Angle W C V. I is the Point of the Sight on the Plane, and X, Z, the Points of Distance, as in the upright Plane. Then, because the Distance of the Cube from the Table at Bottom is C P, if we draw the Radials I C, I B, and the Diagonal Y P, the Point of Intersection F will be the Seat of the Point P; and drawing the Line Y O, it gives the Point H for the Seat of the Point O. Therefore drawing E F and G H parallel to the ground Line B C, the Area E F G H will be the Perspective of the Square Base of the Cube, by the Theory (1487.)

1626. The inclined Table being just the Width of the Cube, we have A D = M N; continue A D to S, which is now to be considered as a ground Line, for the Top of the Cube; therefore taking A R = M N, and drawing R Z it will cut the Radial I A in K; and then drawing K L parallel to A D, it will give A K L D for the Perspective of the Top of the Cube. Lastly, by joining A E, F D, G K, and H L, the whole *Scenography* of the Cube will be compleated.

1627. From this Procedure for the Cube, I presume the Method of finding the Perspective of any *Points*, *Lines*, *Superficies* and *Solids*, will not be difficult to the young Drafts-man; for this one Example includes them all, either contiguous to, or at a Distance from the Table; and parallel and perpendicular to the ground Plane, ground Line, *&c.*

1628. I need not observe, that the Sides of the Cube in Front, are not *Squares* in the Perspective, as they are in upright Planes, but *Trapeziums*; for E F is less than A D; and G H less than K L. And this will be the Case of all Lines and Superficies not parallel to the perspective Table.

1629. The Inclination of the Plane may be such that the *Perspective of a Circle* on the ground Plane shall be a *Circle* also. Let D E (Fig. 3.) be the Diameter of the given Circle; H D the Distance, and H I the Height of the Eye; draw the Rays I D, I E; and let the Plane H F cut them *subcontrarily* in *e* and *d*;

(1510) then ſince all the Angles in the Triangles DHI, and EHI, are known; and by Suppoſition the Angle $HDe = HdE$, and the Angle $HEd = HeD$; therefore ſince $HeD = IHe + HIe$; we have $HeD - HIe = IHe$, the Inclination of the Plane HF required for that Purpoſe.

1630. Hence alſo it is eaſy to aſſign the Inclination of a Plane wherein the Seat or Perſpective of a given Point ſhall be the ſame as in the perpendicular Poſition. Thus (Fig. 2.) ſuppoſe CV the upright Plane, the Eye at Y views the Point O thereon at (*a*); on the Center (C) with the Diſtance C*a*, croſs the Ray YO in (*b*,) and through that Point draw CQ which is the Poſition of the Plane required, ſince $Cb = Ca$.

1631. Let CW be perpendicular to the Ray YO, then is the Angle $VCW = aOC$; and conſequently the Angle $aCb = 2aOC$.

1632. Since the right Line CO occupies the ſame Space C*b*, C*a*, on the inclined and upright Planes, it will follow that any Line CP at a leſs Diſtance will take up a leſs Space, and any Line CX at a greater Diſtance will take up a greater Space on the inclined Plane, than on the ſame in the erect Poſition. Much more might be ſaid on this Head, which we ſhall leave to the Reader's Invention, having premiſed the Principles, which is here all that is propoſed.

CHAP. XII.

Of Theatrical PERSPECTIVE; *or, the* RULES *of* PERSPECTIVE *applied to the* SCENERY *of a* THEATRE.

[PLATE IV. *Of* PERSPECTIVE.]

1633. THE whole Artifice or Deſign of a THEATRE, is founded in PERSPECTIVE. For as we are to be entertained with an Imitation of ſome memorable Actions paſſed, ſo in order to render the Performance more natural and pleaſing, the Scenes on which they were tranſacted ought to be repreſented as they appear to the Eye of a diſtant Spectator, that is, in *Perſpective*. For whether there be a tranſparent Plane, or not, be-

Plate IV.

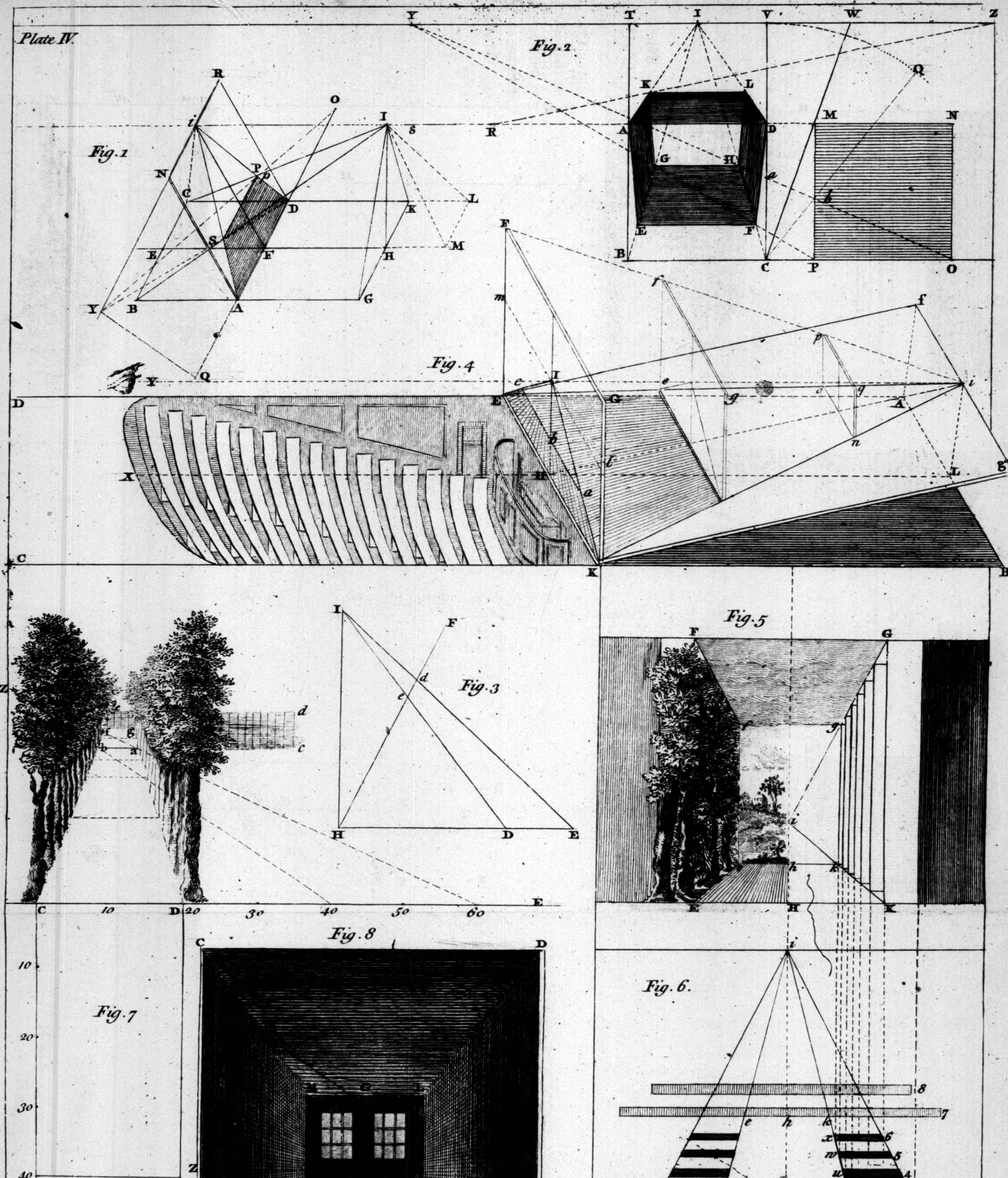
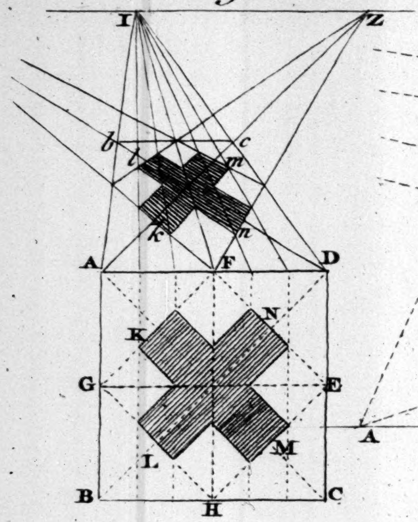


Plate V. of PERSPECTIVE.

Fig. 1



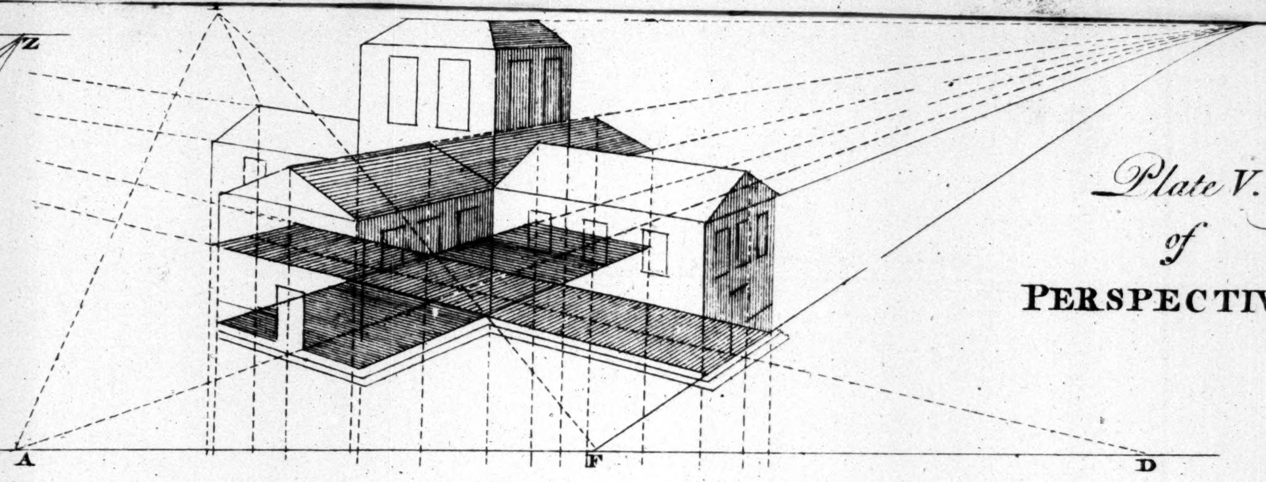


Fig. 3

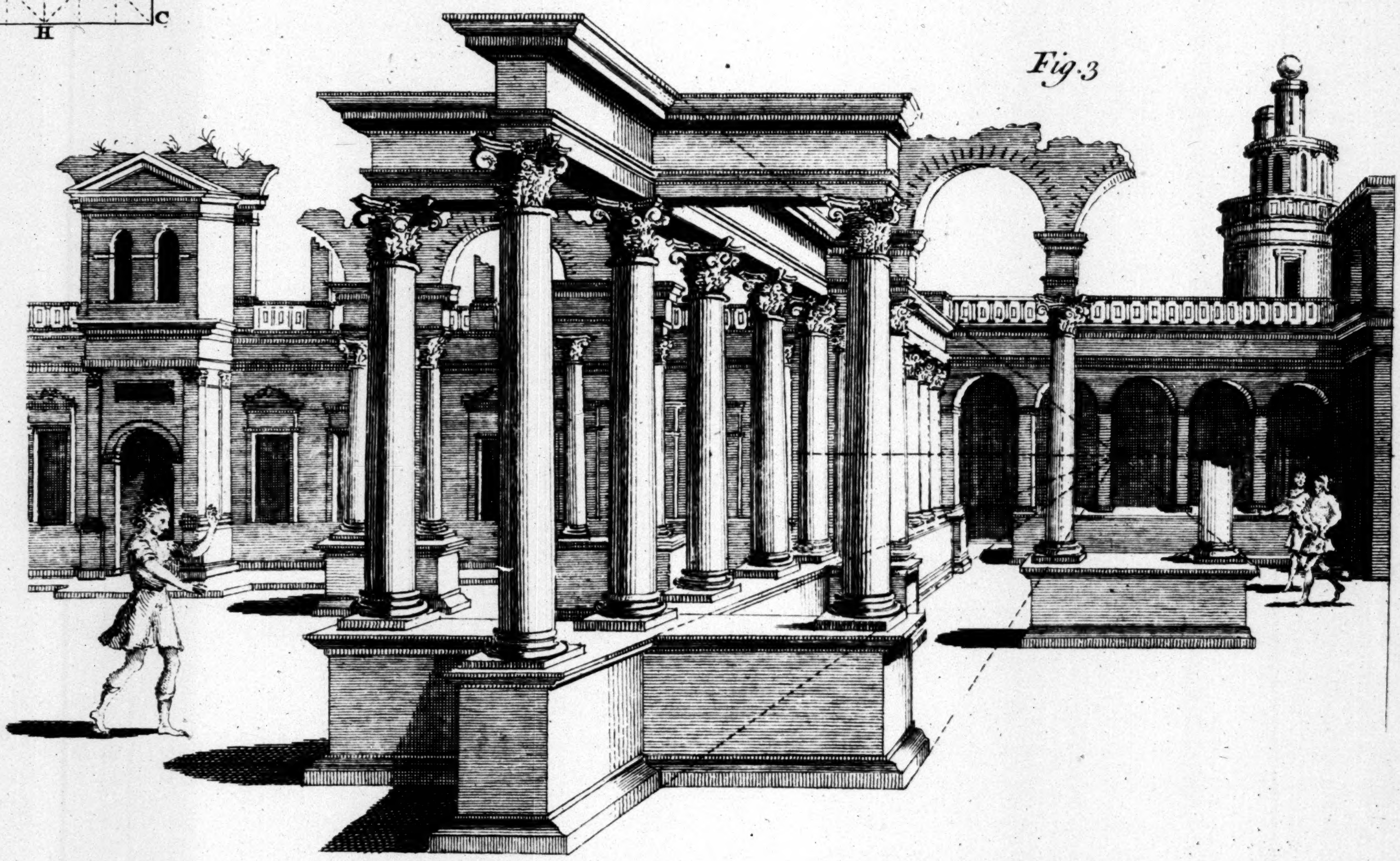


Fig. 4

Fig. 5

between us and the Scene of Action, the Image in the Eye is equally in Perſpective, and excites the ſame Idea in the Mind, as if the whole Affair were viewed on a perſpective Plane.

1634. Hence it is, that the whole HOUSE, internally, is only one large Scenography, or perſpective View of an hollow *Parallelopiped*, or large *Room*, about three or four Times as long as it is Wide or Deep. Hence it is, that you ſee the Body of the Houſe ſo contrived, as to repreſent the Perſpective of the firſt Part of ſuch a large Room, the Ceiling, Sides, and Floor, all Converge, by the Rules of Perſpective, to a Point of Sight at the fartheſt End, which is the remoteſt Part of the *Stage*.

1635. For the STAGE is only the Floor of the other Part of the Scenography of the large Room, ſuppoſed to have been the Scene of Action; and therefore alſo the *Sides* and *Ceiling*, *Proſcenes*, and *Poſtſcenes*, are all in regular Perſpective. But in order to give a more exact Idea of this Matter, we muſt have Recourſe to perſpective Repreſentation alſo.

1636. Therefore let A B C D (Fig. 4.) repreſent the ground Plane or Floor of a long and large Room, on one Part of which toward A B, ſuppoſe ſome great Action or Converſation to have happened, and the ſame to have been obſerved by the Eye Y at the other End of the Room. Then it is evident, if in any intermediate Part a tranſparent Plane E F G K had been erected, the Obſerver would have viewed the Whole tranſacted perſpectively on the Plane E G, in that Part *ac* E K which includes the Scene of Action A E K B. (1487.)

1637. Now though we have ſhewn how ſuch an Event may be truly repreſented in a perſpective View, yet this is ſtill but a Picture of it; there, it is true, we ſee the *Place*, the *Perſonages*, the *Manner*, the *Time*, and ſome other concurring Circumſtances, that altogether, give ſome Idea of the Thing; but real *Life*, *Action*, *Voice*, and *Variety of Scenes*, *Attitudes*, *Paſſions*, *&c.* are wanting to animate and realize the perſpective View, or Picture.

1638. Now all this is effected by the ARTIFICE of a STAGE or THEATRE. For let Y X be the Height of the Eye placed over the middle Line of the Room, and drawing Y I parallel to H X, the Point of Sight will be I in the horizontal Line (*l m*) of he perſpective Plane. And now, inſtead of the upright Plane

E G, we suppose another Plane f E K g placed on the same Line E K, and so far inclined, that its Elevation i L be just equal to the Eye's Height Y X, it is evident, the perspective Horizon (*l m*) will coincide with the End f g of the inclined Plane; the perspective Scene *a c* E K will be projected into the large One *e* E K *k*, and all above it on any other perpendicular Plane at (*e k*) or (*n o*) behind.

1639. This new inclined perspective Plane f E K g is the *Stage* or *Theatre* we speak of, and is a Scene of real Action in a perspective View at the same Time. The Area *e* E K *k* is here the perspective Scene of Action (or *Scena Dramatis*) and instead of Men and Women in the Picture *a c* E K, we have sufficient Room for real Persons (*Personæ Dramatis*) to act the several Parts of the intended Representation (or *Drama*); here now are the *Proscene*, the *lateral Scenes*, the *Postscene*, &c. all in Perspective, and moveable to the most convenient Situations. Not only the whole Scenography of the Stage is by this Means formed and adjusted by the Rules of Perspective, but the very Actors themselves, as they advance or retreat, have their Appearance encreased or diminished to the View of the Spectators (especially those remote in the Pit and Gallaries) according to the Laws of *optical Perspective*.

1640. On this inclined Plane, or perspective Stage, the dramatic Scene may be encreased at Pleasure, and thereby very distinct Views and Prospects agreeably rise to the Sight, by the artful Paintings and Designs on the Lateral and Postscenes; thus *e* E K *k* may be enlarged to *o* E K *n*; and *f e k g* which was before the Postscene, may now be considered as the Proscene, to a distant Dramatic Scene (*o e k n*) whose Postscene is (*n o p q.*) And thus the Stage, or Scene of the *Drama*, may be extended to comprehend the most distant View or Scenes of Action, even to the Horizon itself, for the remote End of the Stage, *viz.* the Line f g, is the horizontal Line in this theatrical perspective Plane.

1641. As the *Dramatic Scene* *e* E K *k* is variable according to the Nature and Circumstances of the particular Parts or Actions of the Play, so the Lateral and Postscenes must be conformable thereto, and consequently variable, and of different Sorts; thus if the Stage represent the Inside of any particular Building, as a

Ca-

Castle, Prison, Church, &c. the Side and Back-scenes must correspond to the same in all the Characters appropriate to such Edifices. If the Stage be a Street, the lateral Scenes represent Houses, &c. on each Side, and the Postscene gives a perspective View of all at the far End, or beyond it. If you suppose the Scene a *Vista*, then Rows of Trees, &c. descend in Perspective on either Side. If *Fields* or *Lawns* be the primary Scene, and Swains, Shepherdesses, &c. the Persons of the Drama, then Trees, Woods, Cottages, &c. make the Perspective of the Sides; and distant Views of Countries, sunny Hills, and horizontal Clouds are portray'd on the Postscene.

1642. With regard to the Scenes, they are variable and different in the different Parts of the Scenography; when the Stage is not open, the *Proscene* is only a plain Curtain, to be drawn up and let down as Occasion requires. The *Postscene* is either a Curtain with Designs or Drawings in Perspective; or else it consists of two sliding Parts, which being put together, from each Side the Stage, make one uniform perspective Piece suitable to the Nature of the Part of the Play then acting.

1643. The lateral Scenes, and those above the Stage, consist of many different Parts; those on the Sides are moveable in Grooves made in the Floor of the Stage; and may by this Contrivance be variously changed, and being placed oppositely one a little before the other, they make together but one united View or perspective Scene, and by this Means produce an agreeable Deception, and at the same Time give an Opportunity for the Actors to come on, and go off, on any Part of the Stage, as the Circumstances of the Action may require. In the same Manner the aerial Scene may be composed of several Parts of particular Curtains, or moveable Pieces, descending one below the other, and uniting in one perspective Design.

1644. But to see more clearly how these Things are contrived and disposed, let Fig. 5. represent a front View of the Stage in Perspective. One Half of which is plain and naked, and the other Half disguised and decorated with perspective Paintings, &c. EFGK is the Front; E*ek*K, the Stage; G*gk*K the plain Side, consisting of several lateral Planes projecting beyond each other in their proper Grooves; (*efgk*) is the Postscene, half plain, and half painted; EF*fe* is the other Side, covered, as it

it were, with perſpective Drawings and Deſigns; and making but one Piece in Appearance. F *fg* G is the aerial Scene, open on one Part, and properly formed on the other; (i) is the Point of Sight by which the Whole is regulated. This Piece of Perſpective is the proper Scenography of that Part of the Stage in Fig. 4. which is denoted by E F G K *k g f e*.

1645. Hence it is evident, that the Heights of the ſeveral Planes, or Sliders, on the Sides of the Stage, are to be adjuſted by the Inclination of the Line F *i*, or G *i*, to the Line E *i*, or K *i*, on the Stage; ſo that the Height EF or G K, of the firſt Sliders, and (*e f*) or (*g k*) of the laſt, in the perſpective Stage Fig. 5. are the ſame as the Lengths of the Lines denoted by the ſame Letters in Fig. 4. And the ſame is to be obſerved, for the Sliders in any other Scene (*e k n o*) in a remote Part of the Stage.

1646. Another Figure is yet neceſſary to ſhew how the Grooves are to be made on the Sides of the Stage, and their Diſtances aſcertained and proportioned. In order to this let K *k* (Fig. 4.) be the Perſpective of the ground Line for the Length of the Stage divided into equal Parts (as directed in 1554, 1555.) Then let Fig. 6. be the whole Length of the Stage; A B the Width of the Houſe, and E *i* K the Area of the Stage, as in Fig. 4. and draw the Lines A *i*, and B *i*. Then from the Points of equal Diviſion in the perſpective Line K *k* (Fig. 5.) raiſe Perpendiculars, and they will terminate the fore Parts of the lateral Sliders reſpectively, in that Perſpective of the Stage.

1647. Let the middle Line H *i* of Fig. 6. be in a right Line with H *i* in Fig. 5. Then from the ſeveral Points of equal Diviſion in the Line K *k* of Fig. 5. let fall Perpendiculars to the Line K *k* in Fig. 6. and they will interſect it in the Points K, *r*, *s*, *t*, *u*, *w*, *x*; through which, if Lines are drawn parallel to the ground Line A B, they will aſſign the Places and Diſtances of the Grooves (*r* 1, *ſ* 2, *t* 3, *&c.*) on each Side the Stage K *k* and E *e*. And the Figures 7 and 8 denote the Grooves of the Poſtſcene. But there is in different Stages, a different Diſpoſition of the Parts and Machinery; what we have ſaid is ſufficient for a *general Rationale of theatrical Perſpective*, which is all that is here intended.

CHAP.

CHAP. XIII.

The Doctrine of PERSPECTIVE DECEPTION *explained, and exemplified.*

[PLATE IV. *Of* PERSPECTIVE.]

1648. WE have already considered the various Illusions and Deceptions which the *visual Sense* is subject to from *Catoptrics* and *Dioptrics*; that is, how different the Magnitude, Position, Place, Distance, *&c.* is of any Object, as it is conceived in the Mind by the *visual Faculty*, from Rays reflected or refracted to the Eye in regard to the Idea of the same Thing excited by direct Rays.

1649. Of these Deceptions some are very useful, witness the *Looking glass*, *Reading-glass*, *&c.* Others afford us a very rational Amusement; as those Lenses and Speculums which give an agreeable *Relievo* to perspective Views, Prints, *&c.* used in *optical Machines*. And some on the other Hand gives us a great deal of Trouble, as in Case of Refractions through the Atmosphere in Matters of Astronomy and Navigation, by which the Celestial *Phænomena* are shewn in very different Circumstances from those which are real.

1650. But the Deceptions we here intend to speak of are purely Perspective, and are designed to impose on the visual Faculty in the most agreeable and advantageous Manner. Indeed all Perspective is a Sort of Illusion, as it represents Things not as they really are, but as they appear on an intervening Plane by refracted or reflected Light. And therefore, if in any particular Case Things are not just as we could wish to have them, we can, at least, by this Art, make them appear to be such, and to an inadvertent Observer, the Difference from the Truth will not be discovered, and that which is only *Perspective* shall be taken for the *real Thing* it represents.

1651. We shall illustrate this Matter by an Example or two. Let A B C D be the ground Plane of the perspective *Vista* C *b a* D, (Fig. 7.) at the far End of which, suppose a high Wall (*b c d e*) at right Angles thereto; and let the Length of the Walk

BC

B C be 40 Yards. Now if it be desired to make this Vista or Walk to appear longer than it really is, it may be thus effected. Suppose it were required to lengthen it by one Half, or to make it appear to be 60 Yards long. Then since on the ground Line C E, the Distance C 40 determines the Length of the Vista C*b*; if from the Point 40 you set off 20 Yards more, it will give the Point 60, from whence drawing a Line to the Point of Distance Z, it will give the Point (f) in the perspective Side C *b* of the Vista continued out; by which Means the perspective C f g D may be compleated for a Walk C F on the ground Plane 60 Yards in Length.

1652. Therefore since the additional Part of the perspective *Vista (abfg)* may be painted on the Wall contiguous to the End *(ab)* and as it will be a perfect Continuation of the said Vista, it will exhibit to the Eye of a Spectator, at a proper Distance, the View of a Walk A F G D just 60 Yards long; nor will he be sensible of the Deception, or that the Part *(abfg)* is drawn or painted on the lower Part of the Wall. The Truth and Pleasure of such perspective Illusions any one may be convinced of by the notable Instances of such Pieces of Art in *Vauxhall Gardens*, and in many other Places.

1653. After a like Manner you may rectify the Appearance of an irregular Room. Thus for Instance, suppose the Floor of the Room was in Form of the Trapezium AEFKB (Fig. 8.) deficient from a Quadrangle by the Part E I F; then in order to make the Room appear of the usual Form to a Person viewing it at some Distance, it will be necessary to compleat the perspective *Trapezium*, or Floor A E F K B into the Rectangle A I K B, which is done by drawing the Line B Z, intersecting the Side A E continued out in I, and joining I F. Therefore on the Side of the Room E F G H, you draw the Perspective of the several Parts deficient, *viz.* (1.) The perspective Triangle EIF for the Floor. (2.) The corresponding Triangle G H M for the Cieling. (3.) The Part E H M I to compleat the Perspective of the Side. (4.) I M G F to compleat the farther End or Side of the Room. And (5.) on these several perspective Supplements, are to be drawn Windows, Doors, Glasses, Pictures, &c. to make the Perspective represent the Room in the Man-

Manner you propose to compleat the View of it; and then it will have the agreeable Effect desired.

1654. We have already taken Notice of the great Use, yea, absolute Necessity of perspective Disguise in all theatrical Purposes; the very Essence of a Theatre consisting in that Sort of Illusion. Here all the Scenery is a Contrivance to represent Things which are not as tho' they really were; and, in short, all the Incidents to a Drama, are one united System of Deception; and by the Price that is paid for it, one would think there was no Pleasure so great as that of *being deceived*; and which, therefore ought to be considered as a high Recommendation of Perspective, which above all other Arts does most agreeably and innocently impose upon our Senses.

CHAP. XIV.

The RULES *of* PERSPECTIVE *applied to* ARCHITECTURE, *in raising the* Perspective ELEVATION *of* BUILDINGS.

[PLATE V. *Of* PERSPECTIVE.]

1655. THERE is certainly no Art in which the Science of PERSPECTIVE is so immediately concerned as Architecture, since whatever Edifice, or Fabric, is proposed, the *Ground Plan* thereof must first be made; then the *Ichnography* or Perspective of the Plan must be drawn; and, lastly, the *Scenography* or perspective Elevation of all the Parts must be compleated, before a proper Idea can be given of the Design; and therefore the Architect, above all Men, is under a Necessity of understanding the *Rules of Perspective*, and the Reason of them likewise, if possible.

1656. We shall illustrate this Matter in regard to the perspective Elevation of an Edifice by the following Example. Let A B C D be a geometrical Square, on the Ground Plane, in which another Square E F G H is inscribed; and in that a CROSS, K L M N, which is the Form of the Plan on which an

an Edifice is to be erected. Suppose it to be viewed obliquely, with an Angle in Front; then, by the Rules delivered (Chap. III.) you find the *perspective Plan (k l m n)* for the given Point of Sight I, and Distance I Z.

1657. Let us now suppose this perspective Plan, or Cross, drawn more at large, and for a less Height of the Eye in Proportion to the Distance, as *(k l m n)* in Fig. 2. Then it is evident, that if from the several angular Points you raise Perpendiculars, and give to each of them their proper Heights (according to their respective Distances from the Ground Line A D) by the *Line of* MEASURES, (as directed in Chap. VIII.) You will then, by joining their Apices in a proper Manner by right Lines, form the Scenography of the Building in its linear Extremities; after which its front Sides, gable Ends, *&c.* are to be filled up, and pannelled with Windows, Doors, and architectural Ornaments, with a just Disposition of Light and Shadow upon the Whole; in all which there can be no Sort of Difficulty, if the foregoing Precepts be understood, as appears by this Example, wherein the several Parts of the Edifice receive their Form and Proportion immediately by the *Radial* and *Diagonal Lines*, drawn to the Points of Sights and Distance, I, Z; it would therefore be superfluous to say any Thing more on this Head.

1658. And because the above Rules of Perspective are in the highest Degree necessary in truly designing and representing the Ruins of Building and Monuments of Antiquity, we thought it would be proper, likewise, to give a Specimen of such a perspective View in all the Variety of Architecture.

1659. In Figure 3. there is a *Side View* of a Colonade of Pillars, all diminishing in just Perspective to the Point of Sight at I. There are also three Pillars in a *Front View*, but entirely out of Perspective, being all of an equal Size, (see 1587) this is one Reason why I made Choice of this Piece (for all the Figures of this Plate except the first, are borrowed,) being willing the Reader should see in what an imperfect State the Practice of Perspective is in at this Day, and how ridiculously the Laws of this Science are transgressed, even by the Professors themselves.

1660. Having

1660. Having ſaid all I think can be neceſſary for perſpective Draughts and Deſigns, either of Superficies or Solids; I have added, for further Illuſtration, two other Figures (*viz.* Fig. 4, and 5.) to ſhew how vaulted Arches, Roofs, the interior Parts of Churches, winding Stairs, diſtant Views thro' Buildings, and many other Particulars, relative to Architecture, appear in perſpective Deſigns. In theſe the Reader will find not a ſingle Article but what is ſtrictly conformable to, and executed by the Methods and Rules of Perſpective above laid down.

CHAP. XV.

Of Sciagraphic PERSPECTIVE, or the ART of SHADOWS.

1661. AS SHADOW is nothing but the *Abſence* of LIGHT, it is a mere *negative*, and therefore can not, properly be the Subject of Art. However, there is what we uſually call the *Art of Shadows*, and as it is a neceſſary Article in Painting and Deſign, and is, for the moſt Part, conducted by the Rules of Perſpective, we ſhall here treat of it in a few Words.

1662. Since whatever intercepts the Rays of Light muſt produce *Shadow*, it is evident, that all Bodies, or Objects which are opake, muſt be attended with Shadow, if placed in the Light which can fall obliquely on them.

1663. Again, it is neceſſary that the Poſition of the Shadow be contrary to that of the luminous Object, whether a *Candle*, the *Sun*, a *Window*, *&c.* becauſe the Rays of all Light are rectilineal.

1664. The Figure of the Shadow depends partly on the Figure of the Object, and partly on the Form of the Rays. For it is evident, that in any Light the Shadow of a Triangle will be different from that of a Circle, or a Square; and the Shadow of a Cone will not be the ſame as that of a Globe or Cube. And, on the other Hand, it is as certain, that the ſame

Body will have its Shadow varied according to the Form of the Rays. For a Cone will caſt a Shadow of a greater Length by intercepting diverging Rays, as thoſe of a CANDLE, than it will do from parallel Rays, as thoſe from the SUN or a *Window*.

1665. But it will be ſufficient here to obſerve, that the *Laws of Shadow*, proceeding from a Privation of diverging Rays, are ſtrictly conformable to, as they reſult immediately from, the *Rules of common Perſpective*; by ſuppoſing a *Candle* inſtead of the *Eye*, and conſidering the Rays of this Light in lieu of the viſual Rays coming from the Object to the Eye.

1666. To demonſtrate this we need only have regard to the foregoing perſpective Figures. Thus, in Plate II. Fig. 1. if a Candle be ſuppoſed to be placed at I, and (*antv*) an opake Object at any Height, P *t*, above the Horizon, or Ground Plane, then, ſince all the Rays which fall thereon are intercepted, it is evident, if the Rays which paſs by the extreme Parts thereof be continued to the Ground-plane, as I *a* A, I *n* N, I *t* T, I *v* V, they will include a Space A N T V, which will be wholly *deprived of Light*; and that *dark Space* is therefore called the *Shadow* of that Object, and gives, of Courſe, its true *Limits*, *Figure*, and *Dimenſions*.

1667. Again, (in Fig. 2.) let a *Candle*, or radiant Point be ſuppoſed at I, and let A *b c* D be any opake Object on the Ground or horizontal Plane, then the Rays I *c* C, I *b* B will project its Shadow into the Space A B C D, as is evident by Inſpection.

1668. The Figure and Dimenſions of the Shadow is determined in the ſame Manner, in any Poſition of the Plane whatſoever. Thus ſuppoſe (*abcd*) (Plate II. Fig. 1.) were an opake Surface; then from a Radiant at I, the Rays, I *a* A, I *b* B, I *c* C, I *d* D, will project its Shadow into the Space A B C D on any Plane ſtanding on the Ground Plane; and if the Plane which receives the Shadow be parallel to that in which the Object is, then the Shadow will be ſimilar to the Object. In ſhort, every Thing, with regard to the Object and its Shadow, will be *inverſely* the very ſame as has been demonſtrated of the various Relations between the Object and its perſpective Repreſentation, and therefore, if theſe be underſtood, the *Doctrine of Shadows* can admit

admit of no Difficulty, unless in that Case, where the Shadow is not projected wholly on one Plane, but partly on several.

1669. For Example suppose *(brtva)* were an irregular, opake Object, and so situated, with regard to the Radiant at I, that its Shadow falls partly on the Ground Plane, and partly on some other Plane elevated above it. Then even in this Case, you have nothing more to do, than, by the Rules of Inverse Perspective, to determine the Prototype of the given Perspective *(brtva)* and how much of it is on one Plane, and how much on the other. *See* Chap. VI.

1670. Or thus; suppose Rays drawn thro' all the angular Points a, b, r, t, v, to the several Planes, they will there determine the Extent and Figure of the Shadow; thus ItT and IvV will determine its Limit T V on the Ground Plane; and the Rays IrR, IbB will mark out the same on the other Plane, as R B; and therefore, all the Space between R B on the elevated Plane, and T V on the Ground Plane, will be occupied by the Shadow.

1671. Or, lastly, by Calculation, thus; let the Distance of the Object Y P $= a$, the Height of the Radiant or Candle Y I $= H$; the Height of the Point t, or P$t = h$, and P T $= x$, the Distance sought of the Point T, where the Shadow commences. Then, by similar Triangles, I Y T and t P T, we have I Y : tP :: T Y : T P; that is, $H : h :: a + x : x$; whence this Theorem $\frac{ah}{H - h} = x$. In Words thus, *multiply the Distance of the Perpendicular* Y P *by the Height* tP *of the shadowing Point*; *and divide that Product by the Difference between the perpendicular Height of the Radiant and given Point* t, *and the Quotient will be the Distance* P T *of the Shaddow* T, *from the Foot of the Perpendicular* P, *as required.*

1672. In the same Manner the Point N is determined for the Shadow of the Point *(u)* in the Object; and from thence the Length of the Shadow T N for the Part of the Object *(tn)*. Also, if the Shadow falls upon a Wall, you find its Height on the said Wall by the same Rule, knowing the Distance of the Wall from the Radiant, and the Point to which the Ray tends beyond it, found as above.

1673. It

1673. It is evident, that the Shadows of all perpendicular Lines converge to a Point perpendicularly under the Radiant; and consequently, if we suppose the Radiant removed to a very great Distance those Shadows will become parallel, which, therefore, is the Case of all such Shadows produced by intercepting the *Rays of the Sun*.

1674. Hence the Shadow of a *Parallelogram* will diverge from the Point under the Radiant, if a *Candle*; but if the *Sun*, the Sides of the Shadow will be parallel. The same may be observed with Regard to the Shadow of a *Cylinder*.

1675. It is also evident, that the Shadow of a round Table on the Floor will be *circular*, both from the Candle standing upon it on any Part, and also from the Sun-beams (1504.) only in the first Case, the Shadow will exceed the Dimensions of the Table; in the latter, it will be just equal to it.

1676. The Shadow of a *Globe* lying on a Table must necessarily be *elliptical*; for the Cone of Rays are intercepted by a Plane perpendicular to its Axis, and passing thro' the Globe in such Position as to make an Angle with the Horizon, or Table, just equal to the Co-Altitude of the Radiant whether *Candle* or *Sun*. In this Case the Cone is never *Scalenous*, and can admit of no Subcontrary Sections from the two Planes, therefore the *Elliptic Shadow* can never become *Circular*. (See Chap. IV.)

1677. As we have shewn a general Method how the Shadow may be determined for any given Line on given Planes, and as the Shadow of any Superficies is determined by that of each bounding Line, and the Shadow of a Solid results from those of its connected Superficies, it is evident that the same general Perspective Rule for Shadows (1671.) of Lines extends to the Determination of the Shadow of any Body whatsoever; the particular Application of which is left to, and is no small Part of the Praxis of the *young* DESIGNER.

1678. The Art of Shadows, in one particular Branch, constitutes a Science of itself alone, viz. GNOMONICS, or the ART of DIALLING; to which therefore we proceed to apply it; but it will be first necessary to teach the *Projection of the Sphere* from the Principles of Perspective above laid down; and then the *Rationale* of making a DIAL will appear in a new Light.

CHAP.

CHAP. XVI.

The RULES *of* PERSPECTIVE *applied to the* Geographical PROJECTION *of the* SPHERE, *for the Construction of* MAPS, CHARTS, *&c.*

1679. THE Method of making *Geographical* and *Astronomical Charts* by a PROJECTION of the Sphere *in Plano* is as ancient as any Mathematical Science; but as it is in that Way limited and intricate, we shall here shew how general and easy it is by the Principles and Rules of *Common* PERSPECTIVE, which is not only new, but the natural Source of such a Doctrine.

1680. For if (Fig. 1. Plate II.) the Plane of Projection HF, the *Vertical* Plane QN, and the *Horizontal* Plane L O, were all of a circular Form, and of an equal Radius, they would represent the Planes of *three great Circles of the Sphere* whose common Intersections would all pass thro' the Center (*i*).

1681. Let the Planes of three such great Circles of the Sphere be EQTR, QKPR, and EPLT (Plate VI. Fig. 1.) of which the first is supposed perpendicular to the Visual Ray proceeding from the Center C to the Eye at Y, and is therefore the *Plane of Projection*. The Second is the *Vertical Plane*, and the Third, the *Horizontal Plane*, as all the three are supposed to intersect each other at right Angles. Consequently QR is the *Vertical Line*, and E T the *Horizontal Line* of the *Projection*.

1682. Let O be the Place of any Object on the Surface of the *Globe* or *Sphere*, then its Distance from the Plane of Projection is the Sine O B of the Arch O G, which measures its *Latitude* or Declination from that Plane. Its Distance from the Vertical Plane is the Sine O D of the Arch O K of a lesser Circle K O L, whose Plane is parallel to the Plane of Projection, and its Distance from the horizontal Plane is the Co-Sine ON of the same Arch O K, or the Sine of the Arch O I. All which Sines or Distances bear a known Proportion to the Radius of the Globe or Sphere, and therefore are to be estimated in that measure, as found in the *Trigonometrical* Canon. (708 to 716.)

1683. The Distance of the Eye C Y is also estimated in the same Measure, viz. in Parts of the Radius: That is, if Radius = 1,

$= r$, then $CY = \frac{1}{2}r$, r, $2r$, $3r$, &c. or nr, which may represent any Distance of the Eye indefinitely, and thereby render the Theory of Projection, this way considered, universal. Therefore let us suppose,

1. *The* EQUATOR, *the* PLANE *of* PROJECTION.

1684. Since the Circle F Q T R, in this Case, is the Equator, the Vertical Circle Q P R, and the Horizontal One E P T will be Meridians; and P the Pole in the Hemisphere remote from the Eye; and if O be the Place of an Object in that Hemisphere, then we have $CG : GH :: CB$ ($= MO$) $: BI$ ($= OD$.) that is, as *Radius is to the Sine of the Difference of Right Ascension or Longitude so is the Co-Sine of Declination or Latitude to the Distance of the Point O from the Verticle Plane.*

1685. Again, we have $CG : GF :: CB$ ($= MO$) $: BS$ ($= ON$.) That is, *as Radius is to the Co-Sine of the Difference of Longitude or right Ascension, so is the Co-Sine of Latitude or Declination, to the Distance of the Point O from the Horizontal Plane.*

1686. Or thus in Symbols; let $HG = p$, $GF = q$, $OB = x$, $OM = y$; Then by (1684.) $DO = \frac{py}{r}$ = Distance of the Point O from the vertical Plane; and by (1685.) $ON = \frac{qy}{r}$ = Distance from the horizontal Plane; and x = Distance from the Plane of Projection. Having these Distances from the three Planes, we determine the Distances from the vertical and horizontal Lines, in the Plane of Projection, by the *common Rules of Perspective.*

1687. Thus, suppose the Equator between the given Point O and the Eye Y, we have $nr + x : nr :: \frac{py}{r} : \frac{npy}{nr + x} = a$ = Distance of the Point O from the vertical Line of the Projection; (1470.) whence we have $npy = anr + ax$; and therefore $nr + x : ny :: p : a$; the Analogy for finding a.

1688. Again, we have $nr + x : nr :: \frac{qy}{r} : \frac{nqy}{nr + x} = b$; (1470.) which gives $nr + x : ny :: q : b$, the Analogy for finding b, the Distance from the *Horizontal Line.*

1689. If

1689. If the Eye be placed in the Pole opposite to P, then $n = 1$; and the two Analogies become (1.) $r + x : y :: p : a$, the Distance from the *vertical Line*; and (2.) $r + x : y :: q : b$, the Distance from the *horizontal Line*; in this Case, the Projections is that of *Ptolomy*; and is represented in Fig. 2.

1690. If the Eye be at an infinite Distance, then n is infinite, and the two Analogies become (1.) $r : y :: p : a =$ Distance from the Vertical; and (2.) $r : y :: q : b =$ Distance from the horizontal Line. In both these Cases the *Meridians* are projected into *right Lines* (1464) and the *Parallels* into *concentric Circles* (1504).

1691. If the Eye were placed in the other Part of the Axis at z, then the Point O is between the Eye and Plane of Projection; and in such a Case the Analogies will be (1.) $nr - x : ny :: p : a$; and (2.) $nr - x : ny :: q : b$; for the Distances from the vertical and horizontal Lines, as is evident from (1470). The first of these Cases is called the STEREOGRAPHIC, and the other the ORTHOGRAPHIC PROJECTION *of the* SPHERE.

II. *Of the* PROJECTION *of the* SPHERE *on the Plane of the* MERIDIAN.

1692. If the Meridian be chose for the Plane of the Projection, as EPTR, (Fig. 3.) then in the Case of common Maps of the World, the Equator EQT is the horizontal Plane; and the Six o'Clock Hour-circle PQR is the vertical Plane. P and R are the two Poles; and the Eye Y is placed in the common Intersection QC of the vertical and horizontal Planes continued out.

1693. Let O be the given Point, as before, through which draw the Meridian POG; then will the Sines and Co-Sines GH, GF, OB, OM, be the same as before, and denoted by the same Letters. Also OD $= \frac{py}{r}$ the Distance from the Vertical Plane, ON $= \frac{qy}{r} =$ Distance from the Plane of Projection as in the last, and OB $= x$ the Distance from the horizontal Plane. Then by the Analogies of Perspective (1687, 1688.) we have $nr + \frac{qy}{r} : nr :: x : \frac{nrrx}{nrr + qy} =$ Distance from the hori-

horizontal Line. And as $nr + \frac{qy}{r} : nr :: \frac{py}{r} : \frac{nrpy}{nrr + qy} =$ Diſtance from the vertical Line.

1694. If the Diſtance of the Eye $nr = r$, as in the common *Stereographic Projection*, then $n = 1$; and the two Expreſſions become $\frac{rrx}{rr + qy}$, and $\frac{rpy}{rr + qy}$; which are to each other, as rx to py. And when $rx = py$; then $r : p :: y : x :: r : t$ (712); that is when of the Tangent Latitude O G is equal to the Sine of Difference of Longitude, then *the Point* O *in the Projection will be equally diſtant from the vertical and horizontal Lines.*

1695. When the Point O is in the Equator, $x = o$ and $y = r$; and the Diſtance from the vertical Plane is then $\frac{rp}{r + q}$. But the Ratio of $r + q$ to p, is the ſame as that of p to $r - q$; as will appear if we place the Arch QG in another View; let QTR (Fig. 4.) be the Plane of Projection, or *primitive Circle*, as it is uſually called. Then by the Eye at R the Point G is projected into the Point O in the horizontal Line CT. Put CR $= r$, GH $= p$, GF $=$ HC $= q$, and QH $= r - q$; and draw the Line RG. Then by the Property of the Circle (658) we have RH : HG :: HG : HQ; that is, $r + q : p :: p : r - q$, and therefore $p = \sqrt{rr - q^2}$.

1696. Now becauſe QH $= \frac{p^2}{r + q}$ and CO $= \frac{rp}{r + q}$, therefore QH : CO :: $p^2 : rp :: p : r ::$ GH : QC; and conſequently QO is the Arch of a Circle. *Therefore all great Circles of the Sphere are projected into Arches of Circles of different Dimenſions.*

1697. Let the Radius of the Circle DOE be $=$ R, the Center B, and Diameter AO. Alſo put CQ $=$ P, and BC $=$ Q, then becauſe CO $= \frac{P^2}{R + Q} = \frac{r^2}{R + Q} = \frac{rp}{r + q}$ (1696) we have $\frac{r}{R + Q} = \frac{p}{r + q}$, and ſo $R + Q : r :: r + q : p ::$ QC : CO; therefore AC : CQ :: RH : GH; and conſequently the Angle QAC $=$ QRG (657) and the Angle QBC $=$ GCH

GCH (642) therefore making the Angle LCT = QCG, the Line LC will be parallel to BQ, and give the Center B.

1698. Or thus; CO is the Tangent of half the Angle QCG (705); whence it appears that the Tangent of half the Angle or Arch GQ set from the Center C, gives the Point O; then through the three Points QOR the Meridian is drawn with Ease (695).

1699. The Projection, as hitherto considered, is the vulgar Stereographic One, which on every Account, is the worst that can be for Maps and Charts; because of the great Disproportion and Distortion it occasions in all the Parts of the Earth's Surface represented upon it, as appears from the very unequal Parts of the horizontal Line or Equator intercepted between Meridians equally distant on the Globe; therefore this Projection must give a very false Idea of the geographical Relations of Places, and is consequently the most unfit for such Purposes. (See Fig. 5.)

1700. To remedy this Imperfection in a considerable Degree, we may suppose the projected Quadrant of the Equator CT, or CE divided into equal Parts, of any Number $= s$, and let CO be any Number of them denoted by t; that is, let CO be the $\frac{t}{s}$ Part of r, then will $\frac{tr}{s} = \frac{npr}{nr + q}$, or $\frac{t}{s} = \frac{np}{nr + q}$; whence $snp = ntr + tq$, and we get $n = \frac{tq}{sp - tr}$ = Distance of the Eye to project that Meridian; for Example, let the Meridian of 10 Degrees from the vertical Plane be projected by the Eye; then $s = 90$, and $t = 10$; $n = \frac{10q}{90p - 10r} = 175$ Parts of which $r = 100$.

1701. In Practice, by dividing the Diameter or horizontal Line into equal Parts; you have the Points given thro' which to draw the Meridians (695). And this is usually called the GLOBULAR PROJECTION; forasmuch as the Meridians and Parallels are circular, and at equal Distances in the Equator and Meridian or vertical Line; and consequently such a Projection is greatly preferable to the former for general Maps of the World; a

a Specimen of which see in that Map we have prefixed to our PHILOSOPHICAL GEOGRAPHY.*

1702. There is another very useful geographical Projection, *viz.* on the *Plane of the Horizon*, of which we have given a Specimen in Fig. 6. and though any particular Point may be determined on the said Plane by the *Rules of Perspective*, yet the Whole of it is with more Ease and Readiness laid down from Calculations founded on the Principles of spherical Geometry hereafter to be explained.

CHAP. XVII.

The RULES *of* PERSPECTIVE *applied to* Astronomical PROJECTIONS *for the Construction of* Celestial PLANISPHERES, *the* ANALEMMA, *&c.*

1703. SINCE the Surface of the *Celestial Globe* may be projected *in Plano*, in the same Manner as that of the *Terrestrial* One, it follows, that the same Theorems which serve for the Construction of MAPS, may also be adapted to CELESTIAL CHARTS or PLANISPHERES.

1704. For this Purpose, instead of the *Latitude* and *Longitude* of Places on the Earth, we must, in regard to the Sun and Stars, use the Words *Declination*, and *Right Ascension*; and then the same Symbols will serve; that is,

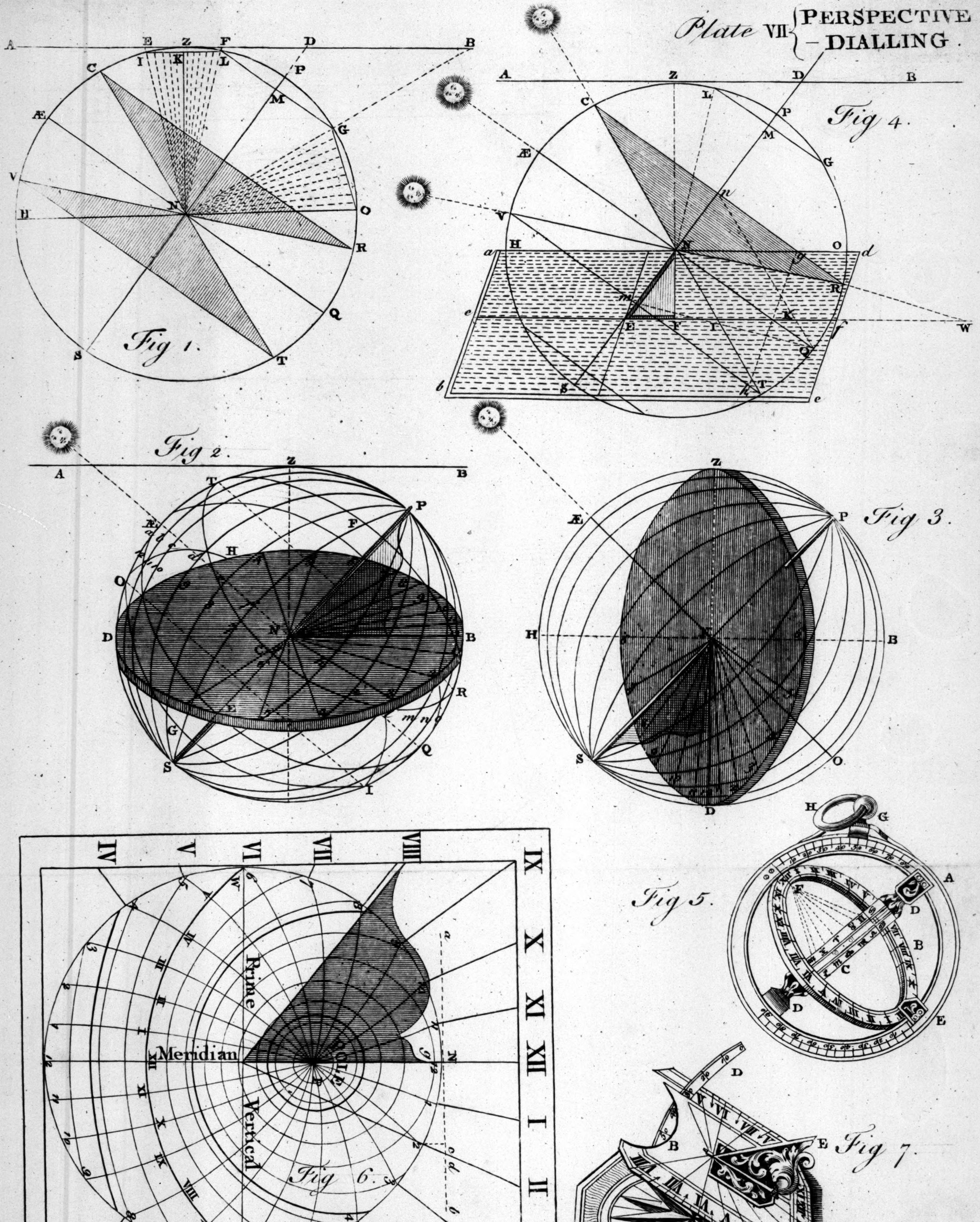
$p =$ Sine of Right Ascension.
$q =$ Co-Sine of that Right Ascension.
$x =$ Sine of Declination.
$y =$ Co-Sine of that Declination.

1705. Also the same Diagrams (in Fig. 1, and 3.) may here be used; if O be the Place of a Star, then its Distance from the vertical Plane is $\frac{py}{r} = DO$; from the horizontal Plane $\frac{qy}{r} = NO$; and from the Plane of Projection $x = OB$; the same as before (1686).

1706.

* Also in a New Map of the World entitled, *The* PRINCIPLES *of* GEOGRAPHY *and* ASTRONOMY *explained*, &c.

Plate VII { PERSPECTIVE - DIALLING



1706. If the Equator be the Plane of the Projection; then the Pole of the World is the Center, and the Circles of Declination are projected into right Lines, also the Parallels of Declination are concentric Circles; all as before (1685.) But in this *Planisphere* the Pole of the Ecliptic is also projected, and all the Circles of Latitude, together with the Ecliptic itself, are projected into circular Arches, by the usual Methods.

1707. In this Projection the *solstitial Colure* is the vertical Plane, and the *equinoctial Colure* the horizontal Plane; the Distance of any Star from the former will be $\frac{npy}{nr + x}$; and from the latter, $\frac{nqy}{nr + x}$. But if the Eye be placed in the Pole on the Globe's Surface, the Distances of the Star becomes $\frac{py}{r + x}$ and $\frac{qy}{r + x}$; and thus all the Stars may be readily laid down from a Table of their Declinations and right Ascensions. Such *celestial Planispheres* we have by the late Mr. *Senex*, of 24 Inches Diameter.

1708. If the ECLIPTIC be the Plane of Projection, then is its Pole the Center, and all the Circles of Latitude are right Lines; also $OB = x$, and $OM = y$, are the Sine and Co-Sine of the Star's *Latitude*; $GH = p$, and $GF = q$, are the Sine and Co-Sine of its *Longitude* (Fig. 1.) and the vertical and horizontal Planes are those of two Circles of Latitude passing through the solstitial and equinoctial Points, as before. And therefore by the same *Formulæ* $\frac{py}{r + x}$ and $\frac{qy}{r + x}$ the Stars may be laid down in this Projection from *Tables of their Latitude and Longitude*. Such Planispheres we have of all the Stars in the *British* Catalogue by the same Hand.

1709. There is also a very useful Projection of the Heavens on the Plane of the Horizon. In this Case the Meridian is the vertical Plane, and the prime Vertical the horizontal Plane. Here the Star's Distance from the vertical Line is $\frac{rpy}{rr + qy}$ as before (1695). Then if a, b, t be the Sine, Co-Sine, and Co-Tan-

Tangent of the Star's Azimuth, we have $a : b :: r : t :: \frac{rpy}{rr + qy}$ $: \frac{tpy}{rr + qy} =$ Star's Diſtance from the horizontal Line of the Projection. But when the right Aſcenſion and Declination of a Star is known, together with the Latitude of the Place, or Pole's Height, then the Azimuth of a Star is known alſo; whence its Place on this Projection is eaſily aſſign'd.

1710. It is not common to project the ſtarry Heavens on the Plane of the Meridian, or Circle of Latitude paſſing through the Poles of the World; but if it be required, it may be done by the *Formulæ* in (1694.) by ſubſtituting as directed (1704). However, this Projection is not confined to *Geography*, but is the Ground-work of that celebrated aſtronomical Inſtrument called the ANALEMMA, or ORTHOGRAPHIC PROJECTION of the SPHERE.

1711. For if the Eye be ſuppoſed at an infinite Diſtance, then the Stereographic Forms (1694) $\frac{nrrx}{nrr + qy}$, and $\frac{nrpy}{nrr + qy}$, become x and $\frac{py}{r}$, for the Diſtances of any Point from the horizontal and vertical Lines of the Projection. And in this Caſe, the *Meridians*, *Parallels*, and other Circles, Oblique to the Plane of Projection, are projected into ELLIPSES. A Specimen of this Projection you have in Fig. 7.

1712. That every Oblique Circle P G S is projected into an Ellipſis P a F S by parallel Rays is thus demonſtrated (Fig. 8.) Let $CP = CG = a$, $CF = b =$ Semi-Conjugate of the Curve P F S. Alſo put $CM = x$, and $OM = y$, and $NM = \text{y}$. Then by the Property of the Circle (658) we have $SM \times PM = OM^2$, that is, $\overline{a + x} \times \overline{a - x} = a^2 - x^2 = y^2$. Hence $\sqrt{a^2 - x^2} = y = MO$. Draw O B parallel to C M, and B b parallel to G F; then $OM : NM :: (BC : bC ::) GC : FC$; that is, $y : \text{y} :: a : b$; therefore $\text{y} = \frac{b}{a} y = \frac{b}{a} \sqrt{a^2 - x^2}$, which is the Property of an Ellipſis (765) but $\text{y} = NM = \frac{py}{r} = \frac{by}{a}$; for $r = a$, $p = b$, (766) therefore the Curve P F S is an *Ellipſis*.

1713. When the Point O coincides with G, or is in the Equator, then $y = r$, and the Distance C F or Semi-Conjugate of the Ellipsis, $\frac{py}{r} = p =$ Sine of right Ascension from the Point Q.

1714. Hence, as $r : p :: p : \frac{p^2}{r} =$ Half the *Latus Rectum* of the Ellipsis (766); but in this Case $\frac{py}{r} = \frac{pa}{r}$; therefore $y = p$ $=$ O M, when M is the Focus of the Ellipsis; whose Distance therefore C M from the Center, is equal to the Co-Sine right Ascension, which is then equal to the Sine ($x =$ O B) of Declination.

1715. Hence if C F, the Semi-Conjugate of the Ellipsis, be made Radius; then, since in this Case $r = p$, we have the Distance from the vertical Line every where $\frac{py}{r} = y$; that is, *all the Semi-Ordinates* N M *of the Ellipsis will be equal to the Co-Sines of Declination, to the Radius* C F.

1716. And whatever has been demonstrated of the MERIDIAN P O G, and its orthographical Projection, may be applied in the same Manner to any PARALLEL A O L when posited obliquely to the Plane of Projection. For in this Case, P C S may be considered as the horizontal Line, and A M L as the vertical One; and then, if M L $= r =$ Radius, O D and O N will be the Sine and Co-Sine of right Ascension from K; and consequently the Distances from the horizontal and vertical Planes. Also O D $= p$ will be the Distance from the horizontal Line, and its Distance from the vertical Line will be $\frac{xq}{r}$ (1711).

1717. Hence when Q coincides with R, then $q = r$; and x, the Sine of Declination (or Obliquity) from the vertical Plane is then the Semi-Conjugate of the Ellipsis of the projected Parallel. And therefore, because $r : x :: x : \frac{x^2}{r} =$ *Latus Rectum* (766) $= \frac{xq}{r}$, it is $q = x$, at the Focus of the Ellipsis; where the

the Diſtance of the Focus, from the Center M, is equal to the Sine of right Aſcenſion, or Co-Sine of Declination.

1718. From all which it appears, that ſince the Diameters, and *Foci*, of the Ellipſis of any projected oblique Circle are given by this Method, therefore ſuch an Ellipſis may be drawn, by Points, or mechanically (785) or from a Table of Sines, or laſtly, by the *Sector*, which is the moſt ready Method of all.

1719. The ANALEMMA (Fig. 7.) is that particular Species of the *orthographic Projection*, where the Eye is placed in the common Interſection of the Ecliptic and Equinoctial, at an infinite Diſtance. Theſe Planes will therefore be projected into right Lines; and the Equinoctial will be the horizontal Line; and the equinoctial Colure the vertical Line; as the other Colure is the Plane of Projection.

1720. Here alſo the *Parallels* of *Declination* are all right Lines, as they ſtand at right Angles on the Plane of Projection. The Meridians here drawn, are thoſe of 15 Degrees interval; and conſequently are the *Hour Circles* of the Sphere. As to the Uſes of the *Analemma* in *Aſtronomy*, ſuch as are conſiderable will appear in the Sequel of this Work, it ſuffices, at preſent, that we have ſhewn its Nature and Conſtruction from the Principles of Perſpective. In the mean Time we refer the Curious to *Sutton*'s *Analemma* of 18 Inches Diameter, in which the Meridians and Parallels are drawn thro' every Degree, and many of the principal Stars laid down; but this Inſtrument is yet capable of great Improvements, as we may hereafter ſhew.

CHAP. XVIII.

The PRINCIPLES *of* PERSPECTIVE *applied to the Computation of* SOLAR *and* LUNAR ECLIPSES.

1721. THE PRINCIPLES of PERSPECTIVE are applicable to Projections for *Solar and Lunar* ECLIPSES; and the ſame Rules ſerve here as in all the foregoing Caſes. For let S be the Center of the Sun A C B (Fig. 9.) E the Center of the

the Earth QW; and LEP a Part of its Orbit. Alſo let XIMF be a Circle deſcribed about the Earth's Center in the Plane of its Motion, or of the Ecliptic, at the Diſtance of the Moon. And let HK be the Orbit of the Moon, making the given Angle HKG, with the Ecliptic, at the Time of a *ſolar Eclipſe*; and K the *deſcending Node.*

1722. Now in Projections of this Kind, the moſt commodious Poſition of the Eye is in the Center of the Sun at S, in which Caſe the *horizontal Plane* will be that of the Ecliptic; the *vertical Plane* will be that Circle of Latitude which cuts the Ecliptic in the Points of Conjunction X, and Oppoſition M; and the *Plane* of *Projection* is that which paſſes thro' the Earth's Centre E, and touches its Orbit, when it is in the Line of Ecliptic Conjunction or Oppoſition SM.

1723. It is evident from the Diagram that the Angle AQS is that under which the Sun's Semidiameter appears at the Earth. The Angle ESQ is that under which the Semidiameter of the Earth appears at the Sun, and is call'd, the Sun's *horizontal Parallax*. The Angle EXQ or EMQ is that under which the Earth's Semidiameter appears at the Diſtance of the Moon, and is call'd the Moon's *horizontal Parallax*. And the Angle XQS is the *Difference of the horizontal Parallaxes* of the *Sun and Moon*. For $XSQ + XQS = QXE$, (632) therefore $XQS = QXE - XSQ$.

1724. By the Word *Parallax*, is only meant the different Place in which the Sun, or Moon appears, when viewed from the Center of the Earth at E, and from any Point of it's Surface as Q. And, therefore, at the Diſtance of the Phænomenon, it muſt be equal to the Angle under which the Semidiameter of the Earth appears. It muſt, therefore, be greateſt of all when in the Horizon of the Place Q; and vaniſh for Objects at an infinite Diſtance; ſo that the *Stars have no Parallax at all*; and that of the Sun ESQ is almoſt immeaſurably ſmall.

1725. Theſe Things premiſed, we may proceed: Let K be the Place of the Node, and O that of the Moon in her Orbit HK, then it is plain, her Diſtance from the *horizontal Plane* is the Sine x of her Latitude OG; her Diſtance from the *vertical Plane* is $OD = \frac{py}{r}$, and her Diſtance from the *Plane of Projec-*

tion is $ON = \frac{qy}{r}$; for here we retain ſtill the ſame Symbols as before (1686, 1704.) only O G is here the Moon's *Latitude* from the Ecliptic, and G K the *right Aſcenſion*.

1726. Then becauſe the Place of the Moon O is between the Eye at S, and the Plane of Projection at E; the perſpective Analogies (1693.) will be for an ECLIPSE of the SUN.

1727. As $nr - \frac{qy}{r} : nr :: \frac{py}{r} : \frac{nrpy}{nrr - qy} =$ to the Diſtance of the Moon from the *vertical Line*. And as $nr - \frac{qy}{r} : nr :: x : \frac{nrrx}{nrr - qy} =$ her Diſtance from the *horizontal Line*. But in Eclipſes, the Latitude and Diſtance of the Moon, from the vertical Plane, are ſo very ſmall, that we have $y = r$, very nearly; therefore in ſuch Caſes the Analogies become $nr - q : nr :: p : \frac{pnr}{nr - q}$; and $nr - q : nr :: x : \frac{nrx}{nr - q}$.

1728. Now becauſe $nr =$ E S the Diſtance of the Sun, and $q = r =$ E X nearly, the Diſtance of the Moon; and ſo $nr - q =$ S X; therefore ſince S X : S E (= S Q) :: Angle S Q X : Angle S X Q or E X Q (718) the firſt Analogy is, in Words; *as the Difference between the horizontal Parallaxes of the Sun and Moon, is to the Parallax of the Moon*; ſo is the Moon's Longitude (at the given Inſtant) *from the vertical Plane, to her Diſtance from the vertical Line.* —— And, *ſo is her Latitude, to her Diſtance from the horizontal Line.* Note, the Longitude and Latitude are here put for their Sines, there being no ſenſible Difference in ſuch very ſmall Arcs.

1729. If therefore from the aſtronomical Tables you take the Moon's horizontal Parallax, and with it, as a Radius, you deſcribe a Circle on E, as Q Y W Z; then will Q W be the horizontal Line, and Y Z, the vertical Line, on the Plane of Projection, which if it be about 10 or 12 Inches in Diameter will be ſufficient for the Delineation of all the Phaſes of a ſolar Eclipſe.

1730. For ſince the Diſtance of the Sun S E is vaſtly great in reſpect of the Earth's Semidiameter E Q, therefore the Hour Circles and Parallels will all be *orthographically* projected, and may be ſo drawn on the ſaid Plane (by 1711 — 1718.) and ſince

by

by the above Analogies, the Path of the Moon over the Disk of the Earth (or Plane of Projection) is determined by finding her Place thereon at two Instants of Time, one just before the Beginning, the other after the End of the Eclipse; it will be easily seen how the Moon, and any particular Place, as LONDON, are situated in respect of each other, and what are their Distances apart at any given Moment, during the Passage of the Shadow.

1731. For if from astronomical Tables you take the Sun's Semidiameter in your Compasses, and with one Foot placed on the given Moment of Time in the Parallel of *London*, you describe a Circle; and then from the said Tables you take the Moon's apparent Semidiameter, and with one Foot of the Compasses, placed in the same Moment in the Moon's Path, you describe another Circle; these two Circles will shew how the Disks of the Sun and Moon are related to each other in the Heaven, in regard to a Spectator at *London*.

1732. Thus if the Circles do not touch, the Eclipse is not begun, or over; if the Circles just touch, the Eclipse is then just beginning or ending. But if the Circle of the Moon lies over that of the Sun, there is an Eclipse. If the Diameter of the solar Disk or Circle be divided into 12 equal Parts, or *Digits*, then the Number of these covered by the lunar Circle, will shew the *Phase* and Quantity of the Eclipse. But on these Particulars we must not enlarge any further at present.

Of a LUNAR ECLIPSE.

1733. When the Moon is so near the Node at the *Opposition* M, that her Latitude is less than the Semidiameter V M of the Earth's Shadow Q T W, it will be more or less immersed into, and eclipsed by it. And then by having the Moon's Latitude, and Longitude (or Distance reduced to the Ecliptic) from the Plane of the Circle of Latitude passing through M, the Point of Opposition, you will by the same Analogies find her Place for any Instants of Time on the Plane of Projection, and thereby be able to delineate her Path. Only here, because the Plane I F is between the Eye and the Object, the *Sum* of the Parallaxes is to be taken; for now it is S M : S Q : : S Q M (or R Q M) :

SMQ; but RQM = QMS + QSM (632) = *Sum of the Parallaxes of the Sun and Moon.*

1734. But to reprefent the lunar Eclipfe, the Diameter of the Earth's Shadow at the Point M muft be known. In order to that, it is to be obferved, that the Angle MQR confifts of two others, *viz.* RQV and VQM; of which RQV = AQM = apparent Semidiameter of the Sun; and the other VQM = apparent Semidiameter of the Shadow at M. But RQM = QMS + QSM; therefore QSM + QMS — RQV = VQM. That is, from the Sum of the horizontal Parallaxes of the Sun and Moon, fubtract the apparent Semidiameter of the Sun, the Remainder is the Semidiameter of the Earth's Shadow at M, and is therefore given from the aftronomical Tables.

1735. With this Semidiameter, as a Radius, defcribe a Circle on the Plane of Projection about the Point E, and fhade it to reprefent the Section of the Earth's Shadow. From this Point E, let fall a Perpendicular to the Path of the Moon, and that will fhew the neareft Diftance of the Centers of the Shadow and Moon. And by defcribing feveral lunar Circles at given Inftants of Time in the Moon's Path, the Phafe and Quantity of the Eclipfe will appear at each particular Inftant (1732.) And as the Shadow of the Earth, at the Moon, is in Diameter near three Times as large as the Moon, therefore a lunar Eclipfe may be total to all the World for a confiderable Time, whereas an Eclipfe of the Sun can be little more than momentarily fo, as the Difk of the Moon but very little exceeds that of the Sun, and that too but really happens in Eclipfes.

1736. We have now explained and applied the perfpective Principles to the Doctrine of Eclipfes, and though there may be readier Methods of conftructing an Eclipfe, there is none fo punctual and exact as this, when applied by the Method of *Interpolation*; but that will be the Subject of a future Part of this Work.

1737. The fame perfpective Analogies are applicable to the TRANSIT of VENUS and MERCURY over the Sun's Difk; and as there is no Difference in the Method of delineating the Path of the Planet, and that of the Moon in a folar Eclipfe, but what

what is merely verbal; and that the Disk of the Sun is to be put for the Disk of the Earth, it is presumed what has been said is sufficient for the Construction of any Transit of those Planets.

CHAP. XIX.

GNOMONICS; *or the* PRINCIPLES *of* PERSPECTIVE *applied to the* ART *of* DIALLING.

1738. IN the Application of PERSPECTIVE to DIALLING, the Position of the *Eye* and *Plane of Projection* are interchanged; or the Eye is in this Case placed in the *Center* of the Sphere, and the *Plane* in *Contact with its Surface*. For in delineating the *Hour-lines* on a given Plane, it must be considered, that the visual Ray which connects the Sun, and the Eye describes the Plane of each *Hour-circle*, and of Course, in its Motion through any Plane on the Surface of the Sphere, it will trace out the *Hour-line* on that Plane.

1739. And it is quite the same Thing whether we consider the Plane of any *Hour-circle* projected on a given Plane at the Surface of the Sphere by Rays from the Sun, or by visual Rays from the Eye in the Center; for since both the Sun and the Eye are in the projected Hour-plane, the Projection itself must be a *Right-line* on the given Plane (1464).

1740. And, in short, since all great Circles of the Sphere pass through the Eye in the Center, therefore they will all be projected into Right-lines on a Plane in any Position whatsoever. But with regard to lesser Circles, they will be of one or other of the *Conic-sections*, as they are posited relative to the Plane.

1741. Let ÆPQS (Fig. 1.) be a great Circle of a Sphere, then will it be projected into the Right-line A B, infinitely continued on a Plane touching the Sphere in the Zenith Z. And any Part or Arch ZP is projected into ZD the *Tangent* thereof.

1742.

1742. Let I L be the Diameter of any leſſer Circle parallel to the Plane A B, then will its Center K be in the Perpendicular N Z, and is projected into the Point Z; its Semidiameters I K and K L into the Tangents Z E and Z F; which as they are equal to each other, ſhews *the Projection of a leſſer Circle parallel to the Plane, is a* CIRCLE *upon the Plane.*

1743. Let L G be a Diameter of a leſſer Circle in ſuch an oblique Poſition to the given Plane A B, that it may be wholly projected upon it within a finite Diſtance Z B; then it is evident the Center M will be projected into the Point D on the Plane, the Semidiameter I M into the Right-line F D, and the Semidiameter M G into the Right-line D B; and ſince it is evident that D B muſt exceed D F, the Projection cannot be a Circle. But conſidering L N G as a right Cone, its *Section* (if continued out) *by the Plane* A B, muſt be an ELLIPSIS (763).

1744. Again, it is as evident, that if the Diameter G O of any leſſer Circle paſſes through O, the extreme Part of the Diameter H O parallel to the given Plane A B; then, becauſe the Cone G N O (infinitely continued) is cut by a Plane parallel to one of its Sides N O, *the Projection of any Circle* G O *on that Plane will be a* PARABOLA (740).

1745. A ſmall Circle poſited in any other Manner than what is above ſpecified, will be projected into an HYPERBOLA, as is manifeſt from the ſame Principles of the Section of a Cone (765).

1746. We have ſhewn that the Interſection of the Plane of any Hour-circle, with a given Plane on the Surface of the Sphere, is the true Hour-line on that Plane (1738) and the Caſe is the ſame if the given Plane paſſes through the Sphere in any Part, provided it be in a *parallel Poſition*; for the Planes of the ſame Hour-circles will interſect two or more parallel Planes in the ſame Manner; and therefore the *Hour-lines* will be the ſame in all; this is evident from (615 and 631.)

1747. Hence it is common to repreſent the given Dial-plane as paſſing thro' the Center of the Sphere, as in Fig. 2. where it is denoted by D W B E. The Sphere with its Hour-circles, is there orthographically projected (1711) Æ Q is the Equator; Æ P Q S is the Meridian, or Hour-circle of XII; and *a*, *b*, *c*, *d*, *e*, *f*, N, are the Elliptic-projections of the Hour-circles of

12,

12, 11, 10, 9, 8, 7, 6, in the Morning, which are all at the equal Diſtance of 15 Degrees on the Sphere; for 24) 360° (= 15° per Hour in the Sphere's Revolution.

1748. The Planes therefore, of theſe Circles, will interſect the given Plane D W B E in the Lines N 12, N 11, N 10, N 9, *&c.* which will therefore be the Hour-lines on that Plane; and thus they are to be conſidered in all the other Quadrants of the Plane.

1749. But when the Hour-lines are aſſigned or delineated on a Plane, it is neceſſary there ſhould be ſome Contrivance to indicate or point out the Moment of Time when the Sun is ſucceſſively in thoſe horary Planes; now if the Plane of the Dial paſs through the Center of the Sphere, then the Expedient we require is found in the AXIS of the Sphere P S. For ſince all the Planes of the Hour-circles interſect each other in the ſaid Axis, the Sun when it comes on any one horary Plane will project that Axis into the Hour-line on the Dial-plane proper to that Hour-circle.

1750. Moreover, as the Axis P S of the Sphere is that Line about which its Motion is performed, it muſt in it's ſelf be conſidered as *abſolutely at Reſt.* For any Line parallel to the Axis, in the Plane of any Hour-circle, has leſs Motion in Proportion as it is nearer to the Axis; and conſequently when it coincides with, or becomes the Axis, it can have no Motion at all.

1751. The Axis therefore of the Sphere retaining always the ſame Poſition, if it be ſuppoſed to conſiſt of a fine inflexible Wire, it will intercept the Sun's Rays in the Planes of the Hour-circles, and therefore its Shadow muſt conſtantly fall on the Hour-lines of the given Dial-plane, correſponding to the reſpective Planes of the ſaid Hour-circles; ſo that *the Axis of the Sphere, by Means of its Shadow, will conſtantly indicate the Moment of Time when the Sun is in any Meridian.*

1752. Now the Plane D W B E is called the horizontal Plane with reſpect to the Place Z, becauſe it is parallel to the horizontal Line A B, touching the Sphere in that Point; and when all the Hour-lines are drawn upon the Plane for all the Meridians or Hour-circles the Sun can be in above the Plane, then

then this Plane with its Hour-lines, and Semiaxis N P compleats the HORIZONTAL DIAL in the Sphere.

1753. The *Semiaxis* N P is called the GNOMON or STILE of the DIAL; and its Elevation or Height B N P above the Plane of the Dial, is always equal to Æ Z the Latitude of the Place. For Æ P = Z B = Quadrant; from each of which take away the common Arch Z P, and there will remain Æ Z = P B. Therefore alſo the Stile's Height is equal to the Elevation of the Pole in the Latitude Z.

1754. If the Plane be in a vertical Poſition, or paſſes thro' the Zenith and Nadir Points Z and D, and is the Plane of the prime Vertical (as Fig. 3.) then ſince the Sun cannot be on or above ſuch a Plane, but from VI to VI; no other Hour-lines need be drawn upon it. And in this Caſe, the Oppoſite or South Pole S is elevated above it in the Angle D N S = Z N P = the Complement of the other Elevation P N B to a Quadrant. Here alſo the other *Semiaxis* N S is the Stile; and this is called an *erect and direct* SOUTH DIAL. And the other Side of the Plane is a *North Dial*, for the Hours before VI, and the Stile is N P.

1755. It is eaſy to obſerve, that this *vertical Dial* is an *horizontal One* in the South Latitude D, which is the Co-Latitude of the Point Z, ſince Z D = 90 Degrees. The Meridian Line N D or Hour-line of XII is directly under the Gnomon or Stile N S, and is therefore called the *Subſtilar-line* or *Subſtile*.

1756. If we ſuppoſe the Plane to continue vertical, but to decline or move towards the Eaſt or Weſt; then it becomes an *Eaſt or Weſt declining* DIAL; and it is evident in ſuch Caſes, the Angle which the Stile N S makes with the Plane is leſſened, and the Subſtile departs from the Hour-line of XII, or rather, this departs from that, towards the Eaſt or Weſt.

1757. When the vertical *Dial* has a Poſition directly *Eaſt* or *Weſt*, then the Axis P S of the Sphere is in the Plane; and conſequently the Stile of an *Eaſt* or *Weſt Dial*, muſt be parallel to the Plane thereof, and the Height of it may be taken at Pleaſure, and will be always equal to the Diſtance of the Hour-lines of III and IX, from the Middle-line of VI. And as the Plane of the Dial

is

is now in the Plane of a Meridian, it is an *horizontal Dial* under the EQUATOR every where, and is therefore called an EQUATORIAL DIAL.

1758. With respect to the horizontal Plane D W B E (Fig. 2.) it is evident, that as the Point Z approaches the Pole P, the Plane will approach to that of the Equator Æ Q; in which Case the Axis P N becomes perpendicular to the Plane, and marks out the Hour by dividing the Plane into 24 equal Parts; and such an One is called a POLAR DIAL, as being an *horizontal Dial* under the Pole P.

1759. This POLAR DIAL (commonly called an *equinoctial Dial)* with a perpendicular Stile, being drawn on a Plane, and that Plane elevated to the Latitude of a Place, the Hour of the Day will be shewn upon it, by the Shadow of the Perpendicular, with the same Exactness as on a common horizontal Dial for that Latitude; and therefore this Dial is, in its own Nature, an UNIVERSAL DIAL, for *all Latitudes* or Parts of the World.

1760. The Eye being in the Center N of the Sphere projects the *Parallels* of the *Sun's diurnal Motion* on a given horizontal Plane into one of the *Conic Sections*; for let Æ C be the Sun's Declination, then C R is the Diameter of the Parallel of its Motion on that Day, and C N R the Section of the Cone described in the Sphere by the visual Ray drawn from the Eye to the Sun. Now since C R is parallel to the Equator Æ Q, the Axis P S will be perpendicular thereto; and therefore upon a horizontal Plane at the Pole P, which is parallel to C R, the Projection of the Parallel, which is a Circle, will be a CIRCLE also (1504).

1761. Produce the Side of the Cone R N to V; then Æ C = Æ V; and on V R erect the perpendicular L N on the Center N. Then upon an horizontal Plane in the Latitude L, which will be parallel to the Side of the Cone N R or V R, the Projection of the Cone's circular Base, or parallel of the diurnal Arch, will be a PARABOLA (1505).

1762. Hence upon every horizontal Plane between the Pole P and the Latitude L, the circular Base, or Parallel must be projected into an ELLIPSIS, more eccentric in Proportion as the

Plane is nearer the Point L (1507) for every ſuch Plane will cut both the Sides of the Cone C N R produced.

1763. And on the other Hand, upon every horizontal Plane, between the Equator and the Latitude L, the Baſe of the Cone will be projected by the Eye into a HYPERBOLA (1506) becauſe ſuch a Plane muſt cut the Side C N of the Cone C N R, and the Side V N of the oppoſite Cone V N T, when produced.

1764. Becauſe ÆC = LP, therefore ÆL = CP = Complement of the Sun's Declination; therefore in all Latitudes leſs than the Co-Declination, the Projection is an *Hyperbola*; in all Latitudes greater, it is an *Ellipſis*; and in the Latitude equal thereto, it is a *Parabola*. In the Latitude 90°, the *Ellipſis* becomes a *Circle*; and in the Latitude 00, or in the Equator, the *Hyperbola* degenerates into a *Right line*.

1765. Hence we have an eaſy Tranſition to the Shadows caſt by a Gnomon on the Plane of any *horizontal Dial*. For the ſame Ray which projected the Sun's diurnal Path or Parallel on a Plane, projects alſo the Shadow from the Point of the Gnomon on the ſame Plane. Thus ſuppoſe a Plane (*a b c d*) placed parallel to A B, below the Center N of the Sphere, cutting the Axis in E, ſo that the Part of the Axis E N becomes the Gnomon on that Plane, then it is evident, the ſame Ray N C projects the Parallel of C R on the Plane at A B on one Hand, and the Shadow of the extreme Point N of the Gnomon on the Plane (*a b c d*) on the other; and that when the Sun is in the Meridian at C, the Shadow of the Gnomon falls in the Subſtile E *f* on the Point I, at the Diſtance F I from the perpendicular F N.

1766. Now becauſe of the Paralleliſm of the Planes, the Curves deſcribed by the Shadow will be exactly the ſame as thoſe of the projected Parallel of the Sun's diurnal Path; that is to ſay, (1.) Under the Pole P, the Shadow of the Point N will deſcribe a CIRCLE. (2.) At the Latitude L, equal to the Co-Declination of the Sun, it will deſcribe a PARABOLA. (3.) On all other Planes between P and L it will deſcribe ELLIPSES. (4.) In all Latitudes from the Equator to the Point L, the Path of the Shadow will be a HYPERBOLA. And (5.) When the Sun is in the Equator at Æ, the Shadow will fall on K, at

Noon,

Noon, and its Path will that Day be the Right-line *g* K *h*. (1760 to 1765.)

1767. Hence it appears, that when the Sun enters *Cancer*, and ÆC = 23° 30′, then L is in Latitude 66° 30′, and consequently, the Shadow can describe Ellipses only in the *frigid Zone* L P, when the Sun is altogether above that Horizon; and its Altitude on the South and North Part of the Meridian unequal.

1768. If the Height of the Gnomon N F be made the Radius, then the Angle N I F = C N H is the Sun's Meridian Altitude, and the Angle I N F its Co-Altitude; then say, as Radius to N F, so is Co-Tantent of the Meridian Altitude, to the Distance of the Shadow F I; when the Sun is in C. If C I be the *Ecliptic*, then the Points I, K, and W, will be found by this Analogy for the solstitial and equinoctial Days.

1769. Hence if the Dial-plane be of a sufficient Length, all the Parallels of the Sun's Declination may be described thereon at his Entrance into each of the 12 Signs. Those of the first Six, or *Summer* half Year, will fall between K and I; and those for the *Winter*, between K and W. In our Latitude of 51° 30′, we have Radius : Tangent of 75° :: FN : FW :: 1 : 3,7 nearly.

1770. Suppose ÆZQS a graduated Brass Meridian, suspended from a Ring at Z, which is moveable and set to the Latitude of the Place; then if C R be the Tropic of *Cancer*, and V T the Tropic of *Capricorn*; and (*n m*) a Plate of Brass in the Axis of the Sphere, with a moveable Piece containing a small Hole which may be placed any where between *n* and *m*; the Ray of Light passing through that Hole, when properly adjusted, will, from the Meridian Sun, always fall on the North Point Q of the Equator, for when the Sun is in *Cancer* at C, and the Hole at (*n*) the solar Ray will be *n* Q parallel to C N; and when the Sun is in *Capricorn* at V, and Hole at (*m*) the Ray will be in Q parallel to V N, and therefore every Day, at Noon, the Ray will pass through the Hole to the Point Q, or Hour of XII in the Equator.

1771. Also when the Sun is on any other Hour-circle before or after Noon, it will still pass through the Hole, (adjusted to the Day,) to the Equator, but just as far from Q, as the Hour-circle

cle is from the Meridian; because the diurnal Motion of the Ray N C, and the Motion of the Ray *n* Q parallel to it, will necessarily be the same; therefore the Ray through the Hole in the Axis will ever point out the Hour on the Periphery of the Equator. And this is the *Rationale* of the Construction of the *universal* RING-DIAL, the Form of which is exhibited in Fig. 5.

1772. We shall finish this Theory of *perspective Dialling*, with observing that the Hour-lines of any Dial are laid off by the Tangents of the Angles which the projected Meridians make on the Horizon of the Place. For let W N E S (Fig. 6.) be the Horizon of LONDON, Latitude 51° 30′, on which project the Sphere, and the horary Meridians will intersect the Horizon in the Points 1, 2, 3, 4, &c. The Eye being in the Center C; let the Plane of Projection touch the Sphere in N; then is (*a b*) the *horizontal Line*, and N C the *vertical Line*; and then to shew from the Rules of Perspective, that the Hour-line of II passes thro' (2) the Point of Intersection of the 2 o'Clock Meridian with the Horizon; we have $CN = r$, $c2 = Ng = v$ = versed Sine, or Distance from the horizontal Line; and $Nc = g2 = s$ = Sine of the Arch N 2, or Distance of the Point (2) from the vertical Plane. Then because that Point is between the Eye and Plane of Projection, the Analogy is (1691) as $nr - v : r :: s : \frac{rs}{nr - v} = \frac{rs}{r - v}$ (because here $n = 1$) $= \frac{rs}{c}$, (putting c = Co-Sine C*g*) which therefore is the Distance N*d* of the Hour-line of II from the vertical Plane CN. But $\frac{rs}{c} = t = Nd$ = Tangent of the Arch N 2; (for $s : c :: r : t$, or $Cg : C2 :: CN : Cd$ (711.) *Therefore the Hour-lines of a horizontal Dial are drawn from the Center* C *thro' the Points of Intersection of the Hour-circles with the Horizon of the Place.**

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* *N. B.* Fig. 7. is here inserted from *Bion*, representing a *general horizontal Dial*; a particular Description of which will hereafter be given.

INSTITUTIONS OF SPHERICAL TRIGONOMETRY.

CHAP. I.

Of the Nature of SPHERICAL TRIANGLES, *with a* SOLUTION *of all their* CASES.

1773. WE have heretofore considered the Dimensions of a SPHERE with regard to its superficial, and solid Contents in the *Institutions of plain Geometry*; that which we call *spherical Geometry* being conversant chiefly about the *Relation*, *Magnitude*, and *Inclination* of the GREAT CIRCLES of the SPHERE; and more particularly the various TRIANGLES that are form'd by their Intersections on its Surface; whose Properties and Affections we now proceed to treat of. And first of those which are *Rectangular*.

1774. A *Spherical Triangle*, is that whose three *Sides* are severally *the Arches of three great Circles of the Sphere*, intersecting each other in the three Angular Points. And the Angles of every Triangle are measured by the Arches of Circles; therefore the Arches of Circles are the Measures of every Part of a *Spherical Triangle*.

1775. And as the Quanity of the Arches themselves is determinable only from the Relations of those right Lines which are called their *Sines*, *Tangents* and *Secants*, as they stand constructed with the Radius in a plain Triangle, (706 to 708.) Therefore the Solution of a spherical Triangle, differs in nothing

thing eſſential to the Operation, from that of a common plain Triangle.

1776. Let ABC (Fig. 1.) be a right-angled ſpherical Triangle on the Surface of the Sphere, and ſuppoſe the Sides continued out indefinitely in the *Baſe* AB to E; the perpendicular AC to H; and the Hypothenuſe BC to I. On the Angle B draw the great Circle GFE thro' D the Pole of the Baſe AB; and on the Angle C, draw the great Circle GHI thro' G, the Pole of Hypothenuſe BC. Then will there be form'd the four Triangles EBF, EDA, ICH and IGF, each having two right Angles, and two quadrantal Sides, as is evident by Inſpection.

1777. There are alſo form'd two other right Angled ſperical Triangles. *viz.* CDF right Angled at F, and GDH, right Angled at H, which may be called *complemental Triangles* with reſpect to the Original, or given Triangle ABC.

1778. For in the firſt CED the Baſe CF is the Complement of the Hypothenuſe BC, and the Angle at the Perpendicular, D = AE, is the Complement of the Baſe AB. And the Hypothenuſe CD, is the Complement of the Perpendicular AC. And, laſtly, the Perpendicular DF is the Complement of EF = B, the Angle at Baſe; the Angle C at the Perpendicular, being the ſame in both.

1779. And in the other Triangle DGH, the Baſe GH is the Complement of HI = C in the given Triangle ABC. The Perpendicular DH = AC, becauſe AD and CH are Quadrants. The Hypothenuſe DG = EF = B the Angle at Baſe. The Angle at Baſe G = IF = BC the Hypothenuſe of the given Triangle. And the Angle at the Perpendicular, D = AE, the Complement of the Baſe AB.

1780. Now by help of theſe Quadrantal and Complemental Triangles, the Canons or Analogies for the Solution of all the Caſes of a right Angle, ſpherical Triangle ABC, may be eaſily derived from the two following Theorems.

Theorem I.

In a right Angled ſpherical Triangle, the Radius (or Sine of 90 *Degrees) to the Sine of the Baſe, as the Tangent of the Angle at Baſe is to the Tangent of the Perpendicular.* For let ÆBP (Fig. 2.) be a fourth Part of the *Orthographic Projection* of the Sphere on the Plane

Plane of the *Meridian* (1710.) And let A B C be a right-angled ſpherical Triangle form'd on the Globe, by the three great Circles Æ B, B F, and A P. Then is B C the Sine of the Hypothenuſe, and A B the Sine of the Baſe (1690.) C H is the Sine of the Perpendicular A C, and F E the Sine of the Angle at Baſe B; alſo A D, and Æ G, are the Tangents of the ſame Parts. And becauſe of ſimilar Triangles A B D and Æ B G, (657) we have *as Æ B Radius is to A B the Sine of the Baſe, ſo is Æ G the Tangent of the Angle at Baſe, to A D the Tangent of the Perpendicular A C.* Q. E. D.

1781. THEOREM II.

The Radius (or Sine of 90 Degrees) is to the Sine of the Hypothenuſe, as the Sine of the Angle at Baſe is to the Sine of the Perpendicular, in every right-angled ſpherical Triangle. For by Reaſon of the ſimilar Triangles C B H, and F B E, we have F B : C B :: F E : C H. *Q. E. D.*

1782. By theſe two Theorems, all the Caſes of a right Angled ſpherical Triangle are ſolved. That is, of the three Sides and three Angles, if any two are given, beſides the right Angle, the other three Parts may be found; and here it muſt be obſerved, that either of the two Sides A B or A C is reckon'd the Baſe, as the Angle B or C is given. And therefore there can be but a ſixfold Variety, or different Caſes in regard to any right-angled Triangle, which here follows.

1783. CASE I.

Given the Baſe A B, *and Angle* B; (Fig. 1.) *to find the Perpendicular* A C. In the quadrantal Triangle E B F; it is *s* B E : *s* B A :: *t* F E : *t* A C; that is R : *s* B A :: *t* B : *t* A C (1780).

1784. *To find the Hypothenuſe* B C; in the quadrantal Triangle A D E, we have *s* D E : *s* D F :: *t* A E : *t* C F; that is, R : *cs* B :: *ct* A B : *ct* B C.

1785. *To find the Angle* C. In the quadrantal Triangle F G I, we have, as *s* F G : *s* D G :: *s* D : *s* G H; that is, R : *s* B :: *cs* A B : *cs* C (1781).

1786. CASE

1786. CASE II.

Given the perpendicular AC, *and Angle at Base* B; *to find the Base* AB. In the quadrantal Triangle FBE we have *t* EF : *t* AC :: *s* BE : *s* BA; ∴ *t* B : *t* AC :: R : *s* BA.

1787. *To find the Hypothenuse* BC; we have *s* EF : *s* AC :: *s* BF : *s* BC; ∴ *s* B : *s* AC :: R : *s* BC.

1788. *To find the Angle* C. In the quadrantal Triangle HCI, it is, *s* DC : *s* DH :: *s* DF : *s* HI; ∴ *cs* AC : R :: *cs* B : *s* C.

1789. CASE III.

Given the Hypothenuse BC, *and the Angle at Base* B; *to find the Base* AB. In the quadrantal Triangle ADE, we have *s* DF : *s* DE :: *t* CF : *t* AE; ∴ *cs* B : R :: *ct* BC : *ct* AB, or R : *cs* B :: *t* BC : *t* AB.*

1790. *To find the Perpendicular* AC. In the quadrantal Triangle EBF, there is *s* BF : *s* BC :: *s* EF : *s* AC; ∴ R : *s* BC :: *s* B : *s* AC.

1791. *To find the Angle* C. In the quadrantal Triangle HCI, it is *s* CF : *s* CI :: *t* DF : *t* HI; that is, *cs* BC : R :: *ct* B : *t* C.

1792. CASE IV.

Given the Base AB, *and Perpendicular* AC; *to find the Hypothenuse* BC. In the quadrantal ADE, there is *s* AD : *s* DC :: *s* AE : *s* CF; that is, R : *cs* CA :: *cs* AB : *cs* BC.

1793. *To find the Angle* B; we have *s* AB : R :: *t* AC : *t* B (1786).

1794. *To find the Angle* C; *s* AC : R :: *t* AB : *t* C; by the same Reason as in the last.

1795. CASE V.

Given the Hypothenuse BC, *and the Perpendicular* AC. *To find the Base* AB. As *cs* AC : R :: *cs* BC : *cs* AB (1792).

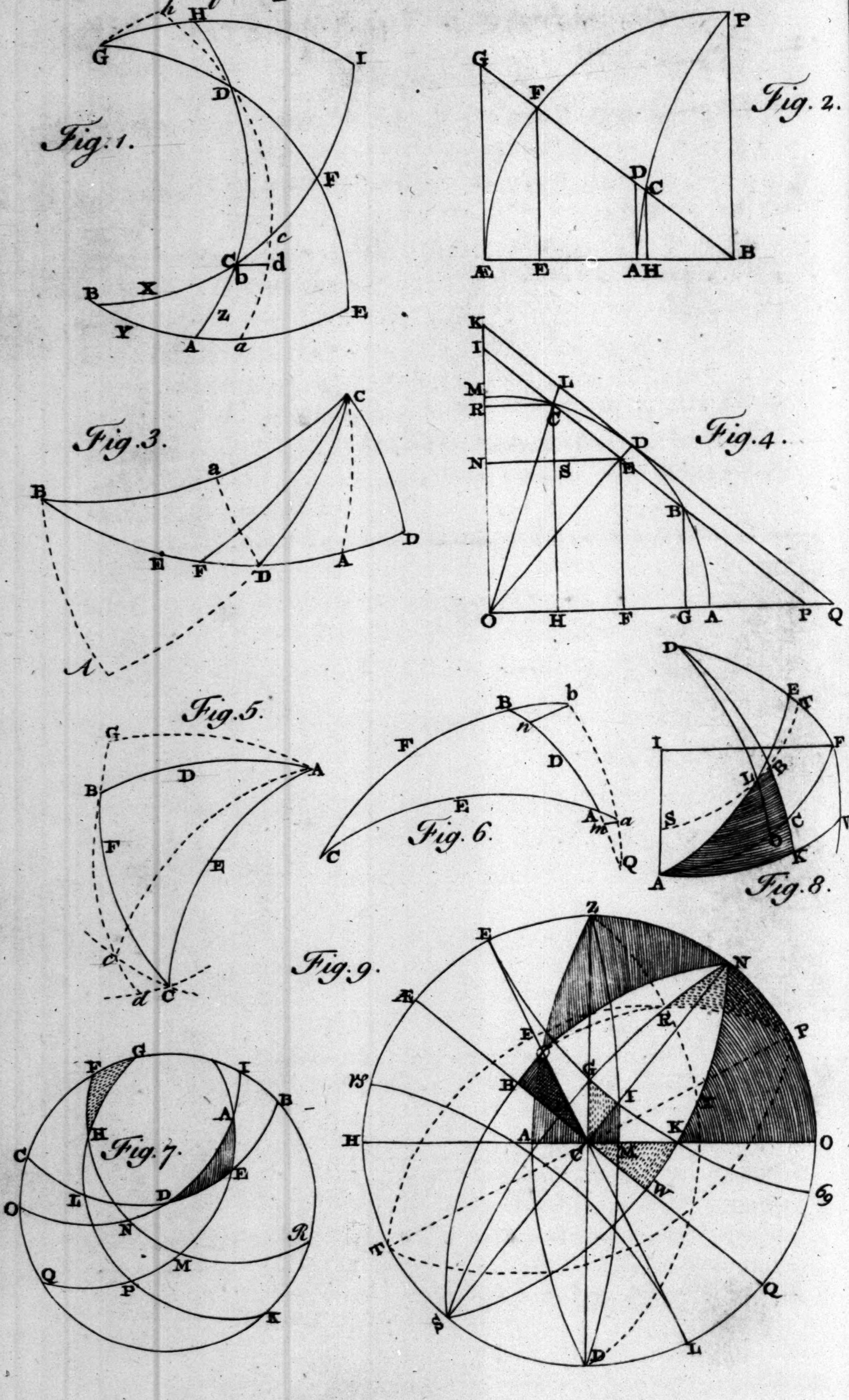
1796. *To find the Angle* B; *s* BC : R :: *s* AC : *s* B (1787).

1797. *To find the Angle* C; in the quadrantal IGF, we have *t* IF : *t* DH :: *s* GI : *s* GH; that is, *t* BC : R :: *t* AC : *cs* C.

1798. CASE

* Because the *Co-Tangents* of any two Arches are inversely as their Tangents, as is shewn hereafter at (1831).

Plate I. Of Spherical TRIGONOMETRY.



1798. Case VI.

Give the Angles B and C; *to find the Base* A B. As *s* B : *cs* C :: R : *cs* A B (1785).

1799. *To find the Perpendicular* A C. As *s* C : *cs* B :: R : *cs* A C (1785).

1800. *To find the Hypothenuse* B C; in the quadrantal I C H, we have *t* H I : *t* D F :: *s* C I : *s* C F; which gives *t* C : *ct* B :: R : *cs* B C.

1801. The two Angles B and C of every right-angled spherical Triangle, are together greater than the right Angle A; for the Angle B = E F = G D; and the Angle C = I H; but I H + D G, is greater than I H + H G = 90 Degrees, or a right Angle, whence the Theorem is evident.

CHAP. II.

The Method of determining the Fluxions *of the* Sides *and* Angles *of a right-angled* Spherical Triangle.

[Plate I. *Of* Spherical Trigonometry.]

1802. IF the Semi-diameter of the Earth bore no sensible Proportion to the Distance of the heavenly Bodies; if we could view them not through a *refracting Medium*; if the Position of the Earth's Axis were *immutable*, and lastly, if *there were no Obliquity of the Ecliptic*, then the Subject of this Chapter would be unnecessary.

1803. But as the Case now stands, we find a sensible Difference between the *true and apparent Place* of the *Sun*, *Moon*, and *Planets*, on Account of their *Parallax* (1724.) *Refraction* (1321.) *Motion* of the *Earth's Axis*, &c. by Means of all which the *Latitude*, *Longitude*, *Right-Ascension*, *Declination*, *Altitude*, *Azimuth*, *Amplitude*, *Hour-Angle*, &c. will all be affected in the same Manner as if the great Circles by which they are ascertained were continually moveable through a very small Space, one Way and the other, about their Axis in the Sphere.

1804. Therefore it becomes neceſſary, where Exactneſs is required, (as in *Aſtronomy*, *Navigation*, *&c.)* to make a proper Correction of the Computations made by ſpherical Triangles, by ſuppoſing ſuch a ſmall Motion in any one of the Circles of which that Triangle conſiſts, and from thence to find the *Fluxionary Increaſe or Decreaſe* of its other Sides and Angles reſpectively, occaſioned thereby; which will expreſs the Quantities ſought from the *Parallax*, *Refraction*, *&c.* given.

1805. Therefore (in Fig. 1.) if we ſuppoſe the Circle H D A to be moveable upon the Pole D; and to change its Situation from H D C A to h D c a; then it is evident, that in the three variable Triangles A B C, D C F, and D H G, there will (beſides the Right-angle) be one Part *conſtant* in each, and the other Parts all variable; thus the Angle B = E F, is conſtant in A B C; the Side D F in D C F; and the Hypothenuſe G D in D G H.

1806. For Diſtinction Sake, let X, Y, Z, denote the *Hypothenuſe*, *Baſe*, and *Perpendicular* of the Triangle A B C, and let C d be drawn at Right-angles to the Circle h D a. Then is A a = $\dot{Y}$, the *Fluxion* of A B; but $-\dot{Y}$ is the Fluxion of its Complement A E; C c = $\dot{X}$, and dc = $\dot{Z}$. Alſo the Fluxion of the Angle C = I H, is H e = $\dot{C}$. Which Fluxionary Parts are thus determined.

1807. In the fluxionary quadrantal Triangle A D a, we have (1780.) s D A : s D C :: A a : C d, or R : s D C :: A a : C d $= \frac{s\,DC \times \dot{Y}}{R} = \frac{cs\,AC \times \dot{Y}}{R}$.

1808. The ſmall fluxionary Triangle c C d may be eſteemed *Rectilineal*, which gives this Analogy. As t C (= c) : R :: C d : c d; that is, $t\,C : R :: \frac{cs\,AC \times \dot{Y}}{R} : \frac{cs\,AC}{tC} \times \dot{Y} = cd = \dot{Z}$ (1807).

1809. Alſo, in the ſame Triangle, as s C : R :: C d : C c :: $\frac{cs\,AC}{R} \times \dot{Y} : \frac{cs\,AC}{sC} \times \dot{Y} = Cc = \dot{X}$.

1810.

1810. Laſtly, $s\,DA : s\,DH :: Aa : He$ (1781); but DH $= AC$; therefore $R : s\,AC :: \dot{Y} : \frac{s\,AC}{R} \times \dot{Y} = He = \dot{C}$, or the Fluxion of the Arch IH which meaſures the Angle C.

1811. In the Triangle CDF, having one Side DF conſtant, the Fluxions of the other Parts are derived from the above Equations by Subſtitution of equal Parts; for becauſe $cs\,AC = s\,DC$, and $+\dot{X} = -\dot{X}$ in Quantity; therefore $\frac{s\,DC}{s\,C} \times \dot{Y} = -\dot{X}$ (1809), the Fluxion of the decreaſing Side or Baſe CF.

1812. For the ſame Reaſon $\frac{s\,DC}{t\,C} \times \dot{Y} = -\dot{Z} = cd$, the Fluxion of the decreaſing Hypothenuſe DC (1808).

1813. Laſtly, we have $\frac{cs\,CD}{R} \times \dot{Y} = \dot{C} = He$, the Fluxions of the increaſing Angle C, or Arch IH. The Fluxion of the Angle D, or Arch AE is negative, or $Aa = -\dot{Y} = -\frac{R}{cs\,CD} \times \dot{C}$.

1814. In the Triangle DGH, the Hypothenuſe DG being conſtant; the Fluxions of the variable Parts are found by Subſtitution of Equals as before. Thus $\frac{cs\,DH}{ct\,GH} \times \dot{Y} = \dot{Z}$, or Fluxion of the Side DH, which is Poſitive (1808).

1815. Alſo, it is $\frac{cs\,DH}{cs\,GH} \times \dot{Y} = \dot{X}$, the Fluxion of the Angle G, which is alſo poſitive (1808).

1816. And then for the Fluxion of the Side GH, (which will be, in this Caſe, negative) we have $\frac{s\,DH}{R} \times \dot{Y} = \dot{C}$.

1817. In the Demonſtrations hitherto, the Letters X, Y, Z, are uſed in Reference to three Triangles, and ſometimes they ſeparately denote an Angle, and ſometimes a Side. But if we have regard to one Triangle only, we muſt uſe ſuch Symbols as will denote the ſeveral Parts abſolutely. Therefore let H be the Hypothenuſe; A and a, the two Sides; and B, b, the two An-

Angles adjacent to them respectively. Then will the Equations in each Case or Triangle be transformed, in the following Manner.

1818. Case I.

When *one Angle* B is invariable, we have $cs\,\mathrm{AC} \times \dot{\mathrm{Y}} = t\mathrm{C} \times \dot{\mathrm{Z}}$ (1808.) become $cs\,\mathrm{a} \times \dot{\mathrm{A}} = t\,\mathrm{b} \times \dot{\mathrm{a}}$, and so of the Rest, whence we derive the following Equations and Analogies.

1. $cs\,\mathrm{a}\dot{\mathrm{A}} = t\,\mathrm{b}\dot{\mathrm{a}}$; $\therefore \dot{\mathrm{A}} : \dot{\mathrm{a}} :: t\mathrm{b} : cs\,\mathrm{a}$ (1808).
2. $cs\,\mathrm{a}\dot{\mathrm{A}} = s\,\mathrm{b}\dot{\mathrm{H}}$; $\therefore \dot{\mathrm{A}} : \dot{\mathrm{H}} :: s\mathrm{b} : cs\,\mathrm{a}$ (1809).
3. $s\,\mathrm{a}\dot{\mathrm{A}} = \mathrm{R}\dot{\mathrm{b}}$; $\therefore \dot{\mathrm{A}} : \dot{\mathrm{b}} :: \mathrm{R} : s\mathrm{a}$ (1810).
4. $t\,\mathrm{b}\dot{\mathrm{a}} = s\,\mathrm{b}\dot{\mathrm{H}}$; $\therefore s\mathrm{b} : t\mathrm{b} :: cs\mathrm{b} : \mathrm{R} :: \dot{\mathrm{a}} : \dot{\mathrm{H}}$.

1819. Case II.

When *one Side* A is invariable.

1. $s\mathrm{H}\dot{\mathrm{B}} = -\,s\,\mathrm{b}\,\dot{\mathrm{a}}$; $\therefore \dot{\mathrm{B}} : -\,\dot{\mathrm{a}} :: s\mathrm{b} : s\mathrm{H}$ (1811).
2. $s\mathrm{H}\dot{\mathrm{B}} = -\,t\,\mathrm{b}\dot{\mathrm{H}}$; $\therefore \dot{\mathrm{B}} : -\,\dot{\mathrm{H}} :: t\mathrm{b} : s\mathrm{H}$ (1812).
3. $cs\mathrm{H}\,\dot{\mathrm{B}} = \mathrm{R}\,\dot{\mathrm{b}}$; $\therefore \dot{\mathrm{B}} : \dot{\mathrm{b}} :: \mathrm{R} : cs\mathrm{H}$ (1813).

1820. Case III.

When the *Hypothenuse* H is constant.

1. $cs\mathrm{A}\dot{\mathrm{B}} = ct\,\mathrm{a}\dot{\mathrm{A}}$; $\therefore \dot{\mathrm{A}} : \dot{\mathrm{B}} :: cs\mathrm{A} : ct\,\mathrm{a}$ (1814).
2. $cs\mathrm{A}\dot{\mathrm{B}} = cs\,\mathrm{a}\,\dot{\mathrm{b}}$; $\therefore \dot{\mathrm{B}} : \dot{\mathrm{b}} :: cs\,\mathrm{a} : cs\,\mathrm{A}$ (1815).
3. $s\mathrm{A}\,\dot{\mathrm{B}} = -\,\mathrm{R}\,\dot{\mathrm{a}}$; $\therefore \dot{\mathrm{B}} : -\,\dot{\mathrm{a}} :: \mathrm{R} : s\mathrm{A}$ (1816).
4. $ct\,\mathrm{a}\dot{\mathrm{A}} = cs\,\mathrm{a}\,\dot{\mathrm{b}}$; $\therefore cs\,\mathrm{a} : ct\,\mathrm{a} :: s\,\mathrm{a} : \mathrm{R} :: \dot{\mathrm{A}} : \dot{\mathrm{b}}$.

1821. From these Equations, it is evident, there is a threefold Value or Expression for the Fluxion of every variable Part; and many Equations to express the Ratio of any two Fluxions, besides those here specified. The great Use of this Doctrine of *Fluxionary Trigonometry* we shall hereafter particularly illustrate and exemplify in all the above-mentioned Cases.

CHAP.

CHAP. III.

THEOREMS *for the* SOLUTION *of the* CASES *of* Oblique Spherical TRIANGLES.

1822. WHEN each of these three Angles of a spherical Triangle is greater or less than a *Right-angle*, it is called an *Oblique Triangle*, as BCD (Fig. 3.) In such an one all the Angles are uncertain, and therefore three Parts out of the Six must here be given, to find the Rest; and consequently there will also be *Six Cases* in the Solution of an *Oblique Triangle*. In order to which, the *Ten* following Theorems must be premised.

1823. THEOREM I.

In every oblique spherical Triangle, the Sines of the Sides are proportional to Sines of the opposite Angles. To demonstrate this; let fall the Perpendicular CA on the Side BD (continued out Fig. 3.) and it will be as R : *s*BC :: *s*B : *s*AC; and again, R : *s*CD :: *s*D : *s*AC (1781). Therefore R $\times$ *s*AC = *s*BC $\times$ *s*B = *s*CD $\times$ *s*D; whence *s*BC : *s*DC :: *s*D : *s*B. *Q. E. D.*

1824. THEOREM II.

In an oblique spherical Triangle BDC, *drawing the Perpendicular* CA, *the Tangents of the Sides* BC *and* CD *are reciprocally proportional to the Co-Sines of the vertical Angles.* For in the right-angled Triangle ACB, it is, R : *cs*ACB :: *t*BC : *t*AC (1789.) and in the Triangle ACD, it is R : *cs*DCA :: *t*DC : *t*AC. Therefore *cs*ACB $\times$ *t*BC = R $\times$ *t*AC = *cs*DCA $\times$ *t*DC; consequently *t*BC : *t*DC :: *cs*ACD : *cs*ACB. *Q. E. D.*

1825. THEOREM III.

In the oblique Triangle BDC, *with the Perpendicular* AC, *the Co-Sines of the Sides* BC *and* CD *are directly proportioned to the Co-Sines of the Parts of the Base* AB *and* AD. For in the right-angled Triangle ACB, it is, R : *cs*AC :: *cs*AB : *cs*BC,

(1792)

(1792) and for the same Reason R : cs A C : : cs A D : cs D C. Therefore (as above) cs A B : cs B C : : cs A D : cs D C. *Q. E. D.*

1826. THEOREM IV.

In the same oblique Triangle B D C, *the Co-Sines of the Angles at Base are directly as the Sines of the vertical Angles.* For as R : s A C B : : cs A C : cs B (1785) and as R : s A C D : : cs A C : cs D. Therefore cs B : cs D : : s A C B : s A C D. *Q. E. D.*

1827. THEOREM V.

In the same oblique Triangle B C D ; *the Sines of the Parts of the Base* A B *and* A D, *are inversely as the Tangents of the Angles at the Base.* For R : s A B : : t B : t A C (1783.) and R : s A D : : t D : t A C. Therefore s A B $\times$ t B ($=$ R $\times$ t A C) $=$ s A D $\times$ t D, whence s A B : s A D : : t D : t B. *Q. E. D.*

1828. THEOREM VI.

As the Sum of the Sines of two unequal Arches is to their Difference ; so is the Tangent of half their Sum to the Tanget of half their Difference. For let A C M be a Quarter of a Circle (Fig. 4.) in which take the two unequal Arches A B, A C ; their Difference is B C, which is bisected by the Radius O D. Then A D is half their Sum, and D C $=$ D B half their Difference. The Sines of the Arches are B G and C H ; and the Co-Sines O G and O H. Through the Points B and C draw the right Line I P, and parallel to it draw K Q touching the Circle in the Point D ; also draw N E parallel to A O, and E F perpendicular thereto. And lastly, through C draw O L, then is D Q the Tangent of half the Sum of the Arches A D, and D L the Tangent of half their Difference C D ; and D K is the Tangent of D M the Complement of A D. Then because C D $=$ B D, or C E $=$ E B, we have also H F $=$ F G, and therefore C H + B G $=$ 2 E F, and C H $-$ B G $=$ 2 C S (221). But 2 E F : 2 C S : : E F : C S (656.) : : E P : E C : : D Q : D L (657.) therefore C H + B G : C H $-$ B G : : D Q (Tangent of A D) : D L (Tangent of D C). *Q. E. D.*

1829. THE-

1829. THEOREM VII.

The Sum of the Co-Sines of two unequal Arches is to their Difference, as the Co-Tangent of half the Sum of the Arches to the Tangent of half their Difference. For $OG + OH = 2OF$, and $OG - OH = 2HF$ (221.) Therefore $2OF : 2HF :: OF : HF$ ($:: EN : SE$) $:: EI : EC :: DK : DL$; therefore the Sum of the Co-Sines $OG + OH : OG - OH$ (their Difference) $::$ DK (the Co-Tangent of AD) : DL, the Tangent of DC. *Q. E. D.*

1830. THEOREM VIII.

In any oblique Triangle BDC, *(having drawn the Perpendicular* AC, *and bisected the Base in* E,*) it will be, as the Co-Tangent of half the Sum of the two Sides is to the Tangent of half their Difference, so is the Co-Tangent of half the Base, to the Tangent of the Distance* (AE) *of the Perpendicular from the middle Point* (E) *of the Base.* For we have $cs\,AB : cs\,AD :: cs\,BC : cs\,DC$ (1824). Whence $cs\,BC + cs\,DC : cs\,BC - cs\,DC :: cs\,AB + cs\,AD : cs\,AB - cs\,AD$, (648, 649.) But $cs\,BC + cs\,DC : cs\,BC - cs\,DC :: ct\,\frac{BC + DC}{2} : t\,\frac{BC - DC}{2}$ (1829.) Also $cs\,AB + cs\,AD : cs\,AB - cs\,AD :: ct\,\frac{AB + AD}{2} : t\,\frac{AB - AD}{2} :: ct\,BE : t\,AE$; and hence by Equality (652) $ct\,\frac{BC + DC}{2} : t\,\frac{BC - DC}{2} :: ct\,BE : t\,AE$. *Q. E. D.*

1831. THEOREM IX.

In any oblique spherical Triangle, the Tangent of half the Base is to the Tangent of half the Sum of the Sides, as the Tangent of half the Difference of the Sides is to the Tangent of the Distance of the Perpendicular from the middle Point of the Base. For since KOQ (Fig. 4.) is a Right-angle, and OD perpendicular to the Base KQ; it is $DQ : DO :: DO : DK$ (660.) Therefore $DK = \frac{DO^2}{DQ}$, or because DO^2 is constant, it is $DK : \frac{1}{DQ}$, or the Co-Tangent of Arches are as the Tangents reciprocally. There-

Therefore $ct\frac{BC+DC}{2} : ctBE :: tBE : t\frac{BC+DC}{2} ::$

$t\frac{BC-DC}{2} : tAE$ (1829.) *Q. E. D.*

1832. THEOREM X.

As the Co-Tangent of half the Sum of the Angles at Base, is to the Tangent of half their Difference, so is the Tangent of half the vertical Angle to the Tangent of the Angle which the Perpendicular makes the Line bisecting the vertical Angle. Let the Circle CF bisect the vertical Angle BCD (Fig. 3.) Then it is, $R : sACB :: csAC : csB$ (1798) and $R : sACD :: csAC : csD$. Therefore by Equality (652) and Permutation, we have $csB : csD :: sACB : sACD$; and therefore it is $csB + csD : csB - csD :: sACB + sACD : sACB - sACD$ $(:: ct\frac{B+D}{2} : t\frac{B-D}{2}$ (1829) $:: tBCF : tACF$ (1828.) *Q. E. D.*

CHAP. IV.

The SOLUTION *of the* SIX CASES *of* Oblique Spherical TRIANGLES.

1833. CASE I.

GIVEN *the two Angles* B *and* D; *and the Side* BC *opposite to one of them*; *to find the other Angle and Sides* (Fig. 3.) The Analogy for the Side CD is, as $sD : sBC :: sB : sCD$, (1823.)

1834. *To find the Angle* BCD; we have $csBC : R :: ctB : tACB$ (by 1791.) Then, it is, $csB : csD :: sACB : sACD$ (1826); whence the whole Angle BCD is known.

1835. *To find the Side or Base* BD; we have $R : csB :: tBC : tAB$ (1784, 1831.) Then, as $tD : tB :: sAB : sAD$ (1827.) Whence $AB + AD = BD$ required.

1836. CASE

1836. CASE II.

Given two Angles B and B C D, *and a Side* B C *between them; to find the other Angle* D. First, R : *cs* B C : : *t* B : *ct* B C A (1791, 1831); whence A C D is also known. Then *s* A C B : *s* A C D : : *cs* B : *cs* D (1826).

1837. *To find the Side* D C. Say as, R : *cs* B C : : *t* B : *ct* A C B (1791, 1831) whence A C D is known. Then *cs* D C A : *cs* B C A : : *t* B C : *t* D C (1824).

1838. *To find the Side* B D. The Process is here the same, as for the Side C D, if from the End B of the given Side B C, you let fall the Perpendicular B A opposite to the given Angle B C D.

1839. CASE III.

Given two Sides B C *and* C D, *and an Angle* B *opposite to one of them; to find the other Side* B D. Say, as R : *cs* B : : *t* B C : *t* A B (1789, 1831). Then *cs* B C : *cs* D C : : *cs* A B : *cs* A D (1825). Whence A B $\pm$ A D $=$ B D sought.

1840. *To find the included Angle* B C D. First say, as R : *cs* B C : : *t* B : *ct* A C B (1791, 1831). Then *t* D C : *t* B C : : *cs* A C B : *cs* A C D (1824). Whence A C B $\pm$ A C D $=$ B C D.

1841. *To find the opposite Angle* D. Say, as *s* D C : *s* B : : *s* B C : *s* D (1823).

N. B. It must here be observed, that when the Side D C is less than B C, this Case will be ambiguous in all its Parts; since it will be uncertain from the *Data*, whether the Angle D is Obtuse or Acute; since the Sine of an Angle, and of its Complement to 180 Degrees, is the same.

1842. CASE IV.

Given two Sides B D *and* B C, *and the included Angle* B; *to find the other Side* C D. Say, as R : *cs* B : : *t* B C : *t* A B (1789, 1831.) whence A D is also known; then *cs* A B : *cs* A D : : *cs* B C : *cs* C D (1825).

1843. *To find the Angle* D. Say, as R : *cs* B : : *t* B C : *t* A B (1791, 1831.) whence A D is known; then *s* A D : *s* B D : : *t* B : *t* D (1827).

1844. *To find the Angle* C. Let fall the Perpendicular Da on the given Side BC, and opposite to the given Angle B; then R : cs B :: t BD : t Ba, and then aC is known; consequently saC : saB :: tB : tC as above (1843).

1845. Case V.

Given all the three Sides BD, BC, *and* CD; *to find an Angle, suppose* B. Say $t\,\frac{1}{2}$ BD : $t\,\frac{1}{2}\,\overline{BC + \frac{1}{2}CD}$ :: $t\,\frac{1}{2}\,\overline{BC - \frac{1}{2}CD}$: t AE, the Distance of the Perpendicular from the middle Point E of the Base BD (1831.) whence AB is known. Then t BC : t AB :: R : cs B (1797).

1846. Case VI.

Given all the three Angles, B, C, *and* BCD; *to find a Side, suppose* BC. Then say, as $ct\,\frac{1}{2}\,\overline{B + D}$: $t\,\frac{1}{2}\,\overline{B - D}$ :: t BCF : t ACF = Angle included between the Perpendicular AC, and the Line CF, which bisects the vertical Angle C; from whence the Angle BCA is known. Then say, as t B : ct BCA :: R : cs BC (1800, 1831).

N. B. *There are other Methods of solving the Cases of oblique spherical Triangles, but they are more complex and difficult both in Theory and Operation, and therefore are not to be expected in an* Elementary Institution. *Nor shall we here insist on those Solutions of Triangles which depend on the given* Sums *and* Differences *of* Sides *and* Angles, *or of their* Sines, Tangents, Versed-sines, *&c. as they are very intricate, and rarely necessary in Practice.*

CHAP.

CHAP. V.

The Method of determining the FLUXIONS *of the* SIDES *and* ANGLES *of Oblique Spherical* TRIANGLES.

1847. CASE I.

LET ABC be an *oblique spherical Triangle*, (Fig. 5.) and let the Side AC move on the Point or Angle A, and by its Motion describe a small Portion of a Parallel Cc; at the same Time let Cd be Part of a Parallel described by BC on the Pole B; and draw the great Circle Bc intersecting Cd in d. Then let D, E, and F, denote the three Sides AB, AC, and BC respectively, of which AB and AC are *constant*, and all the other Parts variable.

1848. Then we shall have $R : sE :: sCAc : sCc$ (1781) for the fluxionary Triangle CcA is right-angled at (c); and because the Sine of a very small Arch is nearly equal to the Arch itself, therefore $R : sE :: CAc\ (= \dot{A}) : Cc = \frac{sE}{R} \times \dot{A}$.

1849. Also, $R : sF :: CBd\,(= \dot{B}) : Cd = \frac{sF}{R} \times \dot{B}$. Again because $ACc = dCB =$ Right-angle (1847); therefore $ACB = dCc$, and because of the Right-angle at d, and the fluxionary Triangle dCc Rectilineal, we have $R : s\,dCc\ (= ACB) :: Cc : cd :: \frac{sE}{R} \times \dot{A} : \dot{F} = \frac{sE \times sC}{R^2} \times \dot{A}$ (1848).

1850. But $sC : sD :: sB : sE$ (1823); therefore $sE \times sC = sD \times sB$; consequently it is also $\dot{F} = \frac{sD \times sB}{R^2} \times \dot{A}$.

1851. Again; $R : cs.\,dCc\ (= C) :: Cc : Cd :: \frac{sE}{R} \times \dot{A} : \frac{sE \times csC}{R^2} \times \dot{A} = \frac{sF}{R} \times \dot{B}$ (1849). Whence we have $\dot{B} = \frac{sE \times csC}{R \times sF} \times \dot{A}$.

1852. And lastly, $ct \,.\, cCd\,(=C) : R :: Cd : cd :: \frac{sF}{R} \times \dot{B} : \dot{F} = \frac{sF}{ct\,C} \times \dot{B}$.

1853. In the same Manner it is shewn (by making the like Construction at B) that $\dot{C} = \frac{sD \times csB}{R \times sF} \times \dot{A}$; and $\dot{F} = \frac{sF}{ct\,B} \times \dot{C}$

1854. Let $\int =$ Secant, and $c\int =$ Co-Secant of an Arch E B (Fig. to Inst. 705) then it is, C H ($=$ E D) : C E :: C G ($=$ C E) : C F; that is, $sEB : R :: R : \int GE$, or $c\int BE$; therefore $R^2 = sBE \times c\int BE$.

1855. Hence because $R^2 = \frac{sD \times sB}{\dot{F}} \times \dot{A}$ (1850) $= sD \times c\int D$ (1854.) Therefore we have, from the above Equations, the following Analogies for the Fluxions of the Sides and Angles of this Triangle, *viz.*

1. $\dot{A} : \dot{F} :: c\int D : sB$.
2. $\dot{A} : \dot{B} :: R \times sF : sE \times csC$ (1851).
3. $\dot{A} : \dot{C} :: R \times sF : sD \times csB$ (1853).
4. $\dot{B} : \dot{F} :: ctC : sF$ (1852).
5. $\dot{C} : \dot{F} :: ctB : sF$ (1853).
6. $\dot{B} : \dot{C} :: ctC : ctB$ (1851, 1853).
$:: tB : tC$ (1831).

Case II.

1856. *If one Side* D, *and an adjacent Angle* B *be invariable*, then this Case will be reduced to that of a right-angled Triangle, by letting fall the Perpendicular A G (from the End of the given Side, and opposite to the given Angle B) on the Side B C continued out. For then there will be found the right-angled Triangle A C G, in which the Side A G (and the Segment G B) is constant; and so this Case becomes the same with Case II. (1811, 1819).

1857. Case III.

If one Side D, *and an opposite Angle* C, *be invariable* (Fig. 6.); then are the other Parts variable by the Motion of the given Side D or A B. Let the Side A B move into the Situation (*ab*) and be-

because it is ever $ab = AB$, it is impoſſible that (ab) ſhould be parallel to AB (631); therefore ab will meet AB, continued out, in ſome Point Q. On the Point Q, as a Center, deſcribe the ſmall Arches bn, am; then becauſe $AB = ab = nm$, if from the firſt and laſt, you take away the common Part An, there will remain $Bn = Am$.

1858. Then ſince the fluxionary Triangles Ama and Bnb may be conſidered in their *Naſcent* State as *Rectilineal*, and right-angled at m and n; therefore we ſhall have $Am : Aa :: csA : R$; and $Bn : Bb :: csB : R$; whence $Am \times R = Aa \times csA = Bn \times R = Bb \times csB$; therefore $Aa : Bb :: csB : csA :: \dot{E} : \dot{F}$.

1859. Again, we have $sC : sD :: sB : sE$ (1823) $:: \dot{s}B : \dot{s}E$. For becauſe the flowing Quantities or Sines of B and E are in the conſtant Ratio of sC to sD (1856). The Fluxions of the Sines, *viz.* $\dot{s}B$ and $\dot{s}E$ will be in the ſame conſtant Ratio (788). But the Fluxion of the Sine is to the Fluxion of its Arch as Co-Sine to Radius (874). Therefore $\dot{s}B = \frac{csB}{R} \times \dot{B}$, and $\dot{s}E = \frac{csE}{R} \times \dot{E}$; Whence $sC : sD :: csB \times \dot{B} : csE \times \dot{E}$, which gives $\dot{B} : \dot{E} :: sC \times csE : sD \times csB$.

1860. In the ſame Manner it is proved that $\dot{A} : \dot{F} :: sC \times csF : sD \times csA$.

1861. By comparing the Analogies in (1858, and 1859,) we get $\dot{B} : \dot{F} :: sC \times csE : sD \times csA$.

1862. And by comparing this with the Analogy in (1860) we have $\dot{A} : \dot{B} :: csF : csE$.

1863. The Analogies in (1859, 1860) may be more ſimply expreſs'd by Tangents; thus, it is $\dot{B} : \dot{E} :: \frac{sC}{csB} : \frac{sD}{csE}$; but $sC : sD :: sB : sE$; therefore $\dot{B} : \dot{E} :: \frac{sB}{csB} : \frac{sE}{csE} :: tB : tE$, becauſe the Sine has the ſame Ratio to the Co-Sine as the Tangent has to Radius $= 1$. (ſee *Fig.* to 705.) Thus alſo, it is $\dot{A} : \dot{F} :: tA : tF$.

1864.

1864 CASE IV.

When two Angles A, *and* D, *in the Triangle* A E D (Fig. 7.) *are invariable.* But this is reduced to the firſt CASE (1847) by conſidering, that the Angles of the Triangle A E D, are equal to the Sides of another Triangle F G H, formed by Circles connecting the Poles G, F, H, of the Sides of the Triangle A E D. For let the Sides D E, D A be produced to Quadrants in O and C, and on the Point D deſcribe a Circle B K O G. Let G be the Pole of the Circle C D B, F the Pole of the Circle O D I, and H the Pole of the Circle A E Q. Then is G C = F O = 90°, from which take the common Part F C, and there remains F G = C O = Angle D. In the like Manner F N = H M = 90°; Subduct H N, and there remains F H = N M = Angle A. Laſtly, G L = H P = 90°, take away H L, and we have G H = L P = Angle P E L, the Complement of the Angle A E D.

1865. Hence alſo it appears, that the *Sixth Caſe* of oblique Triangles (1846) is reducible to the *Fifth*, by changing the given Angles A, E, D into Sides of another Triangle F G H.

1866. I have now premiſed all that I judge does properly belong to the *Elementary Part* of the *Doctrine of ſpherical Triangles*. As to the *Five circular Parts* of Lord *Neper*, I think it an Artifice of more Ingenuity in the Invention, than of Uſe in Practice, and have here omitted it. Alſo thoſe Methods of ſolving oblique ſpherical Triangles by *given Sums, Differences, Products, &c.* of their ſeveral Parts, are not to be conſidered as *firſt Principles*, but rather the Inventions of Art reſulting from thoſe Principles, and may be explicated by them when ever they occur in Practice. However we cannot think theſe Elements compleat without ſuch as determine the *Area* of a ſpherical Triangle, which therefore we ſhall add in the next Chapter.

CHAP.

CHAP. VI.

The Method of determining the AREA *of a* SPHERICAL TRIANGLE.

1867. IN *Fig.* 8. Let A C F be a Quadrant of the Equator, the Pole D; S B T a Parallel to the Equator; A B E an oblique Circle croſſing the Equator in A; A K V an oblique Circle below the Equator; D B K a Meridian, and D L O another drawn indefinitely near to it. By this Means there will be form'd an oblique ſpherical Triangle A B K which is divided into two right angled Ones A B C, and A C K, by the Equator A F.

1868. Let $p =$ Periphery of a great Circle, and $x = s$ B C, or *Sine* of B C, the Latitude of the Zone A H E F, whoſe Surface is $x \times AF = x \times \frac{1}{4}p$; becauſe $xp =$ Surface of the whole Zone continued round the Globe, (838.) therefore alſo the Surface of the indefinitely ſmall Part L O C B is $= x \times OC = sBC \times OC$. But this is alſo the Fluxion of the Triangle A B C; for that of the Zone, and of the Triangle at the Point B will be the ſame (792) as is evident from the Reaſoning there uſed.

1869. The Fluxion of the Angle A B C is $= \frac{sBC}{R} \times OC$ (1810.) But $sBC \times OC : \frac{sBC}{R} \times OC :: 1 : \frac{1}{R} :: R : 1$. That is, the Fluxion of the Triangle A B C is to the Fluxion of the Angle B, *is in the conſtant Ratio of Radius to Unity.* And therefore the contemporaneous Fluents will be in the ſame Ratio.

1870. In the *Naſcent State* of the Triangle, when the Sides may be conſidered as *Rectilineal*, the Angle at B is equal to the Complement of the Angle A to a Right-angle (633). But as the Triangle flows, or encreaſes, this Angle B alſo flows and encreaſes to a larger Quantity, till at laſt the Triangle A B C becomes the quadrantal Triangle A E F, and the Angle B becomes a Right-angle at E, the whole Increaſe therefore of the Angle

Angle B is equal to the Angle A. And it is this increasing Part of the Angle B, which is the Fluent of the Fluxion $\frac{s\dot{B}C}{R} \times$ O C. Therefore *the Area of the Triangle* A B C *is to the Increase of the Angle* B, *or the Excess of* A + B *above a Right-angle, in the constant Ratio of* R *to* 1.

1871. The same Thing is shewn with regard to the Triangle A C K; therefore it is evident, *that the Area of any oblique spherical Triangle* A B K *is constantly proportional to the Encrease of the three Angles above two Right-angles.*

1872. Hence in regard to the Triangles A B C and A E F, it will be as A : A E F :: A + B — 90 : A B C; and in the same Manner in the Triangle C A K and F A V, it is A : F A V :: A + C — 90 : A C K. Therefore with regard to the whole oblique Triangle A B K, we have the Angle B A K to the whole quadrantal Area E A V, as A + B + K — 180° to the Area of the Triangle A B K.

1873. Let S = Surface of the Globe, T = Area of the Triangle A B K, Q = Area of the quadrantal Area E A V, N = Angle B A K = E V, and M = Sum of the three Angles of the Triangle = N + B + K, then it is self-evident, *that the quadrantal Space* E A V *is such a Part of a Hemisphere as the Arch* E V *is of a great Circle*; that is, $Q : \frac{1}{2}S :: N : p$; but $N : Q :: M - 2p : T$ (1870); therefore $p : \frac{1}{2}S :: M - \frac{1}{2}p : T$, or $2p : M - 2p :: S : T$; that is, *as* 720 : N + B + K — 180 :: *the Surface of the Globe* S : *Surface of the Triangle* A B K.

1874. But the Surface of a Globe or Sphere is equal to four Times the Area of its great Circle, or S = 4 A (839); therefore $\frac{1}{2}p : \frac{1}{4}S :: 180 : A :: M - 180 : T$. But $A = \frac{1}{2}pr$, or half the Radius (r) multiplied into the Periphery (830); therefore ($180 : 180r ::$) $1 : r :: M - 180 : T$.

1875. The Radius (r) expressed in Degrees is = 57°,2957795 (884); therefore if the three Angles of a Triangle lessened by 180°, be multiplied by 57,2957795, the *Product will be the Area of that Triangle in Square Degrees.* And because in one *Square Degree,* there are 3600 *Square geographical Miles,* or 4830,25 *English Miles,* therefore the Area of the Triangle may be expressed in Miles of either Sort.

CHAP.

Plate II. Of Spherical Trigonometry.

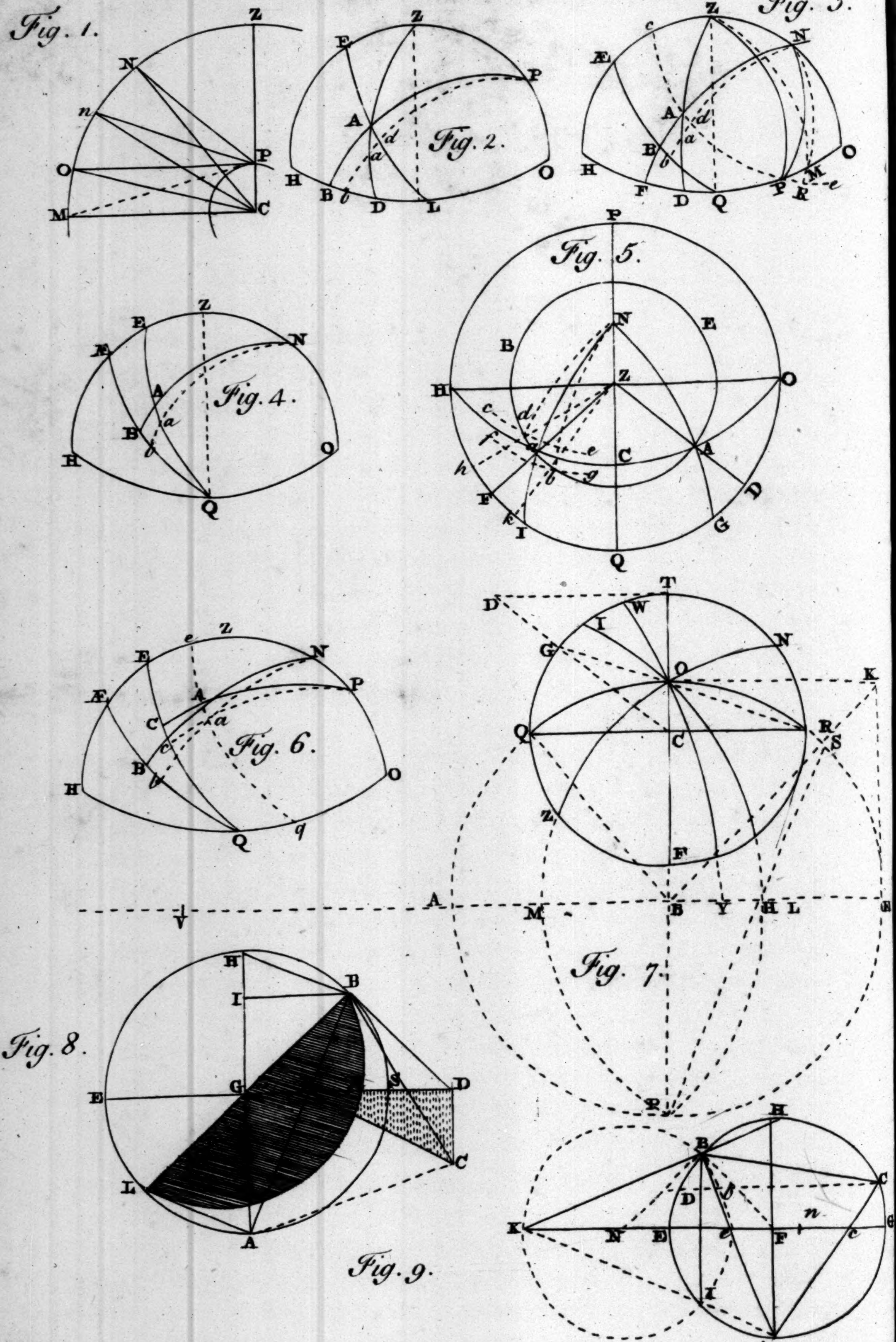


Fig. 1. Fig. 2. Fig. 3. Fig. 4. Fig. 5. Fig. 6. Fig. 7. Fig. 8. Fig. 9.

CHAP. VII.

The foregoing PRINCIPLES *applied to the* SOLUTION *of* PROBLEMS *in* ASTRONOMY, GEOGRAPHY, NAVIGATION, DIALLING, *&c.*

1876. THE Construction of a Diagram for presenting at one View the greatest Numbers of *spherical Triangles*, in order to the Solution of *astronomical Problems*, is that of Fig. 9. which is a *stereographical Projection of the Sphere* on the Plane of that Meridian ÆNQS, which is called the *solstitial Colure* (1707).*

1877. In this Scheme, ÆQ is the EQUINOCTIAL, and its two Poles N, S. EL is the ECLIPTIC, and its Poles P, T. HO is the HORIZON, its Poles Z, the *Zenith*, and D the *Nadir*. ZD, the *prime* VERTICAL, and its Poles H, O. E ♋ is the Tropic of *Cancer*; ♑ L, the Tropic of *Capricorn*. ☉ is the *Sun's Place* in the Ecliptic; N☉S a *Hour-circle*, and Z☉D a *vertical Circle* passing through the Sun ☉. PRT a *Circle of Longitude*, and NRS a *Circle of Declination* passing through a *Star* at R.

1878. Now these various Circles of the Sphere by their Intersections, form many *spherical Triangles*, both *Rectangular* and *Oblique*. In a right-angled Triangle, if *any two Parts are given*, it is a *Problem* to find the rest. And in oblique Triangles, the Problem requires three different Parts to be given, as we have shewn (717—721). But *one Angle* in most Triangles, Right or Oblique, is of no Consequence in astronomical Problems, and therefore does not enter the *Data*.

1879. In a right-angled Triangle, there are, therefore, four significant Parts, *viz. three Sides*, which call *a*, *b*, *c*; and *one Angle*, which let us denote by (*n*). Then since there are six Combinations of two Quantities in four, *viz.* *an*, *bn*, *cn*, *ab*, *bc*, *ac*, there will be at least *six Problems* resulting from every right-angle Triangle.

* It is here supposed, the Reader is acquainted with the Names and Uses of the CIRCLES of the SPHERE, as they have been at large explained in the GENTLEMAN and LADIES PHILOSOPHY.

1880. Again, in an oblique Triangle there will be five significant Parts, *viz.* *three Sides*, *a*, *b*, *c*; and *two Angles*, *m*, *n*; and there will be *ten Combinations of three Quantities in five*, viz. *abm*, *bcm*, *acm*, *abn*, *bcn*, *acn*, *amn*, *bmn*, *cmn*, *abc*; therefore *ten astronomical Problems* will arise from every oblique Triangle.

1881. We shall now specify those right-angled Triangles that are of most Use in *Astronomy*, which are as follow, *first*, the Triangle B C ☉, right-angled at B; in which B C is the Sun's *right Ascension*, or its Complement to the Point C. C ☉ is the Sun's *Longitude*, or Complement to C. B ☉ is the Sun's *Declination*; and the Angle B C ☉ the *Obliquity of the Ecliptic.*

1882. *Secondly*; the Triangle A ☉ C, right-angled at A; in which the Side C ☉ is the Sun's *Longitude* in the Ecliptic from C; the Side A ☉, its *Altitude* above the Horizon. A C is the *Azimuth* from the Point C. And the Angle A C ☉ is the Sum of the *Co-Latitude* and *Obliquity of the Ecliptic*, *viz.* Æ H + Æ E.

1883. *Thirdly*; Let the Sun be in the Tropic of *Cancer*, and in the Point G in the prime Vertical, when due *East* or *West*. Then in the Triangle V C G, right-angled at V, the Side V C is the *Hour from Six.* The Side V G = Æ E, the Sun's *Declination.* The Side C G the Sun's *Altitude when East or West*. And the Angle V C G = Æ Z, the *Latitude of the Place.*

1884. *Fourthly*; On the same Day the Sun is at I, at the Hour of Six precisely; then in the Triangle I C M, right-angled at M; the Hypothenuse C I is the Sun's *Declination*; I M, the Sun's *Altitude at Six*; and C M the *Azimuth* from C. And the Angle I C M = N O = Æ Z the *Latitude* of the Place.

1885. *Fifthly*; On the same Day the Sun is in the Horizon at K, and there is formed the Triangle W C K right-angled at W; in which W K is the Sun's *Declination.* C W is the *ascensional Difference*, or the *Hour* from *Six* of its *Rising and Setting*; C K, the *Amplitude* of Rising and Setting from the *East or West.* And the Angle W C K = O Q, is the *Co-Latitude* of the Place.

1886. *Sixthly*; In the Triangle N K O, right-angled at O, the Side N O is the *Latitude*; the Side N K the *Co-Declination*; and

and the Side K O the *Co-Amplitude* or *Azimuth* from O. And the Angle K N O = W Q the *Hour* from Midnight.

1887. What we have ſaid with Reſpect to the Sun in *Cancer*, is the ſame for its Place in any other Part of the Ecliptic; and the ſame Triangles are to be drawn for the ſame Purpoſes in the Winter Signs, as we have here done for the Summer. And further, it is to be obſerved, that ſince any Star may be ſuppoſed at ☉, it is evident all the ſame Triangles ſerve for a Star as for the Sun. Therefore in the *ſix Triangles* already enumerated, no leſs than *ſeventy-two Problems* may be ſtated in Reference to the Sun and Stars; beſides many other Triangles that might be added to theſe, to encreaſe the Number of aſtronomical Problems.

1888. In the oblique Triangle ☉ Z N, the Side ☉ Z is the *Co-Altitude* of the Sun; ☉ N, the *Co-Declination*; Z N, the *Co-Latitude* of the Place. The Angle Z N ☉ is the *Hour* from Noon, and the Angle ☉ Z N or ☉ Z Æ is the *Azimuth* from the North or South Part of the Horizon.

1889. This Triangle, therefore, affords *ten Problems* more in reſpect of the SUN (1878) and if we conſider the oblique Triangle Z R N, we ſhall find the Sides and Angles the ſame in regard to a STAR at R, and conſequently other *ten Problems* will thence be produced.

1890. Again; in the oblique Triangle R N P, the Side R N is the Star's *Co-Declination*; the Side R P, its *Co-Latitude*; and the Side N P = Æ E the *Obliquity of the Ecliptic*. Alſo the Angle N P R is the Star's *Longitude* from the Point E; and the Angle R N P its *Azimuth*. From hence we have ten Problems more; and therefore this and the laſt Triangle afford *twenty principal Problems about the Stars only*.

1891. In the laſt Place, let ☉ be the Moon or a Planet, then if we draw a great Circle through ☉ and R, the Place of a Star; we ſhall have an oblique Triangle ☉ Z R, in which the Side ☉ Z is the Moon's *Co-Altitude*; Z R, that of the Star, and ☉ R the *Diſtance* of the Moon from the Star. Alſo the Angle ☉ Z R the *Difference of the Azimuths* of the Moon and Star. This Triangle affords *four* very uſeful Problems relative to the Moon and Stars. Upon the Whole, in theſe few Triangles in this one Projection only, more than *one hundred aſtronomical Problems* are contained.

1892. In regard to GEOGRAPHY, if we ſuppoſe Z and R to repreſent two Places on the Surface of the *Terreſtrial Globe*; then in the Triangle ZRN, the Side ZN is the *Co-Latitude* of the Place Z; the Side RN is the *Co-Latitude* of the Place R; and the Side ZR is the *Diſtance of the two Places* on the Surface of the Globe. The Angle ZNR = ÆV, is the *Difference of Longitude* of the two Places, and the Angle RZN is the *Bearing* of the Place R from Z. This one Triangle therefore furniſhes *ten principal geographical Problems*, which depend on the Doctrine of the Sphere.

1893. In NAVIGATION, particularly that Part of it called ORTHODROMICS, which treats of the *Art of Sailing* on a *great Circle* of the Sphere, a ſpherical Triangle is concerned. Thus let Z be the Port from whence a Ship is to ſail to another Port X; then in the Triangle ZNX; the Side ZN is the *Co-Latitude* of the Port Z; the Side NX is the *Co-Latitude* of the Port X; the Side ZX is the *neareſt Diſtance* between them or the *Ship's Way*. Alſo the Angle NZX is the *Courſe* to be ſteered at Z; and the Angle ZNX is the *Difference of Longitude* between the Ports.

1894. But in ſailing from Z to X, the Courſe is perpetually altering, or the Angle made by the Ship's Way on the great Circle ZX with every Meridian is different, or the Latitudes always changing, therefore it is neceſſary to calculate this for every 5 or 6 Degrees of Longitude, that the Ship may be kept upon, or near the circular Arch ZX, every where; and this is the whole Art of great *Circle-ſailing*, which is entirely converſant in the *Solution* of *ſpherical Triangles*. A great Variety, therefore, of *nautical Problems* here offer themſelves from this one View of a ſpherical Triangle.

1895. As to LOXODROMICS, or ſailing upon a *Spiral*, or *Rhumb-line*, as the Theory of that Curve has not yet been conſidered, nor the *Rationale* of that Sort of Sailing been fully explained by Writers on Navigation, we have determined to treat of that Subject more particularly in another Chapter.

1896. In DIALLING, the Triangle NOK, right-angled at O, is uſed in finding all the Angles NKO, which the ſeveral Hour-circles NKS make with the Horizon. For NO being the Latitude of the Place, and the Angle ONK being given,

the

the Arch of the Horizon O K is found by Caſe I. of right-angled Triangles (1783). But this is only touched upon here, to ſhew how very uſeful and extenſive the Doctrine of *ſpherical Trigonometry* is in moſt Arts and Sciences. What relates to the Conſtruction of the *Scale of Latitudes* and *Hours* in practical Dialling, as alſo of a new *Dialling Sector*, and a *new univerſal* HORIRONTAL DIAL, ſhewing the Hour by the ſame Gnomon in all Latitudes, will be treated of more fully hereafter.

CHAP. VIII.

The Application of FLUXIONARY SPHERICS *in* aſtronomical COMPUTATIONS, *relative to* PARALLAXES, REFRACTIONS, equal ALTITUDES, PRECESSION *of the* EQUINOXES, *&c.*

1897. WE have ſhewn how the *Places* of the heavenly Bodies, the *Times* of their *Riſing*, *Setting*, *&c.* and their various *Phænomena* in regard to their *Right-aſcenſion*, *Declination*, *Amplitude*, *Azimuth*, *Longitude*, *Latitude*, &c. are to be calculated by the common Proceſſes of *ſpherical Trigonometry*. But the minute Variations and Alterations which happen in thoſe Quantities by Means of a *Parallax*, *Refraction*, *Receſſion* of the *Equinoxes*, *ſpiral Motion of the Sun*, &c. are of too much Moment in Aſtronomy not to be moſt ſcrupulouſly attended to; and as they are beſt of all computed by *Fluxionary Spherics*, the Principles of which have been explained, we therefore now illuſtrate that Method by ſome Examples.

1898. As the moſt conſiderable of theſe Variations are of that Sort which ariſe from a *Parallax*, and greatly affect the moſt intereſting Subjects of this Science, *viz.* the MOON and the PLANETS, it will be neceſſary to premiſe the following Theorem, *viz.* The *Parallax of a Planet is always proportioned to the Sine of its apparent Diſtance from the Zenith*, which is thus ſhewn. Let C be

be the Center of the Earth, P the Place of a Spectator on its Surface; Z the Zenith, and let N and *n* be two Places of a Planet in an Azimuth-circle ZO; draw PN and CN, also P *n* and C *n*, then are the Angles PNC, P *n* C, the *Parallaxes* of the Planet at the Altitudes NO, *n*O, (1724). To compare which, we have PC : CN :: *s* PNC : *s* NPC = *s* NPZ. Also PC : C *n* (= CN) :: *s* P *n* C : *s* CP *n* = *s* *n* PZ; therefore it is *s* PNC : *s* P *n* C :: *s* ZPN : *s* ZP *n*. But these parallatic Angles at N and *n*, being very small, *are as their Sines*; whence the Proposition is evident.

1899. Hence it follows, *that any Parallax at* N *is to the horizontal Parallax at* O (*viz.* that of the Planet seen from P in the Horizon at O) *as the Sign of its Zenith Distance (nearly)* to the Radius. Also it appears, that the parallatic Angle vanishes at the Zenith Z; and also when the Distance PN or CN becomes immeasurably great. So that the *fixed Stars can have no Parallax* of this Kind. Lastly, we have, *the Distance of a Planet* NC *from the Sun, to the Semidiameter of the Earth* CP, *as the Sine of the Parallax* CNP, *to the Sine of the Zenith Distance* ZPN.

1900. As the Angle ZPN is greater than ZCN by the Quantity of the Parallax PNC, it is evident the *apparent Zenith Distance of a Planet exceeds the true*, by just that Quantity. Therefore let A be the true Place of a Planet; ZA its Zenith Distance, P the Pole of the Ecliptic; and PA a Circle of Latitude passing thro' the Planet. Also let (*a*) be the apparent Place, and draw the Circle of Latitude P *a* *b* thro' the Planet A. Let EAL be drawn parallel to the *Ecliptic*; and let HLO be the Horizon. (Fig. 2.)

1901. Then in the oblique Triangle AZP, the Side ZP, and the Angle Z adjacent, are constant (1855). And the fluxionary Parts are (1.) A *a* = $\dot{a}$ (1817, 1818) the Parallax in Altitude. (2.) The Parallax in Longitude = AP *d* = $\dot{B}$. And (3.) The Parallax in Latitude *ad* = $\dot{H}$. Then because the Parallax in Altitude is known from the *horizontal Parallax* (1898) in astronomical Tables; therefore we have *s* H : *s* b :: $\dot{a}$: $\dot{B}$ = AP *d* (1819). That is, in Words, *as the Co-Sine of the Planet's Latitude, is to the Sine of the Angle* ZAP, *so is the Parallax in Altitude to the Parallax in Longitude.*

1902.

1902. Again, to *find the Parallax in Latitude*; it appears from the fluxionary Triangle A$a\,d$, right-angled at d, that A$a : ad :: R : saAd = sEAZ$, that is, $R : csZAP ::$ A$a : ad :: \dot{a} : \dot{H}$; which is in Words, *Radius is to the Co-Sine of the Angle* ZAP, *as the Parallax in Altitude, is to the Parallax in Latitude.*

1903. If E L be the Ecliptic itself, and the Planet be in, or very near it, then $sH = sAP = R$; and the $sb = sZAP = csZAE$; therefore $sH : sb :: R : csZAE$ (1899) $:: tZA : tEA$ (1797) $: \dot{a} : \dot{B}$; that is, *as the Tangent of the Planet's Zenith Distance is to the Tangent of its Distance from the* Nonagesima *Degree* E, *so is the Parallax in Altitude, to the Parallax in Longitude.*

1904. Also, in this Case, we have $R : csb :: R : sZAE :: sZA : sZE$ (1795) $:: \dot{a} : \dot{H}$; that is, *the Sine of the Zenith Distance, is to the Co-Sine of the Altitude of the* Nonagesima *Degree as the Parallax in Altitude to the Parallax in Latitude.*

1905. If we put $M =$ horizontal Parallax; then, because (1898) $\frac{sZA}{R} \times M = \dot{a} = \frac{tZA}{tAE} \times \dot{B}$, (1903) we have $\dot{B} = \frac{sZA \times tAE}{R \times tZA} \times M = \frac{sZA \times sZAP}{R^2} \times M =$ (1823) $\frac{sZP \times sZPA}{R^2} \times M$. Therefore $R^2 : sZP \times sZPA :: M : \dot{B}$. That is, *the Square of Radius is to the Rectangle of the Sines of Altitude of the* Nonagesima *Degree, and the Planet's Longitude from thence, as the horizontal Parallax is to the Parallax of Longitude.*

1906. Again, $\frac{sZA}{R} \times M = \dot{a} = \frac{sZA}{sZE} \times \dot{H}$ (1904) whence $\dot{H} = \frac{sZA \times sZE}{R \times sZA} \times M = \frac{sZE}{R} \times M$. Which gives $R : sZE :: M : \dot{H}$; that is, *Radius is to the Co-Sine of the Altitude of the* Nonagesima *Degree, as the horizontal Parallax to the Parallax in Latitude.* Hence because R and M are constant Quantities, the Parallax in Latitude will ever be as the Co-Sine of the Altitude of the Nonagesima Degree.

1907. As to the *Parallax* in *Right-ascension* and *Declination*, the Analogies are the same as before; for H Z O being the Meridian, ÆQ, the Equinoctial, its Pole N, H O the Horizon, A the

the Place of the Planet, Z D a vertical Circle, and N F a Hour-circle paſſing through it (Fig. 3.) it is evident the Triangle Z A N, and its fluxionary Parts A *a*, *a d*, A *d*, are all the ſame as in the foregoing Figure, and therefore the ſame Analogy as was uſed (1903) for finding the Parallax of *Longitude*, finds here the Parallax of Right-aſcenſion, *viz.* *s* H (= *s* A N) : *s* b (= *s* Z A N) : : $\dot{a}$ (= A *a*) : $\dot{B}$ (= B N *b*). Alſo for the *Parallax of Declination* (*a d*) it is (1904) as R : *cs* b : : $\dot{a}$: $\dot{H}$ = *a d*.

1908. The Effects of *Refraction of Light thro' the Atmoſphere*, are next to be conſidered; the general Nature of refracted Light has been ſhewn at large (1321, &c.) and that Effect is to elevate Objects in Appearance, or to make them appear higher than their true Places, which is juſt contrary to the Effect of the Parallax which depreſſes them (1899).

1909. It appears alſo from the *Theory*, that the more oblique the Rays are, the more they will be refracted; or the Rays will be more refracted as they are nearer to the Horizon; therefore the horizontal Refractions are greateſt of all, and in the Zenith there is no Refraction at all, which is the Caſe alſo in regard to the *Parallax*.

1910. Therefore if (*a*) be the true Place of a Planet (Fig. 2.) then by Refraction it will be elevated to A in the vertical Z D, which Refraction *a* A in Altitude being found by Obſervation, you will from thence find the Diminution *d* P A in *Longitude*, and the Alteration (*a d*) of *Latitude*, correſponding to the ſame by the Analogies for the *Parallax* (1903, 1904). And alſo thoſe of *Right-aſcenſion* and *Declination* (Fig. 3.)

1911. The horizontal Refraction makes a Difference in the *Time of Riſing and Setting* of the heavenly Bodies, and alſo of their *Amplitude* from the Eaſt or Weſt Points of the Horizon. Thus let (*c c*) be a Parallel of the Sun or Star's Declination, then without the Refraction P would be the Point in which it would aſcend the Horizon, Q P the *true Amplitude*; and P N O the *Hour* from *Midnight*. But if the horizontal Refraction be equal to R M, then will the Sun or Star be thereby elevated to the Horizon in the Point M, and its *apparent Amplitude* will be Q M, and the *apparent Time of Riſing* will be M N O.

1912. In the Triangle ZNP, the two Sides ZN and NP are conſtant (1847). The Side ZN = D, the Side NP = E, the Side ZP = F; the Angle PZN = B, the Angle PNZ, or PNO = A, and the Angle ZPN = C. And we have $sB : c\!f\!D :: \dot{F} : \dot{A}$ (1855) that is, *the Co-Sine of the true Amplitude* QP *is to the Secant of the Latitude* ÆZ, *as the horizontal Refraction* RM = $\dot{F}$ *is to the Angle* PNM, *which is the Difference in Time of the Riſing or Setting of the Sun or Star thereby occaſioned.* Or otherwiſe thus; becauſe $sB : sE :: sA : sF = R$ (1828) therefore $sB = \frac{sE \times sA}{R}$. Again, $c\!f\!D = \frac{R^2}{sD}$ (1854) therefore $sB : \frac{R^2}{sD} :: \dot{F} : \dot{A}$, hence $sB = \frac{R^2 \times \dot{F}}{sD \times \dot{A}} = \frac{sE \times sA}{R}$ which Equation gives this Analogy $sD \times sE \times sA : R^3 :: \dot{F} : \dot{A}$, as before.

1913. Again, for the fluxionary Variation of the *Amplitude* PM, we have $sF : ctC :: \dot{F} : \dot{B}$, (1855) but in this Caſe, $sF = R$, and $ctC = tNPO$, therefore $R : tNPO :: sB$ $(= sPO) : tNO$ (1780). Therefore $sB : tNO :: \dot{F} : \dot{B}$; that is, *the Co-Sine of the true Amplitude is to the Tangent of the Latitude, as the horizontal Refraction is to the Difference* PM *between the true and apparent Amplitude.*

1914. This apparent Amplitude is neceſſary to be known, eſpecially by Navigators, becauſe the *Variation of the Needle* depends upon it, ſince that important Article is nothing more than the Difference between the obſerved Amplitude on the real Horizon, and that of the magnetical Card.

1915. It appears from what we have formerly ſhewn, that the Orbit of a Planet about the Sun is elliptical; and ſince that is the Caſe of the Earth, its Motion will be variable, ſometimes quicker, and ſometimes ſlower, ſuch as is expreſſed in *Seconds* of a Degree in the following Table *per Hour*, for the ſeveral Months of the Year.

January ——	*December* ——	153″
February ——	*November* ——	152
March ——	*October* ——	150
April ——	*September* ——	148
May ——	*Auguſt* ——	145
June ——	*July* ——	143

1916. In Fig. 4. let Æ Q be a Quadrant of the Equinoctial, E C of the Ecliptic; B A N an Hour-circle passing through the Sun at A, and in one Hour after let its Place be at (a). In the right-angled Triangle A B Q, the Angle at Q is constant; the Declination is A B; whose Fluxion is found from the Equation $t\,\mathrm{b}\,\dot{\mathrm{a}} = s\,\mathrm{b}\,\dot{\mathrm{H}}$ (1818) which gives $t\,\mathrm{b} : s\,\mathrm{b} :: \mathrm{R} : cs\,\mathrm{b} :: \dot{\mathrm{H}} (= \mathrm{A}a) : \dot{\mathrm{a}} = \mathrm{AB} - ab$. In Words, *Radius is to the Co-Sine of the Angle* Q A B, *as the horary Motion in Longitude* A a *to the horary Variation in Declination.*

1917. Since $\dot{\mathrm{a}}$ is always as $cs\,\mathrm{b}$, it is evident that the hourly Increase or Decrease of Declination will be least of all when the Sun enters *Cancer* and *Capricorn*, and greatest of all at the Equinoxes Q, where it amounts to a *whole Minute of a Degree.*

1918. Therefore suppose H P O Q (Fig. 5) be the Horizon, Z the Zenith, N the Pole of the *Equinoctial* H C O; also let A B E be an *Almicanter* or Circle of Altitude intersecting the Equinoctial in the Points A and a; and draw Z A D and Z a F; also the Hour-circles N A G and N a I at equal Distances from the Meridian P N Q. Then if the Sun or Star had no Motion in Longitude, its Altitude above the Horizon at equal Times before and after Noon, would be the same, that is A D $=$ a F; also the Declination would continue the same, or N A $=$ N a. But, as we have shewn, the Declination alters hourly; and when the Circle of Declination N A is thereby *lessened*, as in all the *ascending Signs*, it is evident the Sun which crossed the Almicanter at A in the Morning, will not cross it at (a) in the Afternoon, but at a Point more westerly, because N d is by Supposition less than N a; therefore the Time or Hour-angle C N d will exceed that of the *Forenoon* C N A. On the other Hand, when the Declination N a from the North *increases* to N b (as in all the *descending Signs*) then the Sun comes sooner to the Almicanter at (b) and since the *Decrease* or *Increase* of Declination is known for any interval of Time in describing the Arch A C a by (1916) therefore the fluxionary Angle a N d, or a N b is known from the Triangle Z a N, where the two Sides Z a and Z N are constant.

1919. For by Case I. (1855) we have this Analogy for finding the Fluxion of the Angle N (there called B) *viz.* $s\,\mathrm{F} : ct\,\mathrm{C} :: \dot{\mathrm{F}} (= ad) : \dot{\mathrm{B}} (= a\mathrm{N}d)$ that is, *as the Co-Sine of the Sun's*

Decli-

Declination ($= s\,a\mathrm{N} = \mathrm{F}$) *is to the Co-Tangent of the Angle* Z*a*N ($=$ C) *so is the horary Difference of Declination* (a d) *in the Time between the two equal Altitudes, to the Seconds contained in the fluxionary Angle a*N*d.* On the equinoctial Days $\mathrm{NA} = 90°$, or $s\mathrm{F}$ becomes Radius.

1920. If the Moments of equal Altitudes at A and *d* before and after Noon, be observed by a Clock or Watch, then the small Equation of Time *a*N*d*, just found, is to be subducted from that interval of Time, and the Remainder will be the Time of moving through A*a*; and the Half thereof will give the Moment of the Sun or Star's Appulse to the Meridian at C, or the *equated Time at Noon*. And this must be done in the six Signs from *Capricorn* to *Cancer*, and in the other Six, the Equation will be *a*N*b* to be added to the observed Time of equal Altitudes at A and *b*.

1921. To find the Variation of the Azimuth *a*Z*d* or *a*Z*b*, we have the first Analogy (1855) *viz.* $s\,\mathrm{B}\ (= a\mathrm{NZ}) : c\!f\mathrm{D}\ (= \mathrm{ZN}) :: \dot{\mathrm{F}}\ (= a\mathrm{N} - d\mathrm{N}) : \dot{\mathrm{A}} = b\mathrm{ZF}$. In Words, *as the Sine of the Angle* ANC, *or Time before Noon, is to the Secant of the Latitude, so is the Variation of Declination in the Interval of equal Altitudes to the Variation of the Azimuth* F*b*, *or* F*k*.

1922. The Poles of the World, or Equinoctial, are found to have a Motion about the Poles of the Ecliptic, contrary to the Order of the Signs, and therefore the Intersections of these two Circles or equinoctial Points must have a real retrograde Motion; and consequently the fixed Stars will have their Distances from the equinoctial Coloure continually increasing. Now this Motion is at the Rate of 50″ *per Annum*, at a Mean; and so much therefore will the Longitude of the Stars be annully augmented. The *physical Cause* of this Motion will be hereafter explained.

1923. Let HZQ be the Meridian N, the Pole of the equinoctial ÆQ, and P the Pole of the Ecliptic EQ. Also let A be the Place of a Star, PAC a Circle of Latitude, and NAB a Circle of Declination passing thro' it, make $\mathrm{CP}c = 50''$, and let *eq* be parallel to the Ecliptic EQ, cutting *c*P in (*a*). Then is A*a* the Space through which the Star advances in one Year in its Parallel, and C*c* its Difference of Longitude in the Ecliptic. And the Difference of Declination, *viz.* $\mathrm{AB} - ab$ is

found by the first Analogy (1855) *viz.* $c\,s$D (= NP) : sB (= ANP) :: $\dot{A}$ (= Cc) : $\dot{F}$ = AB — ab, the Difference of Declination required.

1924. And to find the *Difference of Right-ascension*, we make use of the fourth Analogy of (1855) *viz.* sF (= AN) : $c\,t$C (= A) :: $\dot{F}$: $\dot{B}$ :: *Difference of Declination* : *Difference or Increase of Right-ascension.*

N. B. Most of those fluxionary Quantities, which we have given Rules for calculating here, are exemplified in Numbers, and applied to Use, in my NEW PRINCIPLES of GEOGRAPHY and NAVIGATION.

CHAP. IX.

Theorems for the Stereographic PROJECTION *of the* SPHERE *in* Plano.

1925. THE Doctrine of *Spherical Projections*, whether *Stereographic*, *Globular*, *Orthographic*, or *Gnomonical*, is wholly derived from the Principles of Perspective as we have largely shewn, and from thence demonstrated the Properties of each particular Species of Projection in a general Manner; but before we can proceed to an Application of this Art, to the Projection of *spherical Triangles*, *Dialling*, *Astronomy*, &c. we must premise a few more Principles, especially with regard to the *stereographic Projection*, in this Place.

1926. If we look back to Fig. 4. Plate VI. of *perspective Projections*, we shall observe, that if any great Circle of the Sphere be oblique to the Plane of Projection, or of the *primitive Circle* QTR; then if the Arch GT measures its Obliquity or Elevation above the Plane of the *Primitive*, that Circle will be so projected by an Eye at R, that its highest Point at G will be in the Diameter at O, distant from the Center C, by the Tangent CO of half the Complement QG of its Elevation (1698).

1927.

1927. Again, becauſe the Arch QL = GT (1697) and QB parallel to CL, the Angle BQC = QCL = Elevation of the Circle; therefore the Diſtance of the Center B, from the Center C of the Primitive, is CB = *Tangent* TD *of the Circle's Elevation* TCD.

1928. Let the Arch QOR (Fig. 7.) be compleated into a Circle AOEP, and draw the Diameter ABE; then this may be conſidered as another *primitive Circle*, on which any oblique Circle may be projected by ſetting off the Tangent of its Elevation from the Center B either Way in the Diameter AE (1927). Thus becauſe the Tangent EK (= BO) of an Arch ES of 45° is equal to Radius BE or AB, therefore the Point A is the Center of the oblique Circle OHP elevated above the Primitive in an Angle of 45° = ES = EOH. If the Elevation be of a leſs Number of Degrees than 45, the Center will fall in the Diameter within the Circle; if greater, without. Thus let BL be the Tangent of 30°, then L is the Center of the Circle OMP, whoſe Elevation is the Angle AOM = 30°. If BV be = Tangent of 75 Degrees, then V is the Center of the Circle OYP, making the Angle YOE = 75° = its Elevation above the Primitive; and ſo for any other oblique Circle.

1929. Hence appears the Method of drawing two oblique Circles through any given Point O in the Primitive, to contain a given Angle with each other, by Means of the Tangents of their Elevations. But the Centers of thoſe Circles are alſo found by the *Secants of their Elevation*, ſet off from the Point O, the extreme Point of the projected Diameter OP; for ſince the Angle QBO = GCQ (1697) the Angle CQB = QBA = DCT; and therefore the Triangles CTD and CBQ are equiangular and equilateral; for CT = CQ = Radius; and TD = CB = Tangent of GT (1927); therefore CD = BQ = OB = Secant of GT the Elevation; whence the Center B is thereby given.

1930. Hence the Angle OQT made by the Interſection of two Circles QT and OQ is equal to the Angle made by their *Radii*; for CQ = Radius of QT, and BQ = Radius of QO, and the Angle CQB = TCD = OQT the Elevation.

1931. Since, therefore, the Angle contained by the Radii of the two Circles in the Projection is equal to the Inclination of the

the Planes of those Circles, it is evident, the Planes themselves in the Projection must contain the same Angle as they do in the Sphere; that is, any Angle O Q T is of the same Quantity in the Projection as it is on the Globe itself.

1932. But as this is a principal Point in the Doctrine of Projections, we shall give another Demonstration of it, thus. Let the Eye at A (Fig. 8.) project the Angle S B R upon the Plane E S placed right before it; and suppose B D a Tangent to the Circle S B, and B C to the Circle R B, in the Point of the Intersection B. Therefore the Plane in which the Tangents B D and B C are contained, as also the Plane of the Circle E S are both perpendicular to the Plane of the Circle B E A S; and so their common Intersection C D will be perpendicular to the Line E S produced. Now the Eye at A projects the Tangent B D into F D, and the Tangent B C into F C, because the Point B is projected into F by the visual Ray A B. Through the Center G draw the Diameters A H and B L, and B I parallel to E S, and join A L, B H. Then is the Angle D B A = B L A (665) = A H B (645) = A B I (659) = B F D (631); that is, D B A = B F D, and therefore B D = D F. Then in the Triangles C D F, and C D B, since the Angles D B C and D F C are subtended at the same Distance by the same Line D C, they must be equal. But the Angle D B C = S B R on the Globe; which therefore is equal to its Projection D F C on the Plane E S continued.

1933. Any Circle B C (Fig. 9.) placed oblique to the Plane of the Circle E G is projected into a *Circle* upon that Plane by an Eye placed at A in the Pole of the Primitive E G. For draw D C parallel to E G, then because the Arch D A = C A, the Angle A B C = A C D (643) and the Angle A is common; therefore the Triangles A B C, and A C *b* or A *c e*, are similar (621). Therefore the visual Cone A C B is cut by the Planes C D and E G *subcontrarily*, and consequently the Projection of its Base B C will be a Circle at *b* C, or *e c* (1510).

N. B. B C is a *small Circle*, but had it passed through the Center F it would have been a *great Circle*, and the Demonstration the same.

1934. Hence as *e* F is the Tangent of half the Arch B H, and F *c* = Tangent of half H C; the Points *e* and *c* are given; there-

therefore the Line *ec* bisected, gives (*n*) the Center of the Circle *ec* in the Projection.

1935. A small Circle B I, perpendicular to the Plane of Projection E G, will be projected by the Eye at A into the Circle K *e*, whose Center N is in the Line E G, produced, Distant from the Center F of the Primitive, by the *Secant of the Arch* B E, (or the Distance of the Circle B I from its Pole E) and the Radius thereof will be equal to the *Tangent* of the same Arch. For draw A B cutting the Line G E in *e*; and through I draw A K intersecting the Line G E (produced) in K; then the Line K *e* bisected, gives the Center N, on which let the Circle K B I be described. Draw N B, K B H, and F B; the two Triangles A B H and K H F being right-angled at B and F, and having one Angle at H common, have also the Angle B A H $=$ F K H. But B A H $= \frac{1}{2}$ B F H (642) and F K H $= \frac{1}{2}$ F N B; therefore B F H $=$ F N B; and because F N B $+$ B F N $=$ B N F $+$ B F N $=$ to a right Angle; therefore N B F is a right Angle; and consequently N B is perpendicular to F B, and is the *Tangent* of the Arch E B; and N F is the *Secant* of the same. *Q. E. D.*

1936. From these Theorems, it is evident, we have certain Rules for describing any Circles great or small, on a given Plane; and therefore the Sphere may be projected on the Plane of any one of its Circles at Pleasure. We have already given a Specimen of three of these Projections in Plate VI. of our *Institutions of Perspective*, in Fig. 2, 5, and 6. with their Rationale, as derived from the Principles of that Science. But the *Praxis* or Method of drawing the Circles in each, are more immediately deduced from the Theorems we have just now premised.

1937. Thus the Projection on the Plane of the Equator (in Fig. 2. of that Plate) consists wholly of *right Lines*, and *concentric Circles*. The first of which are the great Circles, or Meridians, at right Angles to the Plane of Projection, and whose Planes all pass through the Eye; they are therefore all projected into right Lines. The Circles are the *Parallels* of Latitude, whose common Center is the Pole C, or Center of the Primitive, and their Distance from the said Center is equal to the Tangent of half their Distance from the Pole, and are therefore easily drawn by (1934).

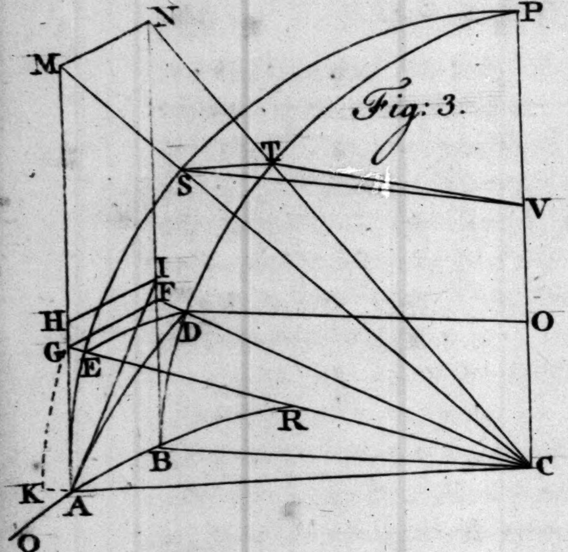
1938. The PROJECTION of the SPHERE on the Plane of a MERIDIAN, is that of Fig. 5. where all the Meridians are *great Circles* oblique to the Plane of Projection, and drawn by finding their Centers as directed in (1927) in the same Manner as the oblique Circles O H P, O Y P, and O M P, were drawn (1928); and all the Parallels of Latitude, being *small Circles*, perpendicular to the Plane of Projection, are drawn by finding their Centers as directed (in 1934 or 1935).

1939. A PROJECTION on the Plane of the HORIZON, is that of Fig. 6. of Plate VI. and the *great Circles* or Meridians are drawn by the same Rules in this as in the former Projection; for let T Q F R (Fig. 7.) represent the Plane of the Horizon where the Elevation of the Pole O is T G = T Q O, then it is evident, the Center of the *Six o'Clock* Hour-circle Q O R is at B, the Center of the Primitive A O E P, on which, if the Meridians or Hour-circles are drawn, as above directed (1938) they will, if continued beyond the Pole O, be the proper Meridians or Hour-circles for the *horizontal Projection* T Q F R. Thus for Instance, the Hour-circle of XII is O P in one, and T F in the other; that of *Six o'Clock* is A O E P in one, and Q O R in the other; the Hour-circle of II. is O Z P in one, and Z O N in the other; and so of the Rest; therefore they are the same in both Projections, and are drawn for both at the same Time, all which is evident by Inspection.

1940. The *Parallels* of *Latitude* are all oblique to the Horizon or Primitive, but are all *Circles* in the Projection (1933) and are drawn by setting off the Tangent of half their least and greatest Distances from the Pole of the Primitive, (1934, or 1935,) or Zenith of the Sphere; and bisecting that Interval, you will have the Center of every Parallel in the Line F T continued out beyond T. The *Equator* is drawn by the same Rule; and the prime Vertical is a right Line Q C R, as its Plane passes through the Eye in the Nadir or lower Pole of the Horizon.

1941. Hence it appears how great the Affinity is between Projections of the Sphere of different Denominations, or rather *that they are all but the same Thing* in different Views, and all performed by the same Rules. It is hoped the Reader will, from the Method we have here taken, have a clear Idea of the *Rationale* and *Praxis* of this most useful Branch of Science.

INSTI-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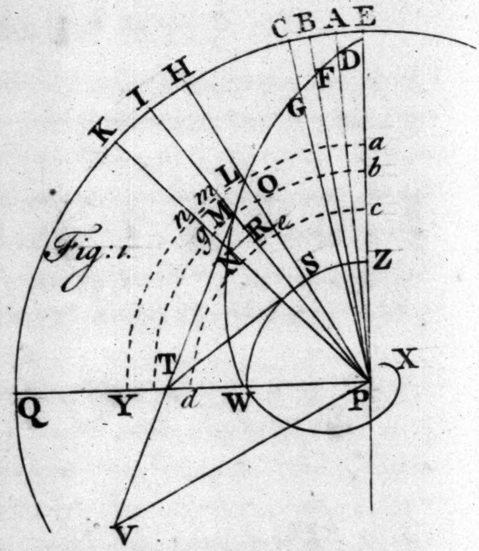
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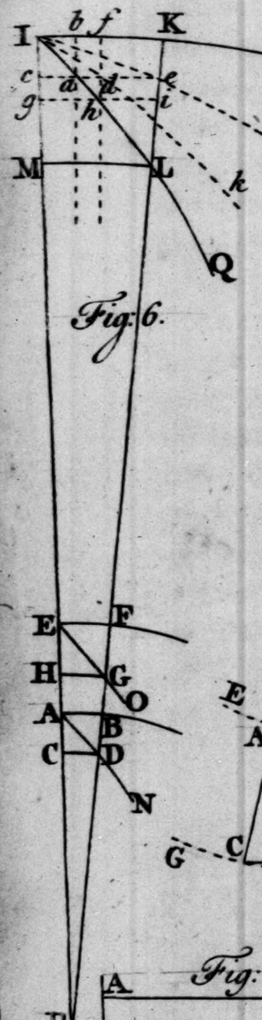
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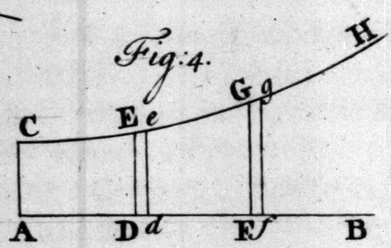
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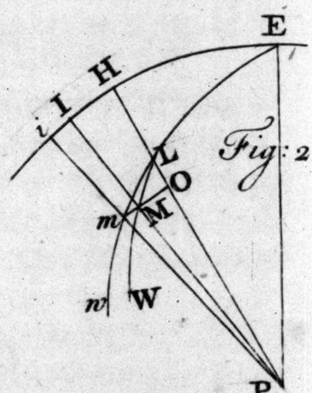
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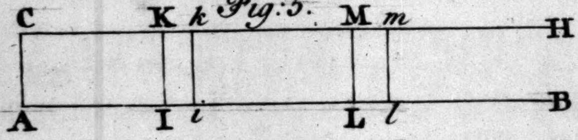
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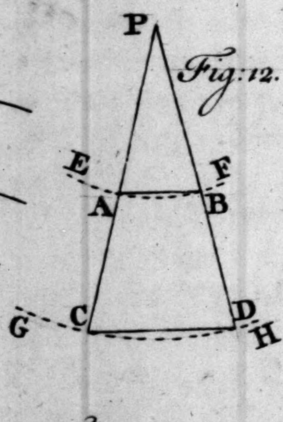
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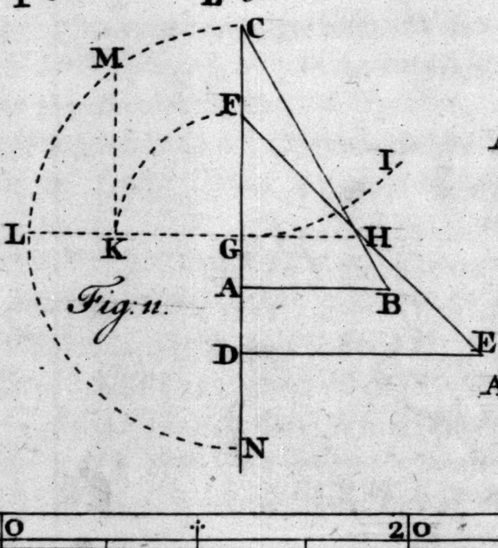
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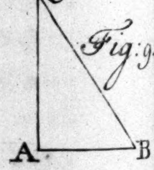
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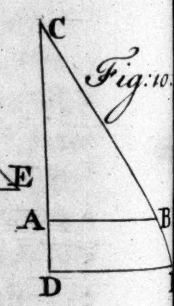
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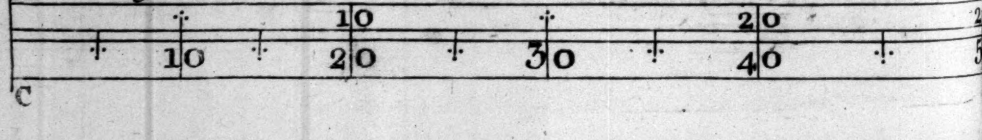


Fig: 8.

INSTITUTIONS OF LOXODROMICS;

OR,

The THEORY of NAVIGATION on a RHUMB LINE demonſtrated. And from thence each particular SPECIES of SAILING is deduced.

CHAP. I.

The THEORY *of* NAVIGATION *on a* RHUMB LINE *demonſtrated, from its* Genuine PRINCIPLES; *with the Nature and Conſtruction of* MERCATOR'S Chart.

1942. THE WAY or COURSE of a Ship, is on the Surface of a GLOBE; and therefore the *ſhorteſt Way* from one Place to another, is in the *Arch of a great Circle*; and when a Ship is conducted on ſuch an Arch, it is called ORTHODROMICS, or GREAT CIRCLE SAILING.

1943. This Method of Sailing, were it *eaſy* and *convenient*, would certainly be the beſt of all others; but ſince a *great Circle* makes different Angles with the Meridians it paſſes over, therefore the *Line* of the *Ship's Courſe* will be perpetually varying; and there will be no *conſtant Rule* for her Conduct. But for eve-

ry ſmall Diſtance ſhe ſails, a new Calculation for the Angle of the Courſe muſt be made, and this would render Navigation extremely difficult and perplexed.

1944. The eaſieſt Method, therefore, of Sailing, is to ſteer her Courſe in ſuch a Manner, that it ſhall every where make the ſame Angle with the Meridians ſhe paſſes over; and then there will be a *conſtant Guide* to direct her Progreſs, *viz.* the COMPASS BOX and NEEDLE.

1945. For ſince by the Property of *Magnetiſm*, the Needle does in every Place make a *given* Angle with the Meridian, if a Ship be ſteered upon a *given Point* of the Compaſs, the Courſe will always make a given Angle with the Meridian alſo; that is, it will croſs every Meridian under equal Angles, while the Needle continues in the ſame Poſition with regard to the Meridian.

1946. But if in different Places, the Needle makes different Angles with the Meridian, that is eaſily diſcovered by the Manner, and the Quantity of its *Variation* obſerved, and then the *Navigator* can readily keep the Ship's Courſe ſtill upon the ſame Point of the Compaſs. And this is the moſt *eſſential Part* of the Art of Sailing.

1947. Therefore it will be neceſſary to enquire into the Nature of that *Line* which a Ship deſcribes in her Courſe, or which interſects all Meridians under equal Angles, and is uſually called the RHUMB, or *Rhumb-Line*. It cannot be a *Right-line*, as being on the Surface of a Globe; it cannot be a *Circle*, as we have ſhewn (2.) it muſt therefore be a *particular Curve* which conſtantly approaches the Pole, and by cutting each Meridian under the ſame Angle, it muſt make perpetual Gyrations about the Pole, but can never terminate in it. The RHUMB, therefore, or Ship's Courſe, is of that *Species of Curves* called SPIRALS.

1948. Of SPIRALS, there are many different Sorts, one of which is called the *Equiangular* or *Logarithmic* SPIRAL; and this is the *Nautical* SPIRAL, or RHUMB-LINE, as we ſhall here demonſtrate. Let E Q be the Quadrant of a Circle, P the Center, and PE, PA, PB, PC, ſeveral Radii drawn very near to each other, and equidiſtant; that is, let EA = AB = BC. Then let the *equiangular Spiral* E W X be drawn, making

making equal Angles with the Radii every where; *viz.* A D E = B F D = C G F, *&c.* Then since the Triangles D P E, F P D, G P F are equiangular, and therefore similar; we shall have E P : D P :: D P : F P :: F P : G P, *&c.* therefore E P, D P, F P, G P, are in *geometrical Proportion*, while their corresponding Arches E A, E B, E C, are in *arithmetical Proportion*; wherefore the latter are *Logarithms* of the former, (140) that is, A E is the Logarithm of the Ratio of D P to E P; and B E is the Logarithm of the Ratio of F P to E P; and C E, the Logarithm of the Ratio of G P to E P, and so on. Therefore the Curve E W X is the *Logarithmic* SPIRAL.

1949. Let a Tangent L V be drawn to the Curve at L, and draw the Radii P L H, P M I, P N K very near to each other, and from the Points M, N, let fall the Perpendiculars M O, N R; and upon P H erect the Perpendicular P V to intersect the Tangent in V. Then because the small Triangles L M O, M N R, are similar, we have L O : L M :: M R : M N :: L O + M R : L M + M N :: L P : L W X :: L P : L V. Therefore the whole Length of the infinite Spiral L W X is equal to the finite Line L V.

1950. From P take any Distance P W, and describe the Arch of a Circle W S Z, and at S erect the Perpendicular S T, cutting the Tangent in T, then in the same Manner it is shewn that the intercepted Part of the Spiral L W is equal to the right Line L T, and of Course the remaining Part W X is equal to T V.

1951. Now let the same Figure represent the common *stereographic Projection* of the Surface of the *terraqueous* Globe on the Plane of the *Equator*, of which E Q is a Quadrant; P the *Pole* of the World; H P, I P, K P, several *Meridians*; and L W X, the *Rhumb-Line*, or Ship's Course, making equal Angles with them all. Then let H L be a given Latitude, and L P will be equal to the *Tangent of half the Complement* of that Latitude (1698, 1926); and a Ship passing from L to M makes her *Difference of Longitude* I H, *Departure* M O; and *Difference of Latitude* L O.

1952. Then it is evident, *that the Differences of Longitude* A E, B E, C E, *are Logarithms of* D P, F P, G P, *the Tangents of half the Complements of the Latitudes* A D, B F, C G; and H E, I E, K E, *are Logarithms of* L P, M P, N P, *the Tangents of*

half the Co-Latitudes of the Places L, M, and N, *in respect to the Radius* P E.

1953. Then, suppose a Ship sails from the Latitude L on the Rhumb E L M W to the Latitude M in the Parallel M O (Fig. 2.) the Logarithm of the Ratio of M P to E P is the Arch I E; but if she sails on any other Rhumb L *w*, she will come to the same Parallel in the Point (*m*,) and then the Logarithm of the same Ratio *m* P (= M P) to E P, will be the Arch *i* E. Consequently as there may be an Infinity of Rhumbs supposed to be drawn through the Point L, there will be an *infinite Variety of Scales of Logarithms* to be found in the Arches of the Equator for expressing the same Ratios.

1954. In sailing on the same Rhumb L W, (Fig. 1.) it is, *as Radius to the Secant of the Angle of the Course* O L M, *so is* L O : L M :: M R : M N :: L O + M R : L M + M N :: L P : L W X; *so is the Arch of the Meridian equal to the Co-Latitude, to the Length of the whole Spiral or Rhumb from the Latitude or Point* L. For the Triangles L M O, M N R, are equiangular and similar.

1955. Hence if L O = M R, then L M = M N; therefore the *Length of the Rhumb between any two equidistant Parallels is the same*; or the Length of the Rhumb L M is always proportioned to the Difference of Latitude L O. For L O : L S :: L M : L W. (1950.)

1956. Again, as Radius to the Tangent of the Course O L M, so L O : M O :: M R : N R :: L O + M R : M O + N R :: L P or Co-Latitude, to the Sum of all the indefinitely small Arches of the Parallels, or whole *Departure* made in sailing obliquely from one Meridian to another.

1957. Hence in very short Courses, where the Distance sailed L M, and the Difference of Latitude L O may be esteemed rectilineal; we have *Radius to the Tangent of the Course, so is Difference of Latitude* L O *to the Departure* M O.

1958. In sailing on different Rhumbs L M and L *m* to the same Parallel *m* O, (Fig. 2.) it will be *as the Tangent of the Course* M L O *to the Tangent of the Course* *m* L O, so is the Departure M O to the Departure *m* O, and so is the Difference of Longitude H I to the Difference of Longitude H *i*.

1959.

1959. And hence it appears, *that the Differences of Longitude made in Sailing upon two different Rhumbs* L W, and L *w*, *are as the Tangents of the Angles, which they make with the Meridians*; and therefore the Differences of Logarithms (I H, *i* H), of the same Ratio (*viz.* of M P or *m* P to E P) in the two different Scales appropriate to the two Rhumbs, will be likewise as those Tangents respectively.

1960. In very short Voyages, the small Triangle L M O may be esteemed *Right-lined*, and used as such without sensible Error, because a small Portion of the Surface of a Globe, near 8000 Miles in Diameter, will differ insensibly from a Plane. Hence *all Cases of Sailing* (except that for the *Longitude*) are solved by a *simple plain Triangle*, this is therefore called PLAIN SAILING, whose Principles we have now premised.

1961. But when the Length of Voyages makes it necessary to consider the said Triangle really as it is, *viz.* a *curvilineal One* throughout; and that every Part consists of a Curve of differing Species, *viz.* L O the Arch of a *great Circle*; M O, the Arch of a *small Circle*; and L M the Arch of a *Spiral*, it will easily appear, that under such Circumstances it will be no easy Matter to resolve *Cases of Sailing* on the *Surface of the* GLOBE *itself*, or by the *Globular Chart*.

1962. Therefore to facilitate the Praxis of so excellent and necessary an Art, Methods have been contrived to delineate the Surface of the Globe on a Plane, in such Manner, that all the *Calculations* relative to NAVIGATION, should still be subject to *plain Trigonometry*, and yet productive of Accuracy and Truth beyond all necessary Degrees.

1963. To illustrate this Affair, let C (Fig. 3.) be the Center of the Globe; P, the Pole; A P, B P, two Meridians standing on the Arch of the Equator Q R. Let E D, S T, be Arches of Parallels whose Radii are D O, T V. At the Points A, B, are erected the two Perpendiculars or Tangents A M, B N; and through the Points E and D, are drawn the Secants G C, F C. Then is the Arch of the Parallel E D thereby projected into the Right-line G F on the Plane between the two Tangents. Therefore G F = A B, in the same Manner the Arch S T is projected into M N = A B, all which is evident from the Doctrine of *Gnomonical Projection* (1738, &c.)

1964.

1964. But these Arches E D, S T, are when compared with the corresponding Arch A B of the Equator, as their respective Radii D O, T V to the Radius A C of the Globe. Or A B is to E D, *as Radius to Co-Sine of the Latitude*; for D O is the Sine of D P the Complement of the Latitude B D or A E.

1965. Again, the Arches E D, S T, are to their Projections G F or M N, as E C or S C, to O C and M C; or *they are enlarged in Proportion of the Secant of the Latitude to the Radius.*

1966. Suppose a Ship sails on the Rhumb A D L from A to D; then is A E the Difference of the Latitude, A B the Difference of Longitude; the Angle E A D the Course; A D the Distance sailed; and D E the Arch of the Parallel at D. Now it is required to delineate this mixtilineal Triangle A E D on a Plane in such Sort, that all the Parts *may have the same Ratio of Magnitude and Position as they now have on the Surface of the Globe.*

1967. In order to this, it must be considered, that in whatever Proportion any one Part is altered by projecting the Triangle E A D on the Plane A M N B, the other Parts must be altered in the same Ratio, to produce *Similarity*, and thereby to keep the given Positions of all the Parts to each other. But by the Projection, the Part or Perpendicular E D is enlarged to G F, in the Ratio of E C to G C (1965.) and therefore the Side or Base A E must be enlarged in the same Proportion on the Line A M.

1968. Now it is evident, the Arch A E described with the Radius A C is to the Arch G K described with the Radius G C, as Radius to the Secant of the Arch A E, or Angle A C E; if therefore in the Right-line A M, we take A H equal to the Arch G K, and draw H I parallel to A B (or Chord of the Arch A B) and draw the diagonal A I; we shall have the right-lined Triangle on the *Plane*, every Way similar and proportional to the Triangle E A D on the Globe: For it is H I (= G F) : E D :: G C : E C :: A H (= G K) : A E :: A I : A D.

1969. Hence the Angle of the Course H A I is the very same as the Angle E A D on the Globe; and therefore the Position of the Point I with regard to the Point A in the Plane, is the very same as that of the Port D in respect of the same Point A on the Globe; so that the *Easting* or *Westing* on a Rhumb A D making

ing any Angle E A D with the Meridians, is truly preserved and represented in the Triangle H A I *in Plano*; and consequently the *whole Affair of Navigation* is by such a Projection reduced to *the Solution of a plain right-angled Triangle*, and thereby becomes a very easy Task as far as Calculations are concerned. And this Projection of the Surface of the Globe on a Plane, is called MERCATOR's PROJECTION, or CHART.

CHAP. II.

The THEORY *of* NAVIGATION *by a* TABLE *of* Logarithmic TANGENTS *demonstrated*.

1970. LET Radius of the Equator P E $= r$ (Fig 1.); the Latitude of the Place H L $= z$; the Radius of the Parallel at L $= x$; and the Longitude or Arch of the Equator E H $= y$; then will H I $= \dot{y}$; and L O $= \dot{z}$; and we have P H : P L :: $r : x :: \dot{y} : \frac{x\dot{y}}{r} =$ L m, the Fluxion of the Parallel at L. And putting $t =$ Tangent of the Angle L M O or L M m, we have L m : m M :: $\frac{x\dot{y}}{r} : \dot{z} :: t : r$. Whence $\dot{z} = \frac{x\dot{y}}{t}$; and so $\frac{t\dot{z}}{x} = \dot{y}$; and $\frac{rt\dot{z}}{x} = r\dot{y}$.

1971. To construct this Equation, let the quadrantal Arch of the Meridian be extended into a Right-line A B (Fig. 4.); at A erect a Perpendicular A C $= t$; and make A D $= z$, the Latitude of the Place; upon D erect the Perpendicular D E, which is to A C $= t$, as r to x, so that it may be every where D E $= \frac{tr}{x}$; and thus by conceiving Perpendiculars to be raised on every Point of the Meridian A B, we shall have the Curve C E H produced, and of such a Nature that the curvilineal Space A D E C, divided by Radius, will be equal to the Arch of the Equator expressing the Longitude E H made in sailing from the Equator E to the Latitude Z on the given Rhumb E L W (Fig. 1.)

1972. For let de be drawn infinitely near, and parallel to DE; then AD $= z$, and D$d = \dot{z}$, and DE $\times$ D$d = \frac{tr\dot{z}}{x}$ $= r\dot{y}$ (1970.) But the Fluent of $\frac{tr\dot{z}}{x}$ is the Space ADEC, and that of $r\dot{y}$ is ry; therefore $\frac{ADEC}{r} = y =$ the Longitude upon the Course EL.

1973. After the same Manner we have $\frac{AFGC}{r} =$ the Longitude answering to the Latitude AF upon the same Course. Consequently $\frac{DFGE}{r} =$ the Difference of Longitude corresponding to the Difference of Latitude DF upon the same Rhumb.

1974. It is easy to conceive the Space ADEC, and its Fluxion DEed may be transformed into equal rectangular Spaces or Parallelograms, as AIKC, and IKki, and because in this Case IK $=$ AC $= t$, therefore I$i = \frac{r\dot{z}}{x}$, which is to Dd $= \dot{z}$, as r to x, that is, *as Radius to the Co-Sine of the Latitude, or as the Secant to the Radius.* But this is the Proportion of the protracted Arch to the Length of the natural Arch of the Meridian (1967, 1968.) And since AD is the Length of the latter, AI will represent the former, *viz.* the *enlarged meridional Arch*, used in *Mercator's Chart.* (1969.)

1975. Hence since the Rectangle AIKC $=$ AI $\times$ AC $=$ ry, or, (putting AI $=$ Z,) $Zt = ry$, therefore $r : t :: Z : y$, or *Radius is to the Tangent of the Course, as the protracted Arch or Degrees of Latitude is to the Arch of the Equator expressing the Longitude.*

1976. Hence when the Rhumb or Spiral ELW intersects the Meridians at an Angle of 45 Degrees, we have $r = t$; and then $z = y$; *the Degrees of Latitude, therefore, in the protracted Meridian of Mercator's Chart, are a* SCALE *of* LONGITUDES *in sailing upon a Course or Rhumb of* 45°.

1977. Because $r : x :: S : r$ (putting S for *Secant* of the Latitude) therefore $\frac{r}{x} = \frac{S}{r}$, and so $\frac{tr\dot{z}}{x} = \frac{tS\dot{z}}{r} = r\dot{y}$ (1970); and

when

$t = r$, as in Case of the Rhumb of 45°, then $S\dot{z} = r\dot{y}$; and if $r = 1$, as in the Canon of natural Sines, Tangents, and Secants; then $S\dot{z} = \dot{y}$; whence it appears, *that the Fluxion of the Longitude* $(\dot{y})$ *is equal to the Fluxion of a Space or Area consisting of Secants successively erected on the fluxionary Points* $\dot{z}$ *of the Meridian.*

1978. Because $\dot{z}$ is constant in the Equation $S\dot{z} = \dot{y}$, we have $S = \dot{y}$, or the Fluxion of the Longitude $(\dot{y})$ will be as the variable Secant S; and so the Sum of all the Fluxions or the *Longitude itself, will be as the Sum of all the Secants* to any given Latitude. And hence the Reason why Mr. *Wright* took this Method to construct his Line of meridional Parts, *viz. by the constant Addition of the Secants,* as they are found in the Canon. See his *vulgar Errors in Navigation* corrected; where the *nautical meridian Line* first made its Appearance.

1979. It has been shewn, that the *Fluxion of the Logarithm of any Number is equal to the Fluxion of the Number divided by the Number itself* (849). But the Longitudes are Logarithms of the Tangents LP, MP, &c. to the Radius EP (1952) such a Tangent therefore is a Number, which put $= n$; then, when $r = 1$, we have $tS\dot{z} = \dot{y} = \frac{\dot{n}}{n}$ (1977) which gives $n\dot{y} = \dot{n}$, and therefore $1 : n :: \dot{y} : \dot{n}$; or PH (= PE) : PL :: IH : Lm (Fig. 1.)

1980. And also, when $t = 1$, as upon a Rhumb of 45°, we have $S\dot{z} = \frac{\dot{n}}{n}$, whence it appears, that the *Fluent* of $S\dot{z}$, which is an Arch of the *nautical Meridian,* (1978) *is equal to the Fluent of* $\frac{\dot{n}}{n}$, *or the Logarithm of the Tangent of half the Co-Latitude of that Arch.* Therefore the *whole nautical meridian Line is a Scale of Logarithmic Tangents of the half Complements of the Latitudes, upon that Rhumb which makes an Angle of* 45° *with the Meridian.* Therefore this we shall call the *nautical Scale of Logarithmic Tangents.*

1981. But it is evident from the original Equation $\frac{ADEC}{r} = y =$ Longitude EH, or Logarithm of the Ratio of PL to PE (1972.) that the Logarithm of the same Ratio will be vari-

able according to the different Values of r, or the Radius of the Globe PE. For HI is the Logarithm of the Ratio of the Tangent PL to Radius PH or PE, when $r = PE = 1$, and the Angle $MLO = 45°$. But if r or PE be expounded by any Number less or greater than Unity, it follows, that the Length of $HI = y$ will be encreased or diminished in the same Proportion; for the Area ADEC being considered as constant, the Logarithm (y) will be always as $\frac{1}{r}$ or *inversely as the Radius.*

1982. From the fluxionary Equation $\frac{tr\dot{z}}{x} = r\dot{y}$ (1970), it appears that near the Beginning of the Rhumb Line in the Equator, where $x = r$, that Equation becomes $t\dot{z} = r\dot{y}$, and then $r : t :: \dot{z} : \dot{y}$. And therefore upon the Rhumb of 45°, where $t = r$, we have $\dot{z} = \dot{y}$. Therefore *the Fluxion* $(\dot{y})$ *of the Logarithm of the Ratio of the first mean proportional Tangent to the Radius is equal to the infinitely small Difference* $(\dot{z})$ *between that proportional and Radius.*

1983. Hence if Radius $PE = 1$ (Fig. 1.) and the first Proportional be $PD = 1 - \dot{z}$, the second $PF = \overline{1-\dot{z}}^2$, the third $PG = \overline{1-\dot{z}}^3$, &c. then will the Logarithms be $AE = \dot{z}$, $BE = 2\dot{z}$, $CE = 3\dot{z}$, &c. therefore $\overline{1-\dot{z}}^n$ is any Proportional $1 - z$, and its Logarithm is $n\dot{z}$. But since $\overline{1-\dot{z}}^n = 1 - z$, it is $\sqrt[n]{1-z} = 1 - \dot{z} = \overline{1-z}^{\frac{1}{n}}$; whence $1 - \overline{1-z}^{\frac{1}{n}} = \dot{z}$. But by the *Newtonian Theorem* for extracting the Roots of Binomials (306) we get $\overline{1-z}^{\frac{1}{n}} = 1 - \frac{1}{n}z - \frac{1}{2n}z^2 - \frac{1}{3n}z^3 - \frac{1}{4n}z^4$, &c. or $\frac{1}{n} \times \overline{z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \frac{1}{5}z^5}$, &c. $= 1 - \overline{1-z}^n = \dot{z}$; whence $z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \frac{1}{5}z^5$, &c. $= n\dot{z} =$ Logarithm of the Number $1 - z$.

1984. This Series is the same with that derived from the *Hyperbola* (829), and when computed in Numbers gives 2302585 for the Logarithm of the Ratio of 10 to 1, or of 1 to 0,1. Consequently

sequently 2302585 $= n\dot{z}$; or so many mean Proportionals there are in the Ratio of 10 to 1 in this Scale of Logarithms which are those first invented by *L. Neper*.

1985. But in Practice it was found much more convenient to put the Logarithm of the Ratio of 10 to 1 $= 1000000 = \text{n}\dot{z}$, as in the common logarithmic Canon. Hence we have $n : \text{n} ::$ $2302585 : 1000000 :: 1 :: 0,43429448$. These Numbers, therefore express the Ratio of the Logarithms in each System respectively.

1986. From what foregoes, it appears, that the infinite Series just now found, answers to the Space ADEC $= y$ (1972) when $r = 1$; and $n\dot{z} = y =$ the Length of the *nautical Meridian* for the Latitude z. Then putting R for the Logarithm of the Tangent of 45° or Radius, and T for the tabular Tangent of any other Arch, as that which denotes the half Co-Latitude of HL; we have, as $EP : LP :: 1 : 1 - z = t =$ natural Tangent of $\frac{1}{2}$ LP. Therefore the Ratio of $\frac{EP}{LP} = \frac{1}{1-z} = \frac{1}{t}$, but the Logarithm of $\frac{1}{t}$ is equal to the Logarithm of Radius 1 less the Logarithm of t. That is, the tabular Logarithm of $\frac{1}{t} = R - T$.

1987. But the tabular Logarithm of any Ratio is to *Neper*'s Logarithm of the same, as 1 to 2,302585 (1985); therefore $\overline{R - T} \times 2,302585 =$ the Length of the protracted meridional Arch HL $= z$, in the Scale of *Neper*'s Logarithms. Put $N = 2,302585$, and $\mathcal{T} =$ Logarithm Tangent of any other Arch, as that of $\frac{1}{2}$ NP. Then because $NR - NT$ is the Length of the Arch HL, and $NR - N\mathcal{T}$ is the Length of the Arch KN, we have $NT - N\mathcal{T} = \overline{T - \mathcal{T}} \times N$, the Difference of the Latitude L and N expressed in Parts, of which the Radius EP is 1.

1988. If the Radius be expressed in Minutes of a Degree, then the *protracted meridian Arch will be had in Minutes likewise.* Now the Circumference of a Circle is to Radius as 6,28318 to 1. But in that Circumference there are 360° or 21600′; therefore say, as $6,28318 : 1 :: 21600' : 3437',747$, which is the

Number of Minutes in the Radius. Put this Number 3437′,747 $= m$; then $mN \times \overline{T - \mathcal{T}}$ will expreſs the Length of the Difference of Latitude in Minutes of the *protracted Meridian*; and thus a Table of *meridional Parts* or *Minutes*, may be computed from the *common Table of logarithmic Tangents.*

1989. For Example, *let it be required to find the meridional Parts in the Arch of the Meridian contained between the Tropic of Cancer and the arctic Circle.* Then the Latitude of the Tropic is 23° 30′, its Complement is 66° 30′, and the Half thereof is 33° : 15′, whoſe logarithmic Tangent is $9{,}816658 = T$; the Latitude of the arctic Circle is 66° : 30′, its Complement 23° : 30′, and the Half of that 11° : 45′, whoſe logarithmic Tangent is 9,318064; then $\overline{T - \mathcal{T}} = 0{,}498594$; and $mN = 7915{,}704$; therefore $mN \times \overline{T - \mathcal{T}} = 0{,}498594 \times 7915{,}704 = 3946{,}722$, the *meridional Parts or Minutes* in that protracted Difference of Latitude.

1990. As in all Circles, the *Radii* P A, P E, P I, (Fig. 6.) are proportioned to ſimilar Arches in their Peripheries A B, EF, I K; and as thoſe Arches are Logarithms of the Ratios of D P to A P, G P to E P, L P to I P, in ſailing on the Rhumbs A N, E O, I Q, making the ſame Angle with the Meridians in all, therefore it follows that the *Radii* are the *Modules* or conſtant Expreſſions of the Ratio of the Logarithms pertaining to their ſeveral Peripheries or Syſtems reſpectively. Thus if I P be *Radius* or the Module for the Logarithms of *Neper*'s Sort, and be expreſſed by *Unity*, then the *Module* or *Radius* for the Logarithms of the common Tables will be $0{,}43429448 = EP$ (1985) and then if the Arch I K be *Neper*'s Logarithm of that Ratio of 10 to 1, E F will be the Logarithm of the ſame Ratio in the tabular Syſtem. So that if $IK = 2{,}302585$, we have $EF = 1$.

1991. Again, if A N be the Rhumb which makes an Angle of 45° with the Meridians, and A B an Arch of *one Minute*; then making $AC = AB$, we have A B D C a Square, and $DC = AC = AB$; and ſince the Radius A P is in this Caſe to be conſidered as divided into *Minutes* and equal to 3437′,747, (1988) which is but a Part of P I; the ſaid Radius P I, if it be divided throughout into the ſame equal Parts or Minutes, will contain juſt 10000. For $IP : AP :: 1 : 0{,}3437747$, which therefore is

is the *Module* for the System of Logarithms pertaining to the Rhumb of 45°, or the *nautical meridian Line*.

1992. The *Modules*, therefore, or *Ratios* of the three Systems of Logarithms are these, *viz.* for *Neper*'s Logarithms PI = 1,000000; for the *Tabular* or *Brigg*'s Logarithms PE = 0,434294; and for the nautical Logarithms we have PA = 0,343775. Therefore if EO and IQ be the same Rhumb of 45°, then by making EH = EF; and IM = IK; the Triangles ACD, EHG, and IML will be similar; and we have AC : DC :: EH : GH :: IM : LM.

1993. Hence because AC = AB = 1 Minute, we have as 0,343775 : 1 (:: PA : PI :: AB : IK) :: 1 : 2,90888 = IK. And this is the Measure of the *natural Sine* and *Tangent* of 1 Minute in the Tables. Again, AP : EP :: 0,343775 : 0,434294 :: (AB : EF ::) 1 : 1,2633114 = EF, the Logarithm of the same Ratio in *Brigg*'s *System*.

1994. Since then the Logarithms of the same Ratio pertaining to the *same Rhumb*, are expressed by the different Numbers 1, 1,263, 2,909, in the three different Systems above-mentioned. It is easy to conceive, that upon different Rhumbs in the same System, the same Numbers will express or be the Logarithms of the same Ratio. For let PI be Radius = 1; and take I*c* = AC, and I*g* = EH, and draw the Parallels *ce* and *gi*; and the Squares I*cab* = ACBD, and IG*hf* = EHGF; then it is I*c* : *ca* :: I*g* : *gh* :: IM : ML = *ce*; whence I*c* : IM :: *ca* : *ce* :: 1 : 2,909 :: Tangent of *a*I*c* = 45° : Tangent of *e*I*c* = 71° : 1′ : 42″. And therefore if a Ship sails on the Rhumb I*el* making that Angle with the Meridians, *Neper*'s *Logarithms will be a Scale of logarithmic Tangents of the half Co-Latitudes every where.*

1995. In the same Manner we have I*c* : I*g* :: *ca* : *gh* = *cd*, that is 1 : 1,2633 :: Tangent of *a*I*c* = 45° : Tangent of *d*I*g* = 51° 38′ 9″. Consequently, if a Ship sails on a Rhumb I*dk* making that Angle with the Meridians, *then (because* cd = EF) *the logarithmic Tangents*, in the common Canon, are *those which every where express the Ratio of the Tangents of half the Co-Latitudes to Radius*; and therefore the Difference *of the logarithmic Tangents of the half Co-Latitudes will be as the Difference of Longitude* cor-

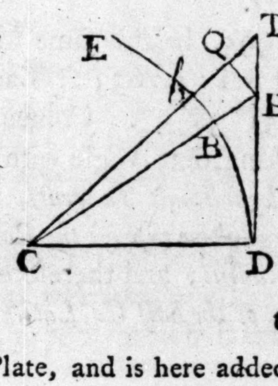
corresponding to the given Difference of Latitudes on the Rhumb of 51° 38′ 9″.

1996. But this Difference of logarithmic Tangents or Longitudes, is an Arch of the Equator, whose Radius, or natural Tangent in the Tables, is 1. Therefore, putting $\overline{T - \mathcal{T}}$ for this Difference of logarithmic Tangents (1987) we must say, as $0{,}434294 : 1 : 4342',94 : 10000' =$ Minutes of a Degree in the Radius or Arch $= 1$ (1990). Then say, as the Arch $1 : 10000 :: \overline{T - \mathcal{T}} : \overline{T - \mathcal{T}} \times 10000 =$ Minutes of a Degree in an Arch of the Equator expressed by $\overline{T - \mathcal{T}}$.

1997. Hence the principal Cases of *Mercator's Sailing* are solvable by the common Tables of logarithmic Tangents; for if the Difference of Latitude be given, the Quantity $\overline{T - \mathcal{T}} \times 10000$, or Difference of Longitude corresponding thereto on the given Rhumb of 51° 38′ 9″ will be known also; and then, if the Difference of Longitude be given on any other Rhumb or Course, the Tangent thereof will be known likewise. For, *as the Difference of Longitude* $\overline{T - \mathcal{T}} \times 10000$ *is to the constant Tangent of* 51° 38′ 9″, *so is the given Difference of Longitudes to the Tangent of the Course on which the Ship sails* (1958, 1959).

1998. From the fluxionary Equation $\frac{rt\dot{z}}{x} = r\dot{y}$ (1970) it appears, that upon the Rhumb of 45°, when $t = r$, the Equation $\frac{r\dot{z}}{x} = \dot{y}$, will then express the *Fluxion of the nautical meridian Line*, which in its nascent State, or very beginning (where $x = r$) will become $\dot{z}$, or the Fluxion of the natural Meridian itself.

1999. Now the *nautical meridian Line* begins from the Radius with the *natural Meridian*; but the *artificial Line* of Tangents begins from the Tangent of 45°. But the Fluxion $\dot{z}$ is equal to $\frac{1}{2}\dot{t}$, or half the Fluxion of the Tangent of 45°, as is thus shewn. On the Center C describe the Circle D B E, and draw the Tangent D P, the



N. B. Fig. 7. was forgot to be put in the Plate, and is here added.

the Secant CP, and infinitely near it CT; and describe the small Arch PQ. Then put Radius $CD = r$, $DP = t$, $DB = z$, $TP = \dot{t}$, $Bb = \dot{z}$; and by similar Triangles we have $TP : QP :: CP : CD$; and $PQ : Bb : CP : CB = CD$. Therefore $TP : Bb :: CP^2 : CD^2$. That is, $rr + tt : rr : \dot{t} : \dot{z}$; and when $t = r$, then $2 : 1 :: \dot{t} : \dot{z}$, or $\dot{t} = 2\dot{z}$.

2000. Because $\dot{z}$ is the Fluxion of the nautical Meridian (1998) it is the *Fluxion of the logarithm Tangents* proper to the Rhumb of 45° (1980). And since the Fluxion of the Logarithm of any Tangent is $\frac{\dot{t}}{t}$, therefore when such a Tangent becomes equal to Radius, or $t = r$, then the Fluxion of its Logarithm is barely $\dot{t}$; but we have shewn that in such a Case $\dot{t} = 2\dot{z}$. *Therefore the Fluxion of our artificial Line of logarithm Tangents is double the Fluxion of the nautical meridian Line of logarithm Tangents.* And, *so the Fluents or Degrees of logarithm Tangents in one will be double to those of the other every where.*

2001. Hence it appears, that any Line of artificial Tangents is the same with the nautical meridian Line, *if we take every half Degree for a whole one, and number them accordingly.* Hence any such *Line of logarithm Tangents* may be used or substituted for a *Line or Table of meridional Parts*, and all the Problems of *Mercator*'s *Sailing* are solved thereby. The Identity or Coincidence of these two Lines is represented in the double Scale (Fig. 8.) where AB is the common Line of artificial Tangents, and CD the *nautical Meridian*; one to 25 Degrees, the other to 50.

CHAP. III.

The different SPECIES *of* LOXODROMIC SAILING, *deduced from the preceeding* THEORY, *and illustrated by proper* DIAGRAMS.

2002. FROM a particular Consideration of the Nature of what is called *Departure* will result another *Species of Navigation* called MIDDLE LATITUDE SAILING. It is evident, by In-

Inspection (Fig. 1.) that which is called *Departure* is only the Arch of a Parallel contained between two Meridians, as P H, P I, passing through the two Latitudes L and M; and is always to be reckoned in that Parallel to which the Ship arrives. Thus if she sails from L to M, the Departure is the Arch M O, passing thro' the Latitude M. But if the Ship sails from M to L, then the Departure is the Arch L *m*, so that there is in Reality, two different Departures belonging to the same Course, Distance, or Difference of Latitude.

2003. Therefore to avoid all Ambiguity and Falacy in Sailing, it is necessary to make constant Computations of the Departure upon Parallels so near together, that the Arches M O and L m may differ but by an infinitely small Quantity, or that the Triangle L O M may be esteemed rectilineal; and then we shall have the several Departures M O, N R, &c. computed as they rise in passing over the several Meridians, as from L to M, from M to N, &c. till at last the Ship arrives at some Distant Port W in Latitude Q W, and the Sum of all these Departures will be the whole Departure made in the Voyage from L to W.

2004. But since the particular Departures M O, N R, &c. proceed continual decreasing from L to W, 'tis evident their Sum will be greater than W S, which is the Sum of the same Number of Departures each of them equal to the least of those made in the Voyage. And on the other Hand if the Ship sails from Latitude W to Latitude L, the Sum of all the particular Departures will be the same as before, but less than the Arch Y L, which is the Sum of the same Number of Departures, each equal to *m* L, the largest of those made in the Voyage.

2005. Consequently, the Departure made in the Voyage from L to W, or from W to L, being the same; and consisting every where of Quantities equidifferent, the greatest and least of which are as L Y to W S; the Sum of all the Departures will be a *Mean* between the two, or as $\frac{LY + WS}{2} = de$, the corresponding Parallel, or mean Arch of the *middle Latitude* H e (136), which therefore will nearly represent the whole Departure of the Voyage either Way between the two Ports L and W. (318.)

2006. Hence

2006. Hence we deduce this *fundamental Theorem*, *The Co-Sine of Middle Latitude* P e *is to Radius* P H *as the Departure* d e *is to the Difference of Longitude* H Q. Note, If $\frac{1}{2}$ the Sum of the Co-Sines of the Latitudes L and S be taken, it will be nearer the Truth than the Co-ſine of Middle Latitude.

2007. Put Radius = R; Middle Lat. = M; Departure = D; Diff. of Lat. = L; Diff. of Long. = G; the Angle of the Courſe = C; and Diſtance run = S. Then by the laſt, we have $cs\,M : R :: D : G$; and by (1957) it is $R : t\,C :: L : D$. Therefore $R \times D = cs\,M \times G = t\,C \times L$, which gives this Theorem, *the Co-Sine of Middle Latitude is to the Tangent of the Courſe as the Difference of Latitude is to the Difference of Longitude.*

2008. Then becauſe the Diſtance ſailed L M is to the Departure M O, as Radius to the Sine of the Courſe; that is, as $S : D :: R : s\,C$; we have $R \times D = s\,C \times S = cs\,M \times G$; which gives this Theorem, *as Co-Sine of Middle Latitude is to the Sine of the Courſe, ſo is the Diſtance run to the Difference of Longitude.*

2009. Having thus premiſed the Principles of each *Species* of *Sailing*, we ſhall illuſtrate the ſame by Figures adapted to them ſeparately. And firſt, as to PLAIN SAILING, we have ſhewn (1957, *&c.*) that it conſiſts only in the *ſix Caſes* of a *plain right-lined Triangle*; which therefore let be denoted by A B C (Fig. 9.) in which A C is the *Difference of Latitude*; A B, the Departure; B C, the Diſtance ſailed; and A C B, the Angle of the *Courſe*; any two of which being known, the reſt are found by *plain Trigonometry* (711 to 716.)

2010. MERCATOR'S SAILING has all its Caſes expreſſed in two ſimilar Triangles in one Figure, *viz.* A C B and D C E (Fig. 10.) in which A C is the Length of the Arch of the Meridian in proper or natural Degrees of the Latitude, or Difference of Latitude between two Ports. C D is the ſame enlarged according to (1968) and is called the *meridional Difference of Latitude*. A B is the *Departure*. D E the *Difference of Longitude*. C B, the *natural*, and C E the *enlarged Diſtance* run, and C the *Angle of the Courſe*. Here are alſo *ſix Varieties* or different Caſes of Sailing from two *Data*, as before.

2011. In MIDDLE-LATITUDE SAILING, two plain right-angled Triangles are alſo uſed, but not *ſimilar* ones, *viz.* A C B

and D F E (Fig. 11.) in which A C = F D is the *Difference of Latitude*. A B the *Departure*, D E the *Difference of Longitude*, C B is the *Distance sailed*; and C the *Angle of the Course*. Two of these being given, the others are found by the Theorems above (2006, &c.)

2012. But that the Reason of this complex Figure of Triangles may appear, it is to be observed, that if G H be drawn parallel to A B or D E, it will be the *Tangent* of the Course C, and G F the *Co-Sine* of the *Middle Latitude* to the Radius C G. For on the Center G describe the Semicircle C L N; and from L setting off the Middle Latitude to M, we have M K the *Sine*, and K G = F G the *Co-Sine* thereof; and therefore by similar Triangles G F H, D F E, we have GF : GH :: FD : DE; that is, the *Co-Sine of Middle-Latitude is to the Tangent of the Course, as the Difference of Latitude to the Difference of Longitude*; as before in (2007).

2013. Again; we have CG : GH :: CA : AB; and GF : GH :: DF (= AC) : DE : consequently it is GF : CG :: AB : DE, or *Co-Sine of Middle Latitude to Radius, as the Departure to the Difference of Longitude*, agreeable to (2006). And so of the other Theorems.

2014. Another Species of Navigation, is called, Parallel Sailing; which consists in conducting the Ship due East or West upon a Parallel to the Equator. The Diagram by which this is represented, is that of Fig. 12. Where P is the Pole of the World; P C, P D, two Meridians; G H an Arch of the Equator, and E F a similar Arch of a Parallel of Latitude. Then we have PC : CD :: PA : AB, or *Radius to the Difference of Longitude as Co-Sine of the Latitude to the Distance sailed in the Parallel* (1964).

2015. This Sort of Sailing therefore admits of three Cases only, *viz.* (1.) When A B and A P are given to find C D. (2.) When A B and C D are given to find P A. And (3.) When A P and C D are given to find A B. All which is evident by Inspection.

CHAP.

CHAP. IV.

The THEORY *of* NAVIGATION *by* Logarithmic TANGENTS *exemplified in a Solution of the ſeveral Caſes of* SAILING *by that* Method.

2016. AS all the above Methods of computing the Way and Courſe of a Ship conſiſt in the Solution of a plain right-angled Triangle, they require no Example; but the Manner of ſolving the Propoſitions of Sailing by the Tables of *Logarithmic Tangents* is not quite ſo common, nor ſo clear, in what we have had hitherto publiſhed on that Subject. It may be proper therefore to illuſtrate the Theory we have given by a Solution of ſuch Caſes in which the *Longitude* is concerned, eſpecially as it will from thence appear that this Invention ſuperſedes the Neceſſity of a Table of *meridional Parts*, and is when joined with *plain Sailing*, a compleat Compendium of practical Navigation.

2017. If we retain the Notation in 1987, *&c.* and, moreover, put L for the Difference of Longitude, C for the Ship's Courſe, and G for the conſtant natural Tangent of $51° : 38' : 9''$; then we have this Analogy $\overline{T - \mathcal{T}} \times 10000 : G :: L : tC$ (by 1997) which admits of *three Caſes*, beſides thoſe of plain Sailing, *viz.*

2018. CASE I.

Given the Difference of Latitude, and Difference of Longitude, to find the Ship's Courſe; the Theorem is $\frac{G \times L}{\overline{T - \mathcal{T}} \times 10000} = tC$.

2019. CASE II.

Given the Difference of Latitude and Courſe, to find the Difference of Longitude; the Theorem is $\frac{\overline{T - \mathcal{T}} \times 10000 \times tC}{G} = L$.

2020. CASE III.

Given the Courſe, and Difference of Longitude, to find the Difference of Latitude; the Theorem is $\frac{G \times L}{tC \times 10000} = \overline{T - \mathcal{T}}$.

2021. For an Example of the first Case, suppose a Ship sails from the *Lizard* in Latitude 49° 55′ North to *Barbadoes* in Lat. 13° 10′ North; the Difference of Longitude being 53° 00′, required the Rhumb on which the Ship was steered?

The Solution of this Case is as follows:

	Lat.	Comp.	Halves	Log. Tang.
Barbadoes	13 10 —	76 50 —	38 25 —	9,8993082 = T.
Lizard	49 55 —	40 05 —	20 2½ —	9,5620477 = *T*.

The Difference is —— $T - \mathcal{T}$ = 0,3372605
Multiply by —— —— —— 10000

The Product is $\overline{T - \mathcal{T}} \times 10000$ — = 3372,605
Then the Tang. of 51° 38′ 09″ = G = 12633210 —— —— —— } 10,1015104
Mult. Diff. of Long. L = 53° 00′ = 3180 — 3,5024271

The Product is G × L —— 13,6039375
Which divide by $\overline{T - \mathcal{T}} \times 10000$ = 3372,605 —— —— — } 3,5279654

The Quotient is the Tangent of the Course required } *t* C = 49° 59′ 10″ — 10,0759721

2022. Then to find the Distance sailed, use this Proportion of *plain Sailing*, viz. *As Co-Sine of the Course is to Radius*, (*or as Radius to the Secant of the Course*) so is the Difference of Latitude A C to the Distance run C B = 3429,38 nautical Miles or Minutes of a Degree. (See Fig. 9.)

2023. Example to Case II.

Suppose a Ship sails from Latitude 49° 55′ *to Latitude* 13° 10′, *on a Rhumb or Course of* 49° 59′ 10″; *to find the Difference of Longitude.* The Solution will stand thus. (See Theorem 2019).

Multiply $\overline{T - \mathcal{T}} \times 1000$ = 3372,605 — 3,5279654
By Tang. of the Course *t* C = 49° 59′ 10″ — 10,0759721

13,6039375
Divide by constant Tang. G = 51° 38′ 9″ — 10,1015104

The Quotient is Diff. of Long. L = 53° 00 — 3,5024271

Thus,

Thus, by two good Obſervations of the Latitude and the Courſe ſteered, the Reckoning of a Ship's Way is beſt aſcertained, eſpecially if you ſail near the North or South.

2024. Example to CASE III.

Admit a Ship ſails from Latitude 49° 55′ *on a Courſe of* 49° 59′ 10″ *ſoutherly, till her Difference of Longitude be* 53° 00′, or 3180 *Miles; to find the Latitude ſhe is then in.* The Solution of this Caſe is thus (by Theorem 2020).

To the Log. of the conſtant Tang. G ——	10,1015104
Add the Log. of the Diff. of Long. L = 3180 —— ——	3,5024271
The Sum is the Log. of G × L ——	13,6039375
Then to the Tang. of Courſe = 49° 59′ 10″ —— —— ——	10,0759721
Add the Logarithm of 10000 =	4,0000000
The Logarithm of t C × 10000 ——	14,0759721
Then $\frac{G \times L}{t\,C \times 10000} = \overline{T - \mathcal{T}} =$ 0,3372605 ——	9,5279654

Now one Latitude being given 49° 55′, the Logarithm Tang. of half its Complement is — 9,5620477 = $\mathcal{T}$.
To which add —— 0,3372605 = T — $\mathcal{T}$.

The Sum is the Log. Tang. 38° 25′ — 9,8993082 = T.
The Double of which is 76° 50′, which is the Complement of 13° 10′, the Latitude of the Ship required. (See 2021).

2025. From hence it is evident, that theſe three Caſes, added to thoſe of *plain Sailing*, are ſufficient for all practical Computations of a Ship's Ways, or for keeping a Reckoning at Sea, which are all performed by a *Table of Logarithmic Tangents*, or by the *common Gunter* properly made, as we ſhall hereafter ſhew, when we come to give the *Theory and Conſtruction of mathematical Inſtruments*.

2026. What has been hitherto delivered is upon the common Hypothesis, that the *terraqueous Globe* is of a *spherical Figure*, but that it is not such is well known; and not only so, but the Difference therefrom is so considerable, as not to be without its sensible Effects in *Astronomy*, *Geography*, and *Navigation*, as I have at large shewn in my NEW PRINCIPLES of GEOGRAPHY and NAVIGATION, and have there constructed large Charts to 70 Degrees Latitude, and new Table of meridional Parts adapted to the true *spheroidical Figure* of the Earth; by which the Solutions of all Cases of Sailing are rendered just as easy as by the false Charts and erroneous Tables in present Use, which are no small Diminution of the Glory of *naval Science* in ENGLAND. What farther relates to this Subject we propose to deliver in the *Theory* and Construction of Instruments used in Navigation (in which there is great Room for Improvement) but this we must defer to a future Part of this Work, and proceed with our Institutions relative to the Theory of the Sciences, which ought to precede the mechanical and instrumental Part.

PHY-

PHYSICO-BALLISTICS:

O R,

INSTITUTIONS of GUNNERY.

CONTAINING

The *philosophical* and *mathematical* ELEMENTS of that Art, connected with, and illustrated by the *Experiments* published by the late Mr. ROBINS.

CHAP. I.

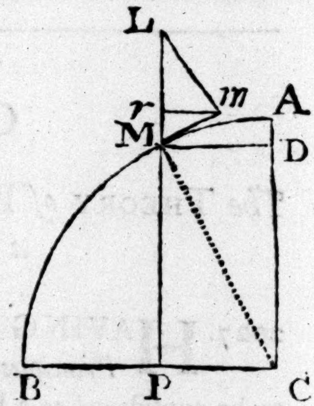
The THEORY *of* RESISTANCE *to* BODIES *moving in a* resisting Medium.*

2027. HAVING in the preceding Part delivered the true *Principles* of the Art of *navigating* a *Ship*, which is to be considered as a Part of *Military Science*, we think it may be very properly connected with its Sister Science, BALLISTICS, or

* It appears, (says Mr. ROBINS) that the modern Writers on the Art of Gunnery have been very much deceived, in supposing the Resistance of the Air to be inconsiderable, and thence asserting, that the Track of Shot and Shells of all Kinds is nearly in the Curve of a Parabola. That by this Means it has happened, that all their Determinations about the Flight of Shot discharged with considerable Degrees of Celerity are extreamly erroneous, and consequently that the present Theory of Gunnery in this its most important Branch is useless and fallacious. *Preface to principles of Gunnery.*

or the *Doctrine* of *Projectiles* (not *in Vacuo*, but) in a *resisting Medium*, especially as nothing considerable has been wrote directly on this Subject in our Language to explain the true PRINCIPLES or THEORY of GUNNERY, nor any Thing of a practical Nature, but a Treatise of Gunnery confirmed by Experiments by Mr. ROBINS, an excellent Performance it is true, but unless the *physical* PRINCIPLES or THEORY on which the greatest Part depends, be explained and connected with the Experiments, the *Art* of GUNNERY will, after all, be deficient in the most essential Part, as we presume will be sufficiently evident from the ensuing Discourses.

2028. Since the Path of Projection is through a *resisting Medium*, the first Thing necessary is to establish the THEORY of RESISTANCE to Bodies moving in a *fluid Medium*, where the Particles, both in the Body and the Medium, are supposed to be absolutely void of any elastic Force, and to act upon each other only by Virtue of the *Vis insita*. We shall here also suppose the Fluid to be quite perfect, or free from all *Tenacity* of Parts, and that the Friction is of Course nothing at all.



2029. Therefore let A M B be any Curve moving in a Fluid in the Direction of the Axis C A; then to any Point M let be drawn a Tangent, in which take $Mm = \dot{z} =$ Fluxion of the Curve; and let L M P be drawn perpendicular to the Base B C, and draw the Perpendiculars Lm and mr; and put $CB = a$, $CP = x$, $PM = y$; then $Mr = \dot{y}$, and $rm = \dot{x}$; and because of similar Triangles C M P and Mrm, we have $CM\ (= a) : PM\ (= y) :: Mm\ (= \dot{z}) : rm\ (= \dot{x})$; whence $\frac{a\dot{x}}{y} = \dot{z}$, and $\dot{z}^2 = \frac{a^2\dot{x}^2}{yy}$.

2030. Let the absolute Force of a direct Stroke of a Particle of the Fluid be represented by L M, and since the Stroke at M is oblique to the Curve, it must be resolved into two, *viz.* Mm, and Lm; the first of which does not affect the Curve, but the latter acts perpendicular to it. And therefore the whole Force is to the oblique Force as L M to Lm, or as Radius to the Sine of In-

Incidence (1033). Wherefore the whole Force being $= 1$, that by which the Point M is ſtruck will be as $\frac{\dot{x}}{\dot{z}}$; for becauſe of ſimilar Trian. LmM, Mmr, we have Mm $(= \dot{z}) : mr (= \dot{x})$ :: LM $(= 1)$: L$m = \frac{\dot{x}}{\dot{z}}$.

2031. But this Force Lm muſt be reduced to another acting againſt the Point M in the Direction in which it moves, that is, it muſt be reſolved into the two Forces Lr and mr; of which the latter being perpendicular to P L does no Ways hinder its progreſſive Motion; but the former Lr acts wholly againſt it. But Mm $(= \dot{z}) : mr (= \dot{x}) :$ Lm $\left(= \frac{\dot{x}}{\dot{z}}\right)$ L$r = \frac{\dot{x}^2}{\dot{z}^2}$, which therefore expreſſes the Reſiſtance to the Particle M moving in the Direction M L.

2032. But as this Force acts perpendicularly (not to the Curve, but) to the ordinate M D, it muſt be multiplied by the Fluxion thereof, and the Product, *viz.* $\frac{\dot{x}^2}{\dot{z}^2} \times \dot{x} = \frac{\dot{x}^3}{\dot{z}^2}$, will be the Expreſſion for the Flux. of the Reſiſtance of the Arch A M. Then ſince $a : p :: x : \frac{px}{a} =$ Circumference of a Circle whoſe Radius is x; if we multiply this Circumference by the Fluxion of Reſiſtance, the Product $\frac{px\dot{x}^3}{a\dot{z}^2}$ will be the Fluxion of Reſiſtance to the Surface of the Solid generated by the Revolution of the Curve A M about its Axis A C.

2033. In order to find the Fluent or Reſiſtance for the Surface of a Globe, we have $\dot{z}^2 = \frac{a^2\dot{x}^2}{yy}$ (2029) therefore $\frac{p\dot{x}^3 x}{a\dot{z}^2} = \frac{pyyx\dot{x}}{a^3}$; but $yy = aa - xx$ (828) whence $\frac{py^2x\dot{x}}{a^3} = \frac{p}{a^3} \times \overline{aax\dot{x} - x^3\dot{x}}$, whoſe Fluent is $\frac{p}{a^3} \times \overline{\tfrac{1}{2}a^2x^2 - \tfrac{1}{4}x^4}$, for the Reſiſtance; and ſo when $x = a$, we ſhall have $\frac{1}{4}ap$ for the Expreſſion of the Reſiſtance to the Surface of the Hemiſphere, or Globe moving in the Fluid.

2034. If a Cylinder were to move in the said Fluid in the Direction of the Axis, since the Stroke of every Particle would be direct, and the whole proportional to the Area of the Base; it is evident (supposing the Diameter of the Cylinder and Globe equal) that the whole Resistance to the Cylinder will be (as the Area of its Base) $\frac{ap}{2}$ (see 830) *whence it appears, the Resistance to the Globe is to that of a Cylinder (of equal Diameter) as* 1 *to* 2.

2035. A Cylinder falling upon such a Fluid, will in the Time of passing through its whole Length, communicate a Motion to the Particles, as will be to the whole Motion of the Cylinder, as the Density of the Medium to the Density of the Cylinder. For let Q, M, V, B, D, be the Expressions for the *Quantity* of *Motion*, *Matter*, *Velocity*, *Bulk*, and *Density* of the Cylinder; and q, m, v, b, d, denote the same Things in the Fluid. Then the Motion of the Cylinder will be $Q = MV$, and that communicated to the Fluid will be $q = vm$ (970); but it is $M = BD$, and $m = bd$ (973). Whence $Q = VBD$, and $q = vbd$; and so $Q : q :: VBD : vbd$; but $VB = vb$, therefore $q : Q :: d : D$.

2036. If the Axis and Diameter of the Cylinder be equal to the same in a Globe; it is evident, since the Resistance to the Globe is but $\frac{1}{2}$ that of the Cylinder, the Globe, that it may communicate the same Motion to the Fluid, must move through twice the Length of its Diameter. If the Velocity and Density of the Cylinder and Globe be equal, (and $\mathcal{Q}$, M, V, B, D, stand for the same Things in the Globe) then, since $V = V$, it is $\mathcal{Q} : Q :: (M : M :: B : B) :: 2 : 3$ (837). Also the Motion of the Cylinder is to that communicated by the Globe in passing through two Diameters, as the Density of the Cylinder (or Globe) to that of the Medium, *viz.* $Q : q :: D : d$ (2035). Lastly, let q be the Motion communicated to the Medium by the Globe in passing through $\frac{4}{3}$ of it's Diameter; then $q : \mathrm{q} :: 2 : \frac{4}{3} :: 3 : 2$. And by compounding these Ratios, we shall have $\mathcal{Q} : \mathrm{q} :: D : d$; that is, the *whole Motion in the Globe is to that communicated to the Medium (equal to the Resistance of the Medium to the Globe) in passing through* $\frac{4}{3}$ *of its Diameter, as the Density of the Globe to the Density of the Medium.*

2037.

2037. If the Particles of the Cylinder, Globe, and Medium be suppos'd perfectly elastic, then the Velocity of the Parts, after Percussion, will be double to what it is in the Medium before mentioned (1015) and therefore the same Effect will be produced in half the Time, or in passing through half the Space. Consequently, in a perfectly elastic Fluid, *a Globe*, in the Time of passing through $\frac{2}{3}$ Parts of its Diameter, will communicate a Motion to the Particles that will be to the whole Motion of the Globe, as the Density of the Fluid to the Density of the Globe.

2038. There is no such Thing in Nature as a *perfect Fluid*, or such whose Parts are absolutely free, and unaffected by any external Force; for all Fluids are heavy Bodies, that is, their Parts gravitate towards the Center of the Earth, and consequently on one another. Therefore the lower Parts will be compressed by the Weight of those above them; and therefore, when a Body is considered as moving through such a compressed Fluid, the Motion or Velocity of the Body will be less than that with which the Particles will rush into the Space relinquished by the Body, by Virtue of the Compression, I say, in this Case we are to conceive the Force of Re-action on the fore Part of the Body, in some Degree abated by the Particles circulating round to fill up the relinquished Space, in order to restore the Equilibrium (that would otherwise be destroyed) by the constant Influx of the Fluid behind the Body.

2039. This complexed Case of real Fluids makes the Computation of their Resistance to Bodies very difficult and tedious, but Sir *Isaac Newton* has fully considered the Thing,* and found that the Resistance to a Cylinder moving in such a compressed Fluid, is but a fourth Part of what we have shewed it would be in a perfectly non-elastic Fluid (2035.) supposing the Velocity of the Body and Density of the Fluid the same in both Cases; that is, *the Motion imparted to such a Fluid in the Time it passes through four Times its Length, is to the whole Motion of the Cylinder, as the Density of the Fluid to that of the Cylinder.*

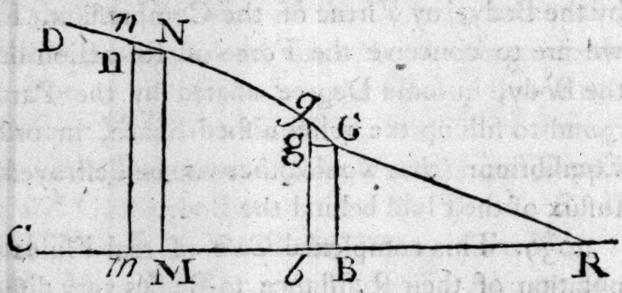
2040. In such a compressed Fluid, Bodies of every Form (if their transverse Sections are equal) will be equally resisted; for if a *Cylinder*, *Globe*, *Cone*, *circular Plane*, &c. of equal Diame-

* See his Principia, Lib. II. Prop. 37, *&c.*

ters were placed in a Canal of ſuch a *current Fluid*, it is evident the Fluid would be equally obſtructed by them all, or the increaſed Velocity of the Fluid paſſing by each will be the ſame; they therefore equally reſiſt the Fluid, and, were they to move in the Fluid, would be equally reſiſted by it. Conſequently, ſince the Motion of a Globe is but $\frac{2}{3}$ of that of the Cylinder circumſcribing it, the Force that will generate or deſtroy the whole Motion of the Cylinder while it moves through 4 Times its Length, will generate or deſtroy the whole Motion of the Globe while it moves uniformly through $\frac{2}{3}$ of 4 Diameters, or $\frac{8}{3}$ Parts of its Diameter, that is, through $2\frac{2}{3}$ Diameters.

2041. It will be neceſſary here to obſerve, that the *Theory of Reſiſtance* in a rare elaſtic Medium is, by the *Newtonian Matheſis*, arrived to its Perfection, ſince thereby it is determined what the Figure or Form of a Body muſt be, which, moving in ſuch a Medium, ſhall meet with the *leaſt Reſiſtance poſſible*. And as this Part of mathematical Philoſophy is of primary Conſideration in *Naval Architecture*, and *Military* Science, we ſhall here explain the Principles of the *Calculus*, as follows.

2042. On the Right-line B C, ſuppoſe the Parallelograms B G g *b*, M N n *m*, of the leaſt Breadth, to be erected, whoſe Heights B G, M N, their Diſtance M *b*, and half the Sum of their Baſes $\frac{1}{2}$ M *m* + $\frac{1}{2}$ B *b* = *a*, are given: Let half the Difference of the Baſes $\frac{1}{2}$ M *m* — $\frac{1}{2}$ B *b* be called *x*: Let G and N be Points in the Curve G N D; and producing *b* g, and *m* n to *g* and *n*, (ſo that g *g* = n *n* = *b*,) the Points *g* and *n* may alſo be in the ſame Curve.



2043. Now if the Figure C D N G B, revolving about the Axis B C, generates a Solid, and that Solid moves forwards in a rare and elaſtic Medium from C towards B, (the Poſition of the Right-line B C remaining the ſame;) then will the Sum of the Reſiſtances againſt the Surfaces generated by the Lineolæ

G *g*,

G g, N n, be the least possible, when $\overline{Gg}^4$ is to $\overline{Nn}^4$ as BG $\times$ Bb to MN $\times$ Mm.

2044. For the Force of a Particle on Gg and Nn, to move them in the Direction BC, is as $\frac{1}{\overline{Gg}^2}$ and $\frac{1}{\overline{Nn}^2}$ (2031); and the Number of Particles that strike in the same Time on the Surfaces generated by Gg and Nn, are as (the *Annuli* described by gg and nn, that is, as BG $\times$ gg and MN $\times$ nn, or as) BG and MN; therefore the Resistances against those Surfaces are as $\frac{BG}{\overline{Gg}^2}$ to $\frac{MN}{\overline{Nn}^2}$, that is (putting y for $\overline{Gg}^2$, and z for $\overline{Nn}^2$,) as $\frac{BG}{y}$ to $\frac{MN}{z}$.

2045. But the Sum of these Resistances $\left(\frac{BG}{y} + \frac{MN}{z}\right)$ is a *Minimum*. Therefore $- BG \times \frac{\dot{y}}{yy} - MN \times \frac{\dot{z}}{zz} = 0$, or $MN \times \frac{\dot{z}}{zz} = - BG \times \frac{\dot{y}}{yy}$: But $y = (\overline{Gg}^2 = \overline{Bb}^2 + \overline{gg}^2 =) \; aa - 2ax + xx + bb$; and $z = (\overline{Nn}^2 = \overline{Mm}^2 + \overline{nn}^2 =) \; aa + 2ax + xx + bb$; therefore $\dot{y} = 2x\dot{x} - 2a\dot{x}$, and $\dot{z} = 2a\dot{x} + 2x\dot{x}$: Consequently $\frac{MN}{zz} \times 2\dot{x} \times \overline{a + x} = \frac{BG}{yy} \times 2\dot{x} \times \overline{a - x}$; or $\left(\frac{MN}{zz} \times \overline{a + x} =\right) \frac{MN}{zz} \times Mm = \left(\frac{BG}{yy} \times \overline{a - x} =\right) \frac{BG}{yy} \times Bb$. (2042.) Therefore $(yy)\, \overline{Gg}^4 : (zz)\, \overline{Nn}^4 :: BG \times Bb : MN \times Mm$.

2046. Consequently, that the Sum of the Resistances against the Surfaces generated by the *Lineolæ* Gg and Nn, may be the least possible, $\overline{Gg}^4$ must be to $\overline{Nn}^4$ as BG $\times$ Bb to NM $\times$ Mm.

2047. Wherefore, if gg be made equal to gG, so that the Angle gGg may be 45°, and the Angle BGg = 135°; also $\overline{Gg}^2 = 2\overline{gg}^2$, and $\overline{Gg}^4 = 4\overline{gg}^4$; then $4\overline{gg}^4 : \overline{Nn}^4 ::$ GB $\times$ Bb

$\times$ Bb : NM $\times$ Mm; and since GR is parallel to Nn, and BG, BR parallel to nn, Nn; also $nn = gg =$ gG; it follows, that (nn $=$ gG $=$) Bb : (Nn $=$) Mm :: BG : BR; therefore $Bb = \frac{BG \times Mm}{BR}$; also ($n$n $=$) gG : Nn :: BG : GR. Consequently, $\left(\frac{4\overline{gg}^4}{\overline{Nn}^4} =\right) \frac{4\,\overline{BG}^4}{\overline{GR}^4} = \left(\frac{BG \times Bb}{MN \times Mm} =\right) \frac{\overline{BG}^2}{MN \times BR}$. Therefore $4\,\overline{BG}^2 \times BR$ is to $\overline{GR}^3$ as GR to MN.

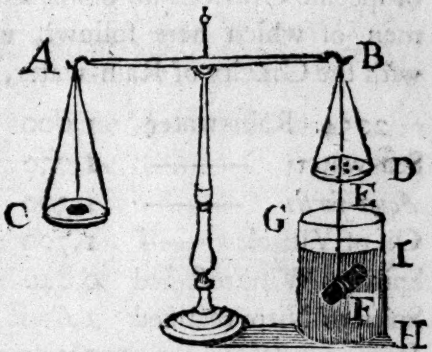
2048. How this Curve DNG is to be constructed by Means of the *Logarithmic Curve*, and from thence a practical Method of forming the SOLID of *least* RESISTANCE, we may hereafter shew, in its proper Place, when we shall have premised and explained the *general Doctrine* of *Curves*.

CHAP. II.

The DOCTRINE *of* absolute, specific, *and* relative WEIGHT *in* BODIES, *explained on the* statical PRINCIPLES *of* resisting FLUIDS, *as* WATER, AIR, *&c.*

2049. BEFORE we proceed any farther, it will be necessary (in the THEORY of BALLISTICS) to premise a few Things relating to the various Distinctions of GRAVITY, or Weight in Bodies, *viz. absolute*, *relative*, and *specific Gravity*. *Absolute Weight* is the *whole Weight* of a Body *in Vacuo*. *Relative Weight* is that which a Body has when weighed in a resisting Medium. *Specific Gravity* is the *comparative Weight* of Bodies of *equal Bulk*. Thus if the Weight of a Cubic Inch of *Water* be to that of a Cubic Inch of *Copper*, as 1 to 9, we say the *Specific* Gravity of *Copper* is 9, or its Weight is 9 Times greater than that of Water.

2050. But to illustrate this Affair, let A B be a Ballance, and from the Scale D let any Body F be suspended by a fine Horse Hair E, and equiponderated in the Air with Weights put into the other Scale C. After this, immerse it in the Fluid I H, contained in the Jar G H, and the Equilibrium will be destroyed. For the Body F will now appear lighter, and ascend, while the Scale C descends.



2051. That the Weight of the Body F should be diminished in the Fluid, is hence evident, that it cannot descend therein without raising at the same Time an *equal Bulk of the Fluid*, which Quantity of the Fluid, thus raised, will resist the Body F with all its Force (equal to its Weight) and therefore will destroy just as much Motion (or Weight) in the Body F (965). And therefore, by putting Weights in the Scale D, till the Equilibrium is again restored, we shall have the *Comparison of Weight between the Body* F, *and the Fluid, in equal Bulks:* For the Weight of the Solid is in the Scale C, and that of an equal Bulk of the Fluid in the Scale D; and hence the specific Gravity of the Solid and Fluid becomes known; for which Reason this Instrument is called the HYDROSTATIC BALANCE.

2052. If from the Weight of the Solid, in Air, we deduct the Weight of an equal Bulk of the Fluid, the Remainder is the *relative Gravity* of the Body; and is all that Force by which the Body *sinks* or descends in the Fluid. But if the Weight of the Solid be less than that of an equal Bulk of the Fluid, it is evident such a Solid cannot be totally immersed by its Weight, but will *swim* with a Part extant above the Fluid. If the specific Gravity of the Solid and Fluid be equal, the Solid will retain any Position in the Fluid, and neither *sink nor swim*. Hence the *Rationale* of SINKING and SWIMMING is evident.

2053. Not only the specific Gravity of Solids, but also of Fluids become known by the same Balance; for if the Solid F be weighed in two different Fluids, the Weights put into the Scale D to restore the *Equilibrium*, each Time will express the Gravities of equal Bulks of those Fluids; and hence a Table

of specific Gravities for Solids and Fluids may be made, a Specimen of which here follows, where the Comparison is made, with the Gravity of Rain-water, put = 1,000.

2054. Rain-water	1,000	Ebony	1,177
Sea-water	1,030	Cork	0,240
Aqua-fortis	1,300	Good Wheat	0,757
Oil of Vitriol	1,700	White Pease	0,807
Spirit of Wine rectified	0,840	Bone	1,656
Spirit of Nitre rectified	1,610	Ivory	1,826
Burgundy Wine	0,953	Horn	1,840
Canary	1,033	Adamant	3,400
Red Wine	0,993	Glass	2,666
Distilled Vinegar	1,030	Flint	2,542
Cow's Milk	1,030	Marble	2,700
Urine	1,030	Mundick	4,430
Mercury Crude	13,593	Chalk	2,370
Amber	1,040	Magnet	1,840
Sulphur	1,800	Newcastle Coal	1,272
Borax	1,720	Oil-stone	2,380
Red Coral	2,689	Slate	2,740
Cinnabar natural	7,300	Alabaster	1,875
Tartar common	1,849	Copperas Stone	4,300
Camphire	0,995	Copper Ore	3,775
Vitriol of *Dantzick*	1,715	Lead Ore	6,800
Sal Gemmæ	2,143	Bismuth	9,700
Allum	1,714	Speltar	7,065
Nitre	1,900	Tin	7,550
Gum Arabic	1,375	Iron	7,645
Verdigrease	1,714	Brass, wrought	8,000
Bees Wax	0,960	Copper	9,000
Pitch	1,190	Lead	10,131
Rosin	1,100	Silver, Sterling	10,000
Honey	1,450	—— Pure	11,091
Dry Box Wood	0,950	Gold, Sterling	17,150
Dry Fir	0,546	—— Pure	19,640
Lignum Vitæ	1,327		

2055. Because it is found by Experiment, a Cubic Foot of Rain-water weighs very exactly 1000 Ounces *Averdupois*, therefore the

the Numbers in the Table will expreſs the Ounces contained in a Cubic Foot of any of the other Subſtances. An Ounce Averdupois $= 437\frac{1}{2}$, and an Ounce Troy $= 480$ Grains. But $437\frac{1}{2} : 480 :: 51 : 56$, nearly. The *Averdupois Pound* is *to the Troy Pound*, as $437\frac{1}{2} \times 16 = 7000$ to $480 \times 12 = 5760$; that is, as 17 to 14 nearly. Therefore a Cubic Foot of Water weighs $62\frac{1}{2}$ *lb.* Averdupois, and 76 *lb.* Troy nearly.

2056. What relates to the *abſolute* and *ſpecific Gravities*, the *Magnitudes*, *Denſity*, &c. of Bodies, will beſt be underſtood by ſymbolical Computation; in order to which let A and B be two Bodies of equal Bulk, but different Quantities of Matter; and let B and C be two other Bodies with equal Quantities of Matter, but of different Bulks:

And let $\left\{\begin{array}{l}\mathrm{D} = \text{Denſity} \\ \mathrm{B} = \text{Bulk} \\ \mathrm{M} = \text{Quantity of Matter}\end{array}\right\}$ in the Body A.

Alſo $\left\{\begin{array}{l}D = \text{Denſity} \\ B = \text{Bulk} \\ M = \text{Matter}\end{array}\right\}$ in the Body B.

And $\left\{\begin{array}{l}d = \text{Denſity} \\ b = \text{Bulk} \\ m = \text{Matter}\end{array}\right\}$ in the Body C.

2057. Then, becauſe the Denſity of any Body is proportional to the Quantity of Matter under equal Bulks, we ſhall have $D : \mathrm{D} :: M : \mathrm{M}$; and, becauſe when the Quantities of Matter are equal, the Bulks muſt be reciprocally as the Denſities, therefore we have $D : d :: b : B$. Whence $D = \frac{\mathrm{D}M}{\mathrm{M}} = \frac{db}{\mathrm{B}}$; conſequently $\mathrm{D}MB = db\mathrm{M}$. But $B = \mathrm{B}$, and $M = m$; therefore $\mathrm{DB}m = db\mathrm{M}$. Whence we have $\mathrm{D} : d :: b\mathrm{M} : m\mathrm{B}$; and $\mathrm{B} : b :: d\mathrm{M} : \mathrm{D}m$; and $\mathrm{M} : m :: \mathrm{DB} : db$.

2058. The *ſpecific Gravity* of Bodies being as the Weights, that is, as the Quantities of Matter, in equal Bulks, will be as the Denſity: Therefore $\mathrm{D} : d :: \mathrm{S} : s$; and by Subſtitution of Ratios, we have the general Theorem above become $\mathrm{SB}m = sb\mathrm{M}$. And ſince the abſolute Weights (A, *a*,) of any two Bo-

dies are as the Quantities of Matter, we have $SBa = Asb$. Wherefore $S : s :: Ab : aB$; that is, the ſpecific Gravities will be as the abſolute Weights directly, and the Bulks inverſely, or as the abſolute Weights divided by the Bulks.

2059. Alſo $A : a :: SB : sb$; that is, the abſolute Weights of Bodies are in the compound Ratio of their ſpecific Gravities and Bulks. Or the abſolute Weight of any Body is had by multiplying its Bulk and ſpecific Gravity together.

2060. Again; becauſe $B : b :: As : aS$, it appears that the Bulk or Magnitudes of Bodies will be as the abſolute Weights directly, and ſpecific Gravities inverſely. Or the Magnitude of any Body is had by dividing its abſolute Weight by its ſpecific Gravity.

2061. From what we have ſaid, it is evident we have not the *abſolute Weight* of Bodies in the Air, but only the Relative. Let $A =$ abſolute Weight, and $B =$ relative Weight; then $A - B = B$ the Weight of an equal Bulk of Air, (2051) and the ſame may be ſaid for any other Fluid. Wherefore let the Motion of a Globe moving in a Fluid be ſuch as will be generated by the Force of its relative Weight falling *in Vacuo* through a Space (S) that ſhall be to $\frac{4}{3}$ of its Diameters ($\frac{4}{3}$ D) as the Denſity of the Globe (D) to the Denſity of the Fluid (d), *then will the Velocity it will acquire by the Fall be the greateſt it can poſſibly acquire by deſcending in the Fluid; and the relative Weight of the Globe will be equal to the Reſiſtance, ariſing from the Medium to the Globe,* as mentioned in (2039, 2040).

2062. For let $R =$ Reſiſtance, $F =$ Force that will generate or deſtroy the Motion of the Globe while it deſcribes $\frac{8}{3}$ Parts of its Diameter; then $F : R :: D : d$ (2039) and ſo $F = \frac{RD}{d}$. And ſince $S =$ Space deſcribed in the Fall, the Globe by an uniform Motion with the Velocity acquired by the Fall, will deſcribe a Space $= 2S$ in the ſame Time (993) and ſo, becauſe $S : \frac{4}{3}D :: (2S : \frac{8}{3}D)\ D : d$. But the Times T, t, in which the Spaces 2 S and $\frac{8}{3}$ D are uniformly deſcribed are as thoſe Spaces, *viz.* $T : t :: 2S : \frac{8}{3}D$, hence $T : t :: D : d$.

2063. Again, ſince in the Times T and t equal Quantities of Motion are produced (for F produces the whole Motion of the Globe in the Time t, and the relative Weight (B) of the Globe pro-

produces the ſame in the Time T, (2061). And ſince the leſs the Time is, the greater muſt be the Force to produce a given Effect; therefore $F : B :: T : t :: D : d$. Conſequently $F = \frac{BD}{d} = \frac{RD}{d}$; whence $R = B$, and the *Reſiſtance being equal to the relative Weight*, the Body can be no longer accelerated, but muſt deſcend with an uniform Motion in the Medium, with a Velocity equal to that acquired by the Fall through S *in Vacuo*.

2064. Hence if the Denſity of the Globe and of the reſiſting Medium be given, as alſo the Velocity of the Globe in the Beginning of its Motion; then the Reſiſtance it meets with may be computed as follows. Let A = abſolute Weight of the Globe *in Vacuo*, and $B =$ Weight of an equal Bulk of the Medium; then $A - B = B =$ the relative Weight of the Globe in the Medium, (2061).

2065. And ſince the Spaces deſcribed in the ſame Time *in Vacuo* are as their accelerating Forces (999), therefore as the Space deſcribed *in Vacuo* by the Weight A is 16,2 Feet in one Second, the Space deſcribed *in Vacuo* by the Weight B in one Second will be known, for as $A : B :: 16{,}2 : \frac{16{,}2\,B}{A} =$ to the ſaid Space.

2066. Moreover, the Time of deſcribing the Space S *in Vacuo* by the Weight B is thus found; as $\frac{16{,}2\,B}{A} : S :: 1''^2 : \frac{AS}{16{,}2\,B}$ the Square of the Time required; or $\sqrt{\frac{AS}{16{,}2\,B}} = T$ the Time. But $S = \frac{4\,DD}{3\,d}$ (2061) whence $T = \sqrt{\frac{4\,ADD}{48{,}6\,Bd}}$.

2067. The Velocity of the Fall will carry the Globe with a uniform Motion over a Space $= 2S$ in the Time of the Fall T; and the Velocity of uniform Motion is always as the Space divided by the Time (991); therefore $\frac{2S}{T} = V =$ the Velocity acquired by the Fall.

2068. The Globe moving in the reſiſting Medium meets with a Reſiſtance $= B$ (2063); if any other Velocity (V) be given the

the Resistance agreeing to that Velocity will be thus found. As $V^2 : V^2 :: B : \frac{V^2 B}{V^2} =$ Resistance to the Velocity *V*. For that the Resistance is as the Square of the Velocity in general, is hence evident; because the Resistance will be greater in Proportion to the Number of Particles which strike against the Globe in a given Time, and also to the Intensity of the Stroke of each Particle; and each of these will be as the Velocity, and therefore both together as the *Square of the Velocity*.

2069. But, as I said, this Ratio of the Resistance to the Velocity is only general, and agreeing to *slow Motions*; because in very swift Motions the Circumstances will be altered, on which the Resistance depends, both in regard to the Medium, and to the Figure of the Body moving in it. For with respect to the Medium, though compressed, yet if the Velocity of the Body be so much greater than that of the Particles rushing in behind the Body, that a *Vacuum* is left, the Case of this compressed Fluid will be nearly the same with one that is free (2028); and consequently the Resistance to a Cylinder moving so very swift as to leave a vacuous Space behind, will be near 4 Times as great as when it moves slowly through the same compressed Fluid (2039).

2070. Again, the Resistance will vary in swift Motions according to the Figure of the Body. In such a Case the vacuous Space will be more or less, and the *compressed Fluid* approaches to the Nature of one that is free, wherein the Resistance to a Globe has been shewn to be but half what it is to a Cylinder (2034); therefore the Resistance to a Globe moving very swiftly, even in a compressed Fluid, (because of the *Vacuum* behind) may meet with little more than half the Resistance of the Cylinder moving with the same Velocity.

2071. But we cannot suppose the Compression of the Medium not at all to affect the Body, for even to a Globe, the Particles which strike it obliquely, will be in some Degree confined by the compressing Force, and therefore give more Resistance than when they are free to move. Also if the Medium be *Elastic*, this Resistance will be thereby farther increased (2037), so that upon the Whole we may conclude that the Resistance to a Globe moving swiftly in a compressed elastic Medium (such as our Air)

will

will be between that of a Globe and of a Cylinder in a free and perfect Fluid, and that arising to either in a slow Motion from a compressed Medium; that is, it is more than *twice* and *less than 4 Times the Resistance* the same Globe would meet with moving slowly in a compressed Medium.

2072. Hence then we may take it for granted, that in very swift Motions, a Globe is resisted about *three Times more*, in Proportion to its Velocity, than when its Motion is slowest; and that the Resistance will decrease as the Velocity decreases, and as the Circumstances of the Medium return to their natural State. Let this suffice at present, for the *Theory of the Resistance of Fluids*, which is a *fundamental Doctrine* in the Science of BALLISTICS.*

CHAP. III.

Of the NATURE *and* PROPERTIES *of* AIR; *its* ELASTICITY, DENSITY, absolute *and* specific GRAVITY *explained by* COMPUTATION *and* EXPERIMENTS.

2073. THE Nature of ELASTICITY, and whence it arises has been already, in general, shewn (975—980). We now propose to consider the *Physico-mathematical Theory* thereof; and apply it to the Air, and other elastic Fluids, particularly that which is generated by *firing* GUNPOWDER, that the ENGINEER may be fully acquainted with the Philosophy of this important Subject.

2074. In order to this, let there be placed any Number of Particles in the right-lined Distance AB at an equal Distance

A . . . . B
C. . . . . D

* It is necessary here to correct an Error in the Table of specific Gravities, Pape 296, where that of a *Magnet* is said to be 1,840, whereas it should be 4,840, as I have found by Experiment. This Error also passed unobserved in the larger Table of the PHILOSOPHIA BRITANNICA, 2d Edition.

Diſtance from each other; and in any other equal Diſtance CD let there be placed twice as many Particles, at equal Intervals alſo, then it is plain the Intervals between the Particles in CD will be but half ſo great as thoſe between the Particles in the Line AB. Hence the Number (N) of Particles in the Lines AB, CD, will be inverſely as the Interval (L) between each, that is N will be always as $\frac{1}{L}$.

2075. In a *Superficies*, ſuppoſing the Intervals of all the Particles equal, we ſhall have N^2 as $\frac{1}{L^2}$ (670); alſo in a Solid it will be N^3 as $\frac{1}{L^3}$ (675). But N^2, and N^3 is as the Denſity (D) of the Surface, and of the Solid (973); therefore D is as $\frac{1}{L^2}$ in a Surface, and as $\frac{1}{L^3}$ in a Solid.

2076. Let us now ſuppoſe each Particle repels the Particles next to it (and thoſe only) with a Force as (F) which is as any Power (n) of the Interval inverſely; that is, let F be as $\frac{1}{L^n}$. Now the Sum or whole Force in the Superficies will be as the Denſity D, and as the centrifugal Force F; or as $D \times F = \frac{1}{L^2} \times \frac{1}{L^n} = \frac{1}{L^{n+2}}$. Which therefore will expreſs the *elaſtic Force* of the Fluid.

2077. But in the Body of the Fluid the Denſity being as $\frac{1}{L^3}$ (2075), we have DL^3 as 1, L^3 as $\frac{1}{D}$, and L as $\sqrt[3]{\frac{1}{D}}$ which ſubſtituted in the Expreſſion of the elaſtic Force $\frac{1}{L^{n+2}}$ gives $D^{\frac{n+2}{3}}$; *therefore the elaſtic Force of the Medium is as the Cube Root of that Power of the Denſity whoſe Index is* $n + 2$.

2078. Hence if the elaſtic Force (E) in any Fluid be as the Denſity (D) then the Expreſſion $D^{\frac{n+2}{3}} = E^{\frac{n+2}{3}} = E$; and ſo $\frac{n+2}{3} = 1$, whence $n + 2 = 3$, and therefore $n = 1$;

where-

wherefore in ſuch a Fluid F will be as $\frac{1}{L}$ (2076); that is, the *Particles repel each other with Forces that are inverſely as the Diſtances of their Centers.*

2079. And this is the Property of our Air whoſe Denſity is always as the compreſſing Force, as is proved by the following Experiment. Let Mercury be poured into an inflected Tube A B C D, open at both Ends, to a ſmall Height B C; then ſtopping the Orifice D very cloſe, meaſure the Length of the confined Air D C very nicely; this done, pour Mercury into the other Leg A B till its Height above the Surface of that in C D be equal to the Height at which it ſtands in the Barometer. Then it is plain the Air in the ſhorter Leg will be compreſſed with a Force twice as great as at firſt; for then it was preſſed only with the Weight of the Atmoſphere, but now it is preſſed with that Weight, and an additional equal Weight of Mercury. With this double Force the Air in D C is now compreſſed into the Space D E = $\frac{1}{2}$ D C, as appears by meaſuring it.

2080. Hence it appears, that the Spaces S = D C, and s = D E, which the ſame Quantity of Air poſſeſſes under different Preſſures p and P, are as thoſe Preſſures inverſely, *viz.* $S : s :: P : p$. And becauſe the Denſities d and D (where the Quantity of Matter is given) are inverſely as the Bulks (973); therefore $d : D :: s : S :: p : P$; that is, *the Denſity of the Air is directly as its compreſſing Force, or as the Elaſticity,* which is equal thereto (982).

2081.

2081. The ſpecific Gravity of Air, and thence its abſolute Weight may be diſcovered by the following Experiment. Let A *B* be a Glaſs Tube, open at both Ends, and the End B immerſed in the Water E D of the Phial C D, and let the Mouth of the Phial C be ſealed cloſe. Then let a little Air be blown through the Tube A B, and it will condenſe the Air C E in the Phial, and thereby encreaſe its Elaſticity, which will raiſe a Column of Water B F in the Tube, whoſe Weight together, with that of the Air, preſſing on the Surface F, will be equal to the encreaſed Spring of the Air C E. If now this Phial be carried to the Height of 72 Feet, or 864 Inches above the Earth's Surface, the Water will riſe from F to G, through a Space F G = 1 Inch; whence it appears, that 1 Inch of Water is equivalent in Weight to 864 Inches of Air, of the Denſity it has at the Earth's Surface; and ſo 1 Inch of Air is equivolent, or of the ſame Weight with $\frac{1}{864}$ Part of an Inch of Water. Whence the Weight of Air is to the Weight of Water (in equal Bulks) as $\frac{1}{864}$ to 1, or as 1 to 864. Sir *Iſaac Newton* has ſtated it as 1 to 860.

2082. Now becauſe it is found by Experiment, a Cubic Inch of Water weighs 253,3 Grains; therefore ſay, as 864 : 1 :: 253,3 : 0,293 = $\frac{3}{10}$ of a Grain nearly, for the Weight of a cubic Inch of Air; and becauſe a Cube is to its inſcribed Sphere as 1 : 0,524 (847); therefore ſay, as 1 : 0,524 :: 0,293 : 0,1535 = $\frac{15}{100}$ of Grain, the Weight of a Globe of Air 1 Inch in Diameter.*

2083. The ſame Inſtrument (*viz.* the Phial with the Tube of Water (2081) being held near the Fire, the Water will aſcend to the Height of ſeveral Inches in the Tube, which ſhews the *Expanſion of the Air*, or its *increaſed Elaſticity* by HEAT. On the other Hand, if the Phial be immerſed in cold Water, the Water in the Tube will deſcend ſome Inches below F, which will demonſtrate the *Contraction of the Air*, or the *Diminution of its Elaſticity*, by COLD.

2084.

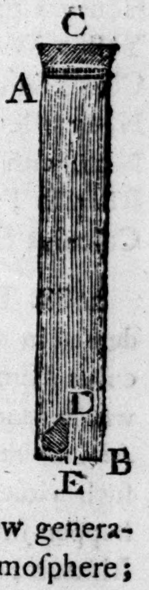
* The Weight of a Pint of Air, Wine Meaſure, or 231 cubic Inches, I have conſtantly found by the *Ballance* to be 8 $\frac{1}{2}$ Grains, which is at the Rate of $\frac{3}{10}$ of a Grain, *per* cubic Inch, as above.

CHAP. IV.

Of the NATURE *and* PRODUCTION *of* artificial AIR; *and a* COMPUTATION *of the* explosive FORCE *of* GUNPOWDER *thence deduced.*

2084. THE Properties of common Air hitherto considered are not so much the Subject of our present Speculation, as the same Properties in another Subject which the modern Philosophy has discovered, and which from its perfect Similarity to common Air, is usually called *artificial* or *factitious* AIR. This Substance, in its natural State, is no other than the common Substance of all Sorts of Bodies, and which being by chymical Operations, or otherwise, set at Liberty, acquires the Form of Air in every Respect, *becoming a fine, transparent, invisible, elastic, ponderous Fluid.*

2085. One easy Method of obtaining this elastic Fluid is thus; let A B be a tall cylindric Glass, with a very small Hole E at the Bottom; in the Glass put any alkalinous Substance D as a Piece of Chalk, Tartar, *&c.* then fill the Glass to the Top with common Water (stopping the Hole E with the Finger) and to the Water put a little *Aqua Fortis*, and then stop the Orifice A close with the Cork C; this done, you will see the Acid act upon the Alkali, and produce a Fermentation which consists in the violent Eruption and Ebullition of this new generated Fluid, which will keep constantly ascending to the Top of the Vessel, in Form of small Bubbles of Air; and by its Elasticity, it will, by Degrees, force all the Water out of the Tube through the Hole E.



2086. By this Experiment the Elasticity of this new generated Air is shewn to be greater than that of the Atmosphere; and by many other Experiments, it appears to be extremely great, and to produce Effects thereby superior to those of any other Force we know of in Nature; but in order to make Estimations of this Kind, we must first know what Quantities and

Weights of this Fluid are produced from Bodies of ſeveral Sorts; and as theſe have been determined by Experiments, by ſeveral eminent Philoſophers, I ſhall here ſubjoin a Specimen thereof from the late Dr. *Hale's Vegetable Statics*.

Several Sorts of Matter.	Cubic Inch of Matter.	Cubic Inches of general Air.	Weight of the Body.	Weight of general Air.	Proport. Part of the Whole.
2087. Dear's Horn	$\frac{1}{2}$	117	241	33	$\frac{1}{7}$
Oyſter Shell ———	$\frac{1}{2}$	162	266	46	$\frac{1}{6}$
Heart of Oak ———	$\frac{1}{2}$	108	135	30	$\frac{1}{4}$
Indian Wheat ———		270	388	77	$\frac{1}{3}$
Peaſe —— ———	1	396	318	113	$\frac{1}{3}$
Muſtard Seed ———		270	437	77	$\frac{1}{6}$
Amber —— ———	$\frac{1}{2}$	135	135	38	$\frac{1}{3.5}$
Dry Tobacco ———		153	142	44	$\frac{1}{3}$
Honey with Calx of Bones	1	144	359	41	$\frac{1}{9}$
Yellow Wax ———	1	54	243	15	$\frac{1}{16}$
Coarſe Sugar ———	1	126	373	36	$\frac{1}{10}$
Newcaſtle Coal ———	$\frac{1}{2}$	180	158	51	$\frac{1}{3}$
Nitre with Bone Calx	$\frac{1}{2}$	90	211	26	$\frac{1}{8}$
Rheniſh Tarter ———	1	504	443	144	$\frac{1}{3}$
Calculus Humanus —	$\frac{3}{4}$	516	230	147	$\frac{1}{2}$

2088. By this Table it appears how greatly this Fluid is condenſed in its natural State in Bodies; thus $\frac{3}{4}$ of an Inch of Calculus Humanus, produced 650 Times its own Bulk of Air, which made nearly one Half of its fixed Subſtance. When this Air, therefore, comes to be ſet at Liberty, no wonder we ſee ſuch violent Expanſions, Exploſions, Incaleſcences, *&c.* as happen in firing the *Pulvis fulminans*, GUNPOWDER, and in the Mixtures of various Sorts of Fluids. For ſince it has been found by many Experiments, that the Elaſticity and Weight of this Air is the ſame with that of the common Air; therefore its *Elaſticity being as its Denſity* (2078, 2079) will be as much greater in its fixed State in Bodies than that of the common Air, as its Bulk

Bulk when set at Liberty (or in a State of Expansion) exceeds the Bulk of the Body which produced it (973).

2089. But to apply this to Gunpowder (with which we are here more immediately concerned) we must first consider its constituent Parts which are *Nitre*, or *Salt-petre*, *Sulphur*, and *Charcoal Powder*. As to Charcoal, it appears, by Experiment, to afford none of this fluid elastic Air; and as to *Sulphur*, it is so far from yielding any of this Fluid, that on the contrary, it absorbs or attracts, and fixes the common Air in which it is fired, and thereby diminishes its Quantity and Elasticity, as is well known by Experiment. The elastic Fluid, therefore, that is produced, by firing Gunpowder, must be derived chiefly from the *Salt-petre*; and this is known to be a Substance imbibed from the Air by the Earth, because those who make it, extract it from the *same Parcel of Earth* many Times one after another, after it has been duly prepared, and exposed to the Air for a proper Time.

2090. But because Nitre produces only 180 Times its Bulk of Air (2087) and Mr. *Haukshee*, by Experiment, found Gunpowder produced 232 Times its Bulk; and by most accurate Experiments, Mr. *Robins* has found it to produce 244 Times its Bulk; therefore it seems, that some additional Quantity of Air is produced by compounding these different Substances together in this Sort of Powder. This Quantity of Air, from Gunpowder, was determined in the following Manner: Mr. *Robins* fired $\frac{1}{16}$ of an Ounce, Averdupois, in a Receiver of about 520 cubic Inches Capacity; this sunk the Mercury in the Gage 2 Inches; and as the Height of the Mercury was then near 30 Inches, therefore $\frac{15}{16}$ of an Ounce would have produced so much Air as would have pressed all the Mercury out of the Gage; that is, $\frac{15}{16}$ of an Ounce would produce 520 cubic Inches of Air of the same Elasticity of common Air. But $\frac{15}{16} : 520 :: 1 : 575$, the cubic Inches that *one Ounce* of Powder would produce.

2091. But as Heat augments the Elasticity of Air, and this Powder was fired upon a red-hot Iron; Part of the Expansion was owing to that Heat. But this Heat was less than that of boiling Water which encreases the Expansion of Air to more than a $\frac{1}{4}$ Part of the Whole; if, therefore, the Number 575 be lessened by $\frac{1}{5}$ Part, it may be nearly the same as would be

produced in the Receiver not heated, *viz.* 460 cubic Inches; but 17 Drams of Powder will fill 2 cubic Inches, therefore 16 : 17 :: 460 : 488 $\frac{3}{4}$ = cubic Inches of Air from 2 cubic Inches of Powder; therefore 1 cubic Inch of Powder will produce 244 of Air.

2092. Mr. *Robins*, to try how much this Elasticity or Expansion of Air would be augmented by the Flame of the fired Powder, supposes the Heat of this Flame nearly the same with that of the extreme (or white) Heat of red-hot Iron; and this he found to augment the Elasticity of common Air in Proportion of 194 $\frac{1}{3}$ to 796. If then we say, 194 $\frac{1}{3}$: 796 :: 244 : 999 $\frac{7}{3}$; this last Number will shew how much the Elasticity of the Air produced from Powder, when inflamed, is greater than that of the Air in its natural State.

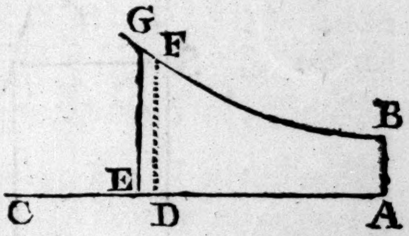
2093. Since 1 cubic Inch of Mercury weighs very nicely 148,1 Ounce Averdupois Weight, a Pillar of Mercury, whose Base is one Square Inch, and Height 29 $\frac{1}{2}$, will weigh 14 Pound 15 Ounces; therefore as the Air sustains by its Pressure or natural Elasticity, a Weight of 15 Pound (at a Medium) upon a Square Inch; the Elasticity of the inflamed elastic Air of Gunpowder, which is 1000 Times as great, will act upon every Square Inch (at its first Accension) with a Force which is = 15000 *lb.* or somewhat above 6 Ton Weight.

CHAP. V.

From the given DIMENSIONS *of any* PIECE *of* ARTILLERY, *the* DENSITY *of its* BALL, *and the* QUANTITY *of its* CHARGE, *are determined the* VELOCITY *which the Ball will acquire from the Explosion; and also the* Length of the Charge *producing the* greatest Velocity possible.

2094. IN the preceeding Article we have shewn what the Force of the Powder is at the Instant of its Accension or taking Fire, but since this Action of the Powder is not instanta-

neous, but continues to urge the Bullet all the Time it is in the Barrel; and ſince this Force is not uniformly the ſame, but decreaſes from firſt to laſt, and is every where inverſely as the Space which the elaſtic Fluid fills, that is, as its Bulk, (2080) it will be neceſſary in order to make a Computation of the Motion or Velocity of the Bullet, to premiſe the following *Lemmata.*



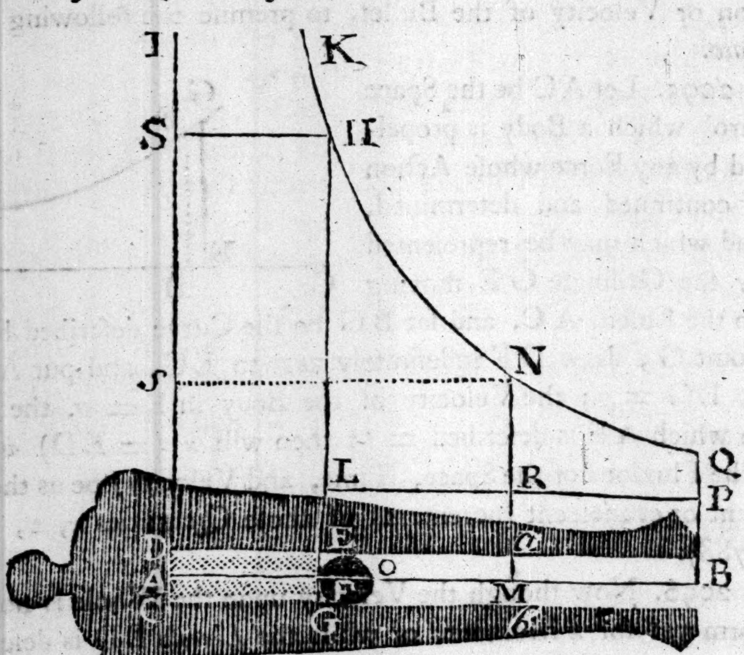
2095. Let A C be the Space thro' which a Body is propelled by any Force whoſe Action is continued and determined, and which may be repreſented by the Ordinate G E moving on the Abſciſs A C, and let B G be the Curve deſcribed by the Point G; draw D F indefinitely near to E G, and put A E = x, E G = y; the Velocity of the Body in E = v, the Time in which A E is deſcribed = t; then will $\dot{x}$ (= E D) $\dot{v}$ and $\dot{t}$ (the Fluxions of the Space, Time, and Velocity) be as the naſcent or evaneſcent Increments of thoſe Quantities x, v, and t (788).

2096. Now though the Velocity upon the Whole is not uniform yet for a Moment, or while the *Lineola* E D is deſcribed, it may be eſteemed ſo (992) and therefore we ſhall have $v = \frac{\dot{x}}{\dot{t}}$ (by 991). Alſo we have the moving Force $y = \frac{\dot{v}}{\dot{t}}$ (998); wherefore it is $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{v}}{y}$, whence $y\dot{x} = v\dot{v}$ = E G × E D = the Fluxion of the Space A B G E; therefore $\frac{1}{2}v^2$ = A B G E (804), and ſo we have $v = \sqrt{2\,ABGE}$; that is (ſince 2 is a conſtant Quantity) *the Velocity will be every where in the ſubduplicate Ratio of the Area* A B G E.

2097. Since the Quantity of Motion Q is always proportional to the moving Force G E, or y; therefore $Q = \frac{v}{t} = vm$ (970) whence $t = \frac{1}{m}$; alſo, becauſe when $Q = 1$, a given Quantity; then, alſo $v = \frac{1}{m}$, and t as v; therefore $\frac{1}{m} = t \times v = v^2$;

con-

consequently $v = \sqrt{\frac{1}{m}}$, *that is, the Velocity of any Body urged through a given Space by a given Power acting with a determinate Force in every Part of that Space will be reciprocally in the subduplicate Ratio of the Quantity of Matter.*



2098. These Things premised, let A B represent the Axis of any Piece of Artillery, A the Breech, and B the Muzzle; D C the Diameter of the Bore, and D E G C a Part of its Cavity filled with Powder; O the Ball that is to be impelled thereby, lying with its hinder Surface at the Line G E. Then, because the Force (F) of the Powder against an Inch Square is known (2092, 2093) the Force (F) exerted on the Area of a Circle, one Inch in Diameter, will be known also; for it will be F : F :: 1 : 0,7854, whence F = 0,7854 F (269). Now let D = 1 Inch, and d = Diameter of any other Circle as that of the Ball O, that is, let d = E G, and (f) the Force of the Powder exerted on this last Circle; then $F : f :: D^2 : d^2$ (840) therefore $f = Fd^2 = 0{,}7854\,F\,d^2$ = the Force of the Powder acting at the first Instant of its Accension on the Ball O in the Direction of the Axis A B.

2099.

2099. Let this Force then be represented by any Line F H, perpendicular to A B in the Point F. Then, because the Density of the elastic Fluid confined in the Space E D C G is to the Density thereof when expanded into any other Space *a* D C *b*, as those cylindric Spaces inversely (2072) or as the Lengths A M, A F; and since the elastic Force is in the same Ratio with the Density; therefore by making M N : F H :: A F : A M; and B Q : F H :: A F : A B; then shall M N and B Q be as the Force of the Powder upon the Ball at M and at B, and the Curve which connects the Points H, N, Q, will be an *Hyperbola* (by 778).

3000. To the Point A draw the Right-line A I parallel to F H, and S H parallel to AB; then will A I, A B be the Asymptotes of the Hyperbola, and the Rectangles $AF \times FH = AM \times MN = AB \times BQ = 1$, the Power of the Hyperbola (779); and from what has been said, it is easy to understand that the whole Force of the Powder exerted on the Ball while it moves from F to B, is as the hyperbolical Space HFBQ, and therefore as the hyperbolical Logarithm of the Ratio $\frac{AB}{AF}$ (829, 850).

3001. Since the Force impelling the Ball at F is known (2098) and the Weight of the Ball is supposed given, the Ratio between the said Force and the Weight of the Ball is known, which let be as F H to F L; then if the Ball were to be impelled to the Distance F B by a Force equal to its Gravity, because the Force of Gravity through so small a Space is uniform, or acts in every Point with the same Tenour, therefore in any Point M or B the Force will be as $MR = FL$, and $BP = FL$, and consequently the Line which joins the Points L, R, P, is a Right-line, and parallel to the Axis A B, and the whole Action or Force of Gravity impelling the Ball through A B will be as the Rectangle $FLPB = FB \times FL$.

3002. But the Velocities acquired by the Ball when propelled by the Force of Gravity, and by the Force of the Powder, thro' the Space F B, are as $\sqrt{FLBP}$, and $\sqrt{FHQB}$ (2096). And since F B is a given Length, the Velocity acquired in falling through that Space is known. Thus suppose the Length of the Bore $AB = 45$ Inches, and the Length of the Charge $AF =$

2 $\frac{5}{8}$

$2\frac{5}{8}$ Inches; then $AB - AF = FB = 42\frac{3}{8}$ Inches (as in Mr. *Robins*'s Example). Then 16,2 Feet : $42\frac{3}{8}$ (=3,53 Feet) :: $\overline{32,4}^2$: V^2 (by 996). Whence $\sqrt{64,8 \times 3,53} = V = 15,07$ Feet *per* Second, the Velocity acquired by the Body falling thro' the Space F B.

3003. Again, to find the Ratio of F L to F H, suppose the Ball O be of Lead, whose Diameter $d = \frac{3}{4}$ of an Inch; then will its Weight be $\frac{1}{12}$ of a Pound Averdupois. Therefore the Gravity of the Ball is to the Force of the Powder at F, as $\frac{1}{12}$ to $0,7854\,F\,d^2$ (by 2098), that is, as 1 to 79521,6; and such is the Ratio of F L to F H, supposing the Pressure of Air on a square Inch just equal to 15 *lb.*

3004. But the Number 79521,6 is somewhat too large, since a Column of Mercury, whose Base is 1 square Inch, and Altitude 29 $\frac{1}{2}$, weighs but 14 *lb.* 15 Ounces; and I find Mr. *Robins* has taken for a mean Altitude 28,2 Inches of Mercury, or 33 Feet of Water, whose Weight is but about 14 *lb.* 4 Ounces, and he also makes the specific Gravity of Lead to that of Water, as 11,345 to 1, consequently a Column of Lead, 34,9 Inches Altitude, will have the same Weight or Pressure; and multiplying this by 1000 (2093) the Product 34900 Inches, will be the Height of a Column of Lead, whose Pressure is equal to that exerted on the Bullet at the Moment of Accension.

3005. Now the Bullet being $\frac{3}{4}$ of an Inch in Diameter, is equal to a Cylinder on the same Base, and $\frac{1}{2}$ Inch in Height; for the Solidity of the Sphere is $\frac{pd^2}{6}$, and that of a Cylinder is $\frac{pdh}{4}$ (by 836, 831). Wherefore if $\frac{1}{6}pd^2 = \frac{1}{4}pdh$, then $4d = 6h$; whence $h = \frac{2}{3}d = \frac{2}{3}$ of $\frac{3}{4} = \frac{1}{2}$, the Height of the Cylinder, equal to the Ball in Weight and Magnitude. Therefore $34900 \times 2 = 69800$; whence the Ratio of the Force of Powder is to the Weight of the Bullet, as 69800 to 1. And so F L : F H :: 1 : 69800.

3006. But we have F B : F A :: $42\frac{3}{8}$: $2\frac{5}{8}$: 339 : 21. Hence the Rectangle F L B P is to the Rectangle A F S H, as $1 \times 339 : 21 \times 69800 :: 1 : 4324$. And since the Rectangle A F S H is equal to the Power of the Hyperbola (853) it is to any hyperbolic Logarithm F H Q B, as 0,43429, &c. to the tabu-

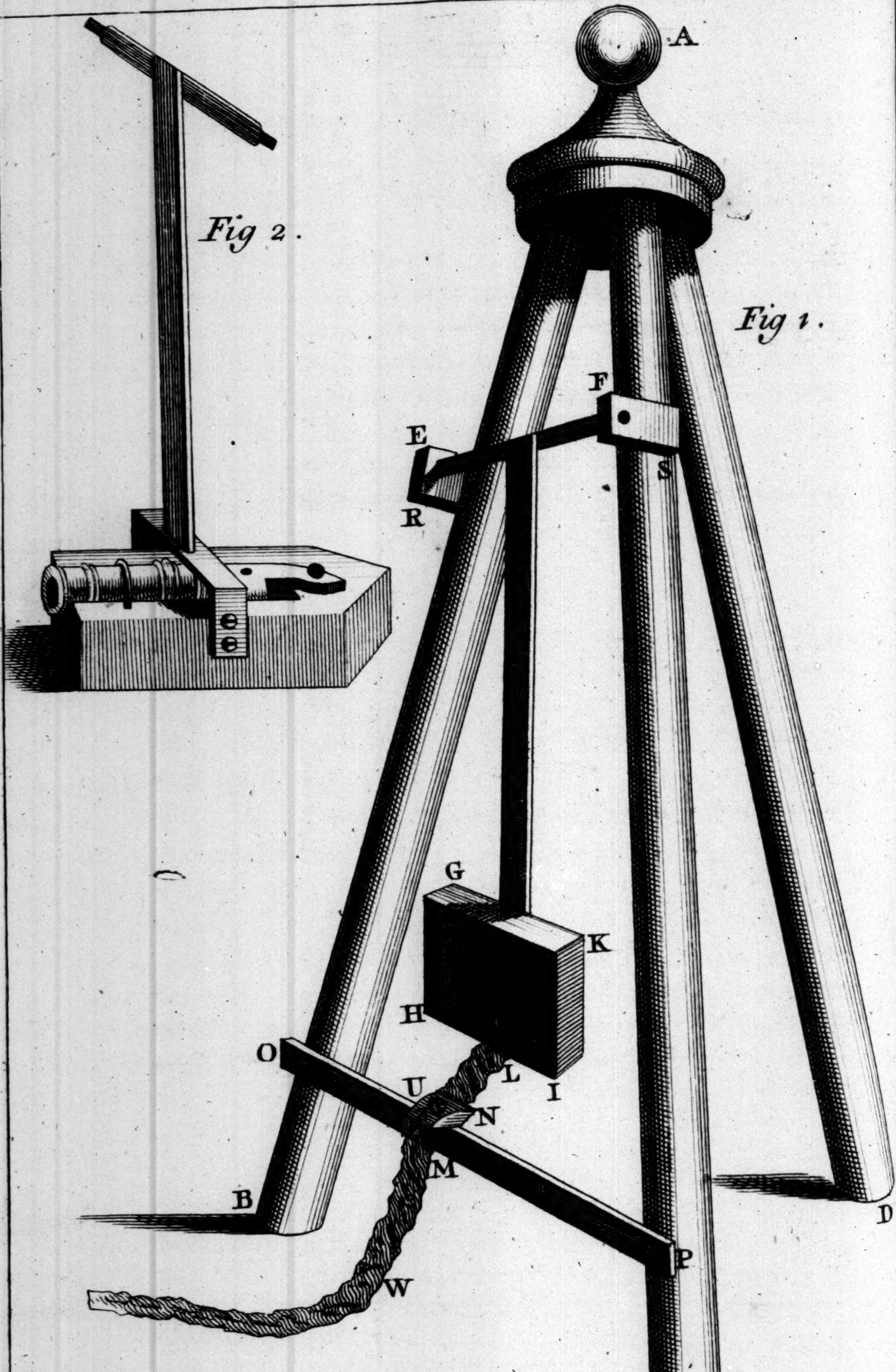


Fig 2.
Fig 1.
A
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C

tabular Logarithm of the ſame Ratio $\frac{AB}{AF} = \frac{360}{21}$, which is 1,2340579 (by 854). Therefore the Rectangle FLBP : Space FHQB :: 1 × 0,43429 : 4324 × 1,2340579 :: 1 : 12263. Conſequently $v : V :: \sqrt{FLBP} : \sqrt{FHQB}$ (2096): $\sqrt{1} : \sqrt{12263}$ :: 1 : 110,7 :: 15,07 Feet : 1668 Feet = V = Velocity of the Ball required; that is, the Velocity communicated to the Ball by the whole Force or Action of the Powder will be that of 1668 Feet *per Second*, of uniform Motion.

CHAP. VI.

The THEORY *for determining the* VELOCITY *of the* BULLET *at the* MUZZEL *of the* GUN *ariſing from the* QUANTITY *of the* CHARGE, *the* LENGTH *of the* PIECE, *and* DIAMETER *of its Bore*; *alſo the* CHARGE *producing the* GREATEST VELOCITY *in a Piece of given Dimenſions.*

3007. AS the THEORY of *military* PHILOSOPHY conſiſts chiefly in two great Points, *viz.* (1.) To ſhew how far the Velocity of the Bullet or Cannon-ball is affected, whilſt in the Piece, by the *Force of the Powder*, the *Diameter of the Bore*, and the *Quantity of the Charge*, under any given Variations; or (2.) after it is out of the Piece, by the *Reſiſtance of the Air*; and having premiſed the neceſſary geometrical Principles, we ſhall now proceed directly to an Illuſtration of them both.

3008. Let a = AB, the LENGTH *of the* PIECE under Conſideration, and b = Length of any other Piece of Artillery. Then if the Length of the Charge AF and the Bore be the ſame in each, we ſhall have the Area FHQB, or $L\frac{a}{AF} : V^2 ::$

$L\frac{b}{AF}$

$L\frac{b}{AF} : V^2$ (3006). But the three first Quantities are known, and therefore $V =$ Velocity of the Bullet through the Length of the Bore denoted by (b) will also be known. Hence it appears, *that the longer the Barrel is, the greater will be the Velocity of the Bullet.*

3009. If the BORE *of the* GUN be varied, all other Things remaining the same; then, putting D = Diameter of the Bore in the present Instance, and D = Diameter of any other Bore; it is evident the *Quantity*, and, consequently, the *Force of the Powder* in each will be in the Ratio of D^2 to D^2, and the Force of the Powder in the Bore, whose Diameter is D, will be resisted in the Ratio of the Weight of the Ball or of its Magnitude, that is as D^3 (by 841). Wherefore the Area F H Q B will be varied in the Ratio of $\frac{D^2}{D^3}$, or $\frac{1}{D}$; that is, $D : \mathrm{D} :: \mathrm{V}^2 : V^2$; and so $\sqrt{D} : \sqrt{\mathrm{D}} :: \mathrm{V} : V =$ the Velocity of the Bullet sought. *Hence the smaller the Bore is, the greater will be the Velocity of the Bullet.*

3010. If A F, or the QUANTITY *of the* CHARGE be varied, other Things remaining the same; then the Rectangle A F S H, and, consequently, the Area F H Q B will also vary, but not in the same Proportion; for there is a certain Quantity of Powder or Length of A F, which with Respect to the Length of the Piece A B, will give a greater Velocity to the Bullet than any other.

3011. Therefore let A B $= a$, and A F $= x$. Then will the Velocity be as the Force of the Powder at the first Instant of Accension, *viz.* as A H, or A F $= x$; and also as the whole Action of the Powder upon the Ball while in the Barrel, or as the Area F H Q B $= L\frac{a}{x}$; consequently, the *Velocity will be* as $xL\frac{a}{x}$, or is $xLa - xLx$. Let $z =$ any other Length of the Charge, then we shall have $xL\frac{a}{x} : zL\frac{a}{z} :: \mathrm{V}^2 : V^2$.

3012. Hence the Velocity of the Bullet at the Muzzle of Guns of a different Length, Bore, and Charge, will be as in the following Analogies, *viz.*

V^2

$V^2 : \mathcal{V}^2 :: L\frac{a}{x} : L\frac{b}{x}$, for Lengths as *a* to *b*.

$V^2 : \mathcal{V}^2 :: \mathcal{D} : D$, for Diameters of Bore as D to $\mathcal{D}$.

$V : \mathcal{V} :: xL\frac{a}{x} : zL\frac{a}{z}$, for Lengths of Charge as *x* to *z*.

Consequently, $V : \mathcal{V} :: \sqrt[5]{\mathcal{D}xL^2\frac{a}{x}} : \sqrt[5]{DzL\frac{b}{x}L\frac{a}{z}}$.

3013. But since it is evident from the Expression of the Velocity $xL\frac{a}{x} = xLa - xLx$, there will be one determinate Value of *x* which will make it a *Maximum*, therefore by putting its Fluxion $= 0$, we have $\dot{x}La - \dot{x}Lx - \dot{x} = 0$ (because the Fluxion of $Lx = \frac{\dot{x}}{x}$, (849); wherefore $La - 1 = Lx$. Now let $a = 10$; then, because, the hyperbolical Logarithm of 10 is 2,302585; therefore $Lx = 1,302585$; but as 1 : 0,4342448 :: 1,302585 : 0,565710 = tabular Logarithm of *x* (853). Therefore $x = 3,679 = AF$, when the Velocity is a *Maximum.*

3014. And because in this Case $L\frac{a}{x} = 1$ (for $La - Lx = 1 = L\frac{a}{x}$) the Velocity $xL\frac{a}{x}$ when a *Maximum*, will be as *x* or 3,679. Also when *x* or $AF = \frac{1}{10}$ of AB, then because $x = 1$, we have the Velocity $xL\frac{a}{x} = La = 2,302585$, which is not quite $\frac{2}{3}$ of the *Maximum* Velocity. And thus the *Velocity acquired from any other Charge of Powder may be compared with the greatest Velocity.*

3015. It is observable, that when the Velocity is greatest, then AF : AB :: (3,679 : 10 ::) 1 : 2,71828; which Ratio is called the *modular Ratio*, because its Measure $\left(L\frac{a}{x}\right)$ is $1 =$ the *Module* of the *System*, or *Power of the Hyperbola* (779).

3016. This *Maximum* Charge is to that in common Use, (in Art. 3003) as $\frac{1}{2.7}$ to $\frac{21}{360} = \frac{1}{18}$, that is near 7 Times as large. The Reasons why we do not, or rather care not to use

this Charge for the greatest Velocity, are many; for *first*, it would make too great a Consumption of Powder, *viz.* 6 or 7 Times more than what we now use. *Secondly*, the Force of the Powder in the Gun, would in that Case, be so great, that the Barrels must be 5 or 6 Times as thick as they now are to avoid bursting; by which Means they would be rendered too expensive and unweildy. *Thirdly*, the Velocity of the Bullet, from common Charges, is sufficient in long Guns; and the greatest Velocity would be much too great. *Lastly*, in short Guns, as Pistols, &c. we may use this *Maximum* Charge very well on many Occasions, for $1 : 2{,}7 :: 2\frac{5}{8} : 7$ Inches, the Length of the Barrel, whose Charge $AF = 2\frac{5}{8}$ Inches.

3017. If instead of one Bullet, there be 2, 3, or 4, laid before the Charge, then will the Velocities of the Bullets, in each respective Case, be as $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{4}}$, (574). Therefore, since $\sqrt{\frac{1}{4}} = \frac{1}{2}$, the Velocity of a Bullet when there is 4, will be but $\frac{1}{2}$ that of a single one. Hence also the Velocity of a leaden Bullet is to that of an Iron one, as $\sqrt{11{,}345}$ to $\sqrt{7{,}645}$, or $\sqrt{3}$ to $\sqrt{2}$.

3018. In this Theory, Mr. *Robins* has supposed two Things, *viz.* (1.) That the Action of the Powder upon the Bullet ceases as soon as it is got out of the Piece; and (2.) That all the Powder of the Charge is fired before the Bullet is sensibly moved out of its Place. The first of these *Postulates* is self evident; and the Second is very nearly true, that the few Grains usually thrown out unfired, can but little affect the Theory; and he found it so by many Experiments upon Guns of different Lengths, and discharging a different Number of Bullets from the same Gun. But this was more particularly confirmed afterwards by many Experiments made by Appointment of the *Royal Society*. The general Result of which proved, that not above $\frac{1}{6}$ Part of the whole Charge was collected in Grains unfired; and this, in Reality, could be reckoned no more than a $\frac{1}{12}$ Part. For these Experiments were made with common Powder, which produced a larger Quantity unfired, than the finest grained Powder will do (such as Mr. *Robins* used) in the Proportion of 5 to 3, as was found by Experience. And again, the *Saltpetre* extracted from the Powder, thrown out unfired, compared with that of the

the Charge, was found to be nearly the Ratio of 7 to 9; and, therefore, though 2 Penny-weight of unfired Grains were collected from a Charge of 12 Penny-weight, which was $\frac{1}{6}$ Part, yet this reduced in the Proportion of 5 to 3, will be but $\frac{1}{10}$; and this again reduced in the Ratio of 9 to 7, is nearly $\frac{1}{12}$ of the Whole. In some Trials, the Deficiency was but the $\frac{1}{18}$ Part of the whole Charge, from whence it will appear, how instantaneuosly the whole Body of the Charge is fired, and how little the Velocity of the Bullet must be affected by the small Part unfired, as will be evident from the ensuing Experiments in Chap. VIII.

CHAP. VII.

The THEORY *of the* MACHINE *contrived by Mr.* ROBINS, *for determining the* VELOCITY *which any* BALL *moves with at any Distance from the* Piece *it is discharged from.*

3019. HAVING ascertained by the THEORY what the Velocity of a Bullet should be under any given Quantities of *Charge*, *Length of Barrel*, and *Diameter of Bore*; it remains in the next Place to confirm this excellent Theory by Experiments. For this Purpose, Mr. *Robins* invented a Method, which by Means of a *compound* PENDULUM, and its *Apparatus*, gives the Velocity of any Bullet issuing either from the Muzzle of the Gun, or at any Distance from it. The *Rationale* of which Machine, and the Manner of applying it for such Uses, are now to explained.

3020. To this End the Machine must be first of all described, of which the Body consists of three strong Poles or Legs B, C, D, spreading at Bottom, and joining at Top in the Head-piece A. On two of these Legs, towards the Top, are screwed on two Sockets R, S, in which a Pendulum E F H I is hung, by Means of the Cross-piece E F, which is the Axis of Suspension on which

the

the Pendulum vibrates very freely. The lower Part of the Pendulum is a thick square Piece of Wood G H I K, fastened to the Back-part (which is of Iron) by Screws. A little below the Bottom of the Pendulum, there is a Brace O P fixed to the Legs B, C; and to the Brace is fixed a Part M N U made with two Edges of Steel, bearing on each other in the Line U N, somewhat in the Manner of a Drawing-Pen; the Strength with which they press on each other being diminished or encreased at Pleasure by a Screw Z going through the upper Piece. At the Bottom of the Pendulum is fastened a narrow Ribbon L N W, which passes between the Steel-Edges, and passing thro' a Hole in the lower Piece of Steel hangs loosely down as at W.

3021. The Artifice of this Contrivance, tho' simple, is admirable; the Bullet discharged from the Gun against this Pendulum puts it into Motion; and both the Bullet and Pendulum are to be considered as *Non-Elastic Bodies*; the Bullet being of Lead, is evidently so; and the Wood tho' to slow Motions it may be considered as somewhat elastic, yet with respect to very swift Motions it cannot be conceived in the least Degree so; since there is no Time for the Parts to act by their natural Resort.

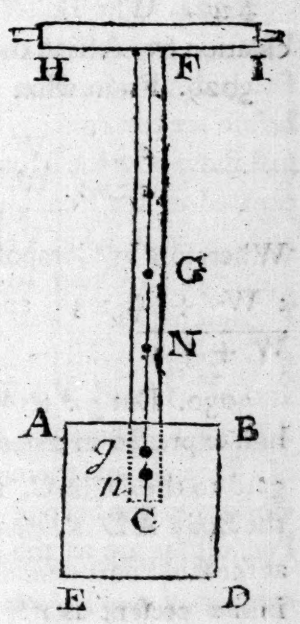
3022. The Bullet and Pendulum, therefore, will in their Motions and Action on each other, observe the Laws of Percussion laid down for Non-elastic Bodies (from Art. 1003, 1012.) But before we can come to apply them in the present Case, we must first show how the Momentum or Quantity of Motion is to be estimated in the Pendulum when put into Motion by the Bullet, and also with what Velocity it moves after the Stroke.

3023. In Order to this we must know that the *Weight* of the Pendulum (Mr. *Robins* made Use of) was 56*lb.* 3*oz.* The *Center of Gravity* was distant from the Axis of Suspension 52 Inches. And 200 of its small Vibrations were performed in the Time of 253 Seconds; wherefore the Time of one Vibration is $\frac{253}{200}$ Parts of a Second; and therefore since the Length of a Pendulum which vibrates in one Second is 39,2 Inches (1125) and the Lengths of Pendulums are as the *Squares of their Time of Vibration* (1116.) Therefore say, as $1''^2 : \frac{253''^2}{200} :: 39.2$ Inches : 62,73 Inches, the Distance of the Center of Oscillation from the Axis of Vibration.

tion. Lastly the Center of the Piece of Wood GHIK (on which the Bullet is supposed to impinge) is distant from the said Axis 66 Inches.

3024. Now if all the Matter of the Pendulum were concentered in the Center of that Piece or Wood, it would resist the Bullet with all its Force, *viz.* of 56*lb.* 3*oz.* because that would then be the Center of both *Gravity*, and *Oscillation*; but since the Center of Gravity is short of this Point, the Resistance will be diminished in Proportion, *viz.* in the Ratio of 66 to 52. And again, another Diminution of the Force will arise on Account of the Velocity of the Motion, because also the Center of Oscillation (by which that is estimated) is short of the Center of the Wood where the Stroke is made in the Ratio of 66 to $62\frac{2}{3}$. Therefore the Force or Resistance will be diminished in the Ratio of 66×66 to $62\frac{2}{3} \times 52$; that is, it will be $66^2 : 62\frac{2}{3} \times 52 :: 56$*lb.* 3*oz.* : 42 *lb.* $\frac{1}{2}$*oz.*

3025. What has been hitherto said, is rather by Illustration deduced from the Consideration of a *compound Pendulum* of the *most simple Form*, but such was not the Pendulum which Mr. *Robins* used in his Machine, the *Theory* of which we shall therefore give with its Demonstration, as follows:



3026. Let FC be the Shaft, and ABED the Weight of the Pendulum, both Parallelopipeds, and HI the Axis of Motion; then is FG $(= \frac{1}{2}$ FC), the Distance of the *Center of Gravity* of the Shaft, and FN $(= \frac{2}{3}$ FC), the Distance of the *Center of Oscillation*, and put the whole Length FC $= a$. And let the End C be the Center of the square Parallelopiped ABDE, whose Weight let us call W, and the Weight of the Shaft w.

3027. The two Centers of Gravity, G and C, will have a common Center at g; and it will be $w : W :: Cg : Gg$. Put $Cg = d$; then is $Gg = \frac{Wd}{w}$

$\frac{Wd}{w} = \frac{1}{2}a - d$; therefore $Wd = \frac{1}{2}aw - wd$; and $Wd + wd = \frac{1}{2}aw$, whence $d = \frac{aw}{2 \times \overline{W + w}}$. Consequently, $Fg = a - d = a - \frac{aw}{2W + 2w} = g = \frac{2aW + aw}{2 \times \overline{W + w}}$, the Distance of the *compound Center of Gravity* from the Axis of Motion.

3028. By the Addition of the Weight AD, the Center of Oscillation will be drawn down from N to some other Point n; to determine which, we proceed thus. Since $FN = \frac{2}{3}a$ (1097) and the *Momenta* of the Weights in the Shaft is $\frac{1}{2}aw$; therefore $\frac{2}{3}a \times \frac{1}{2}wa = \frac{1}{3}aaw =$ *Momenta*, or Sum of all the Forces in the Shaft. Also the whole Force of the Body AD (considered alone, vibrating at the Distance $FC = a$) is aaW, and the *Momenta* of its Weight will be aW. If therefore the *Sum of the Forces* of the Shaft and Body AD, be divided by the *Sum of the Moments*, we shall have $\frac{\frac{1}{3}aaw + a^2W}{\frac{1}{2}aw + aW} = \frac{\frac{1}{3}aw + aW}{\frac{1}{2}w + W} = n = Fn$, the Distance of the Center of Oscillation from the Axis HI (1094).

3029. From what we have shewn, it appears that

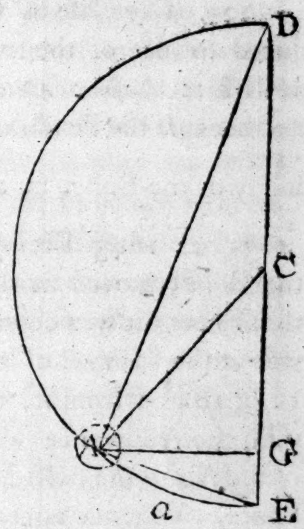
$$a : g :: W + w : W + \tfrac{1}{2}w \quad (3023).$$
$$a : n :: W + \tfrac{1}{2}w : W + \tfrac{1}{3}w \quad (3024).$$

Wherefore by Composition of Ratios, we have $aa : gn :: W + w : W + \frac{1}{3}w$; and therefore $gn \times \overline{W + w} = aa \times \overline{W + \frac{1}{3}w}$.

3030. But $gn \times \overline{W + w} = F$, the Force of the Pendulum expressed in a general Way (1095) and therefore with Regard to the Point C, the Force is $aa \times \overline{W + \frac{1}{3}w}$. But were the Body AD a *simple Pendulum* of the same Weight, its Force at the said Point would be $aa \times \overline{W + w}$, which is to the Force in its present State, as $W + w$ to $W + \frac{1}{3}w$; and which, therefore, is the Ratio of Diminution.

3031. To apply these Theorems to the Machine here used, we have $a = 66$, $g = 52$, $n = 62{,}73$, (or $62\frac{2}{3}$ to use Mr. *Robins*'s Number) and the Weight of the whole $W + w = 56\,lb.\ 3\,oz. = 899$ Ounces. Then we have $66^2\ (= 4356) : 62\frac{2}{3} \times 52\ (= 3258\frac{2}{3}) :: 56\,lb.\ 3\,oz.\ (\doteq 899\,oz.) : 671$ Ounces, or $42\,lb.$ nearly (3025). As $899 - 671 \doteq 228\,oz. = \frac{2}{3}w$; therefore $w \doteq 342\,oz.$ and $W = 557\,oz.$

3032. This compound Pendulum, then, is reduced to the Case of a simple Pendulum whose Length is 66 Inches, and Weight $= 42\,lb.\ \frac{1}{2}\,oz.$ which is to the Weight of the Bullet $= \frac{1}{12}\,lb.$ as 504 to 1. Let this equivalent simple Pendulum then be represented by $AC = 66$; and on the Center C and Diameter DE describe the Semicircle DAE. Let A*a*E be the Arch described by the Pendulum impelled by the Bullet; and AE the Chord of that Arch. Then we have shewn (1113) that the Velocity of the Pendulum by which it shall describe the Arch EA is the same as would be acquired by an heavy Body falling through the same perpendicular Height GE; which Velocity is the next Thing to be determined. In order to this, it is easy to understand, that when the Pendulum is in its perpendicular Position in a State of Rest, and the Ribbon drawn strait, with a Pin put through that Part which is contiguous to the Edges UN, that when the Bullet impinges on the Pendulum, the Ribbon will be drawn out in such Manner as to measure the Chord of the Arch described by the Point L; for it will be equal to the Interval between the Pin, and the Edges UN.



3033. Now by an Experiment made with a Gun 45 Inches in Length, $2\frac{5}{8}$ Inches Charge of Powder (weighing 12 Pennyweight) the Bullet $\frac{3}{4}$ Inch-diameter, and $\frac{1}{12}\,lb.$ Weight; and the Muzzel of the Gun at the Distance of about 16 or 18 Feet; it was found that the Ribbon was drawn out $17\frac{1}{4}$ Inches. Now the Distance of the Point L is $71\frac{1}{8}$ Inches from the Axis, and therefore if we say, as $71\frac{1}{8} : 17\frac{1}{4} :: 66 : 16$ Inches nearly; it

will appear, that the Chord of a ſimilar Arch deſcribed by the Center of the Piece G H I K was 16 Inches nearly.

3034. Wherefore (in the Fig. 3026) $AE = 16$, and $DE = (2\,AC =)\ 132$. And drawing AD, we have $DE : AE :: AE : GE$, that is $132 : 16 :: 16 : 1{,}939 = GE$. If then we ſay, as $\sqrt{193\frac{1}{3}} : 32 :: \sqrt{1{,}939} : 3\frac{1}{4}$ = Number of Feet *per* Second, that a Body would deſcribe by an uniform Motion with the Velocity acquired in falling through G E. The Velocity therefore with which the Pendulum moved after the firſt Moment of Impact was that of $3\frac{1}{4}$ Feet *per* Second.

3035. The Caſe of the Bullet and Pendulum is therefore reduced to that of the two Bodies B and A (in 1003); and ſince $m : M :: \frac{1}{12}\,lb. : 42\,lb.\ \frac{1}{2}\,oz. :: 0{,}083 : 42{,}031 : 1 : 504{,}6$; and becauſe the Pendulum was at Reſt, we have (1009) the Velocity of the Bullet $= v = \frac{M + m}{m}\ (= 505{,}6) \times V = 505{,}6 \times 3\frac{1}{4} = 1643$ Feet *per* Second. That is, by Experiment, the Bullet moved at the Rate of 1643 Feet *per* Second, and by the Theory it was determined 1668 Feet (in 3006); and as this was at the Muzzel of the Gun, and the other at the Diſtance of 16 or 18 Feet from it, the ſmall Difference of 15 Feet is no more than the Allowance which ought to be made for the Reſiſtance of the Air to the ſwift Motion of the Bullet paſſing through that Space. Whence appears the wonderful Degree of Exactneſs between the Theory and Experiments made quite independent of it.

3036. Having thus determined the Velocity for the Length of one Chord; the Velocity of the Pendulum for any other Length of a Chord (or Ribbon drawn out) will be known; becauſe the Velocities have the ſame Ratio with thoſe Chords, for ſince $DE \times GE = AE^2$, and DE is a ſtanding Quantity; it will always be GE as AE^2, and conſequently AE as $\sqrt{GE}$, that is, as the Velocity acquired in falling through G E (by 991). And hence, becauſe the Velocity of the Bullet $v = V \times 505{,}6$; and this Number 505,6 (for the ſame Bullet and Pendulum) is conſtant, we have the Velocity (v) of the Bullet always proportional to (V) the Velocity of the Pendulum, and conſequently to the Chord of the Arch in any Vibration.

CHAP.

CHAP. VIII.

An Account of the EXPERIMENTS *made for comparing the* actual VELOCITIES *with which* BULLETS *of different Kinds are discharged from their respective* PIECES, *with their* VELOCITIES *computed from the* THEORY.

3037. THE THEORY for determining the Velocities of Bullets discharged from given Pieces of Artillery having been fully described; and in the last Chapter it has been shewn how the *actual Velocities* of Bullets are measured and ascertained by Means of a Machine; it now remains to compare the Result of the Theory with Experience, and thereby to evince how accurately this Theory agrees with the real Motions of Bullets though founded on Principles no ways connected with those Experiments. Some of these we shall select for illustrating the Theory by a particular Application; and then give a general View of the Experiments as we find them tabulated by Mr. *Robins* himself.

3038. By Experiments made on a Barrel only 12,375 Inches, the Theory was farther confirmed. For (*cæteris paribus*) we have by the Theory (3008) the L $\frac{45}{2,625} : \overline{17,25}^{2} :: \text{L}\ \frac{12,375}{2,625} : 162,37 =$ Square of the Chord whose Length is sought in the present Case, which therefore would have been 12,74 Inches, had the Weight of the Pendulum been the same, but as it was some small Matter lighter, than that in (3020), the Chord was by the Theory a little larger, *viz.* 12,8 Inches; and by Experiment it was 12,7, 12,6, 12,4; on three different Trials. Here the very small Differences are little more than what must result from the Bullets lodging in the Pendulum and making it thereby heavier each Time.

3039. By Experiments made with a Barrel 24,312 Inches in Length, all other Things the same; the Length of Chord, by the Theory, agreed exactly with the Length of the Ribbon drawn

out being both = 14,4 Inches. Whence the Velocity of the Bullet from the present Gun was to that from the Gun of 45 Inches, as 14,4 to 17,1; that is, as 17,1 : 14,4 :: 1643″ : 1544″ nearly. And for the Velocity of the Bullet from the Gun of 12,375, we have by Theory 17,1 : 12,8 :: 1643″ : 1432″. Whence it appears, the Velocity in a ſhort Gun is much greater (*cæteris paribus*) than in a long one.

3040. Again, by varying the Quantity of Powder in the ſame Space or Cavity D E G C the Theory is ſtill farther confirmed; for if inſtead of 12 *dwt.* you put only half that Quantity, *viz.* 6 *dwt.* Then by the Theory (2096) the Velocity by 6 *dwt.* is to that of 12 *dwt.* in the ſubduplicate Ratio of 6 to 12 (becauſe thoſe Numbers are as the Forces or Elaſticities impelling the Ball) that is, as $\sqrt{12} : \sqrt{6} :: 17,1 : 12$; the Chord or Length of Ribbon therefore ſhould be nearly 12 Inches when the Bullet was diſcharged with 6 *dwt.* of Powder, and by two Experiments the Length of the Ribbon drawn out was 11,2; 12,2; and the Deficiency was owing to this, that the Heat in firing ſmall Quantities of Powder is not proportionally ſo great as in firing larger Quantities; and therefore the Impulſe is not to be expected quite ſo great as by the Theory. Thus 1 *dwt.* of Powder by the Theory gives a Velocity of 482 Feet *per* Second, but by repeated Trials with that Quantity, the Velocity was not quite 400 Feet *per* 1″.

3041. The leſs Space the Fire or inflamed Powder has to act in the more Denſe it is, and conſequently the ſtronger its Action; therefore when the 6 *dwt.* of Powder was rammed into but half the Space D E G C, it was found in two Experiments to produce Lengths of Ribbon which meaſured 13,2 and 13,9 Inches, whereas by the Theory it ſhould be 13,6 Inches.

3042. Let us now apply the Theory to determine the Velocity of a Ball of 24 *lb.* ſhot from a Cannot of 10 Feet = 120 Inches Length, with $\frac{2}{3}$ of its Weight (*viz.* 16 *lb.*) of Powder. The Diameter of an Iron Globe of that Weight (and conſequently of the Bore of the Piece) will be found to be 5,325 Inches, allowing the ſpecial Gravity of Iron to be as in (2054). Alſo 16 *lb.* of Powder will fill a Cylinder of that Diameter to the Height of 21,63 Inches, allowing 8 $\frac{1}{2}$ Drams to a Cubic Inch.

Inch. Hence (in the Fig. of Art. 2098.) we have A B = 120; A F = 21,63; and F B = 98,37. Whence (computing as in 3002, 3003,) we shall find F L : F H :: 1 : 14732,4.

3043. Therefore F P : F S :: 1 × 98,37 : 21,63 × 14732,4 :: 1 : 4078,1. Also, we have E S : F Q :: 0,43429 : $L\frac{AB}{AF}$ = 0,744125. Consequently it is F P : F Q :: 1 × 0,43429 : 4078 × 0,744125 :: 1 : 6987,4. The subduplicate of this Ratio is that of 1 to 83,58. And the Velocity acquired in falling through the Height F B = 98,37 Inches is 23,05 Feet *per* 1″. Whence 83,59 × 23,05 = 1926,6 Feet *per* 1″ which the said Bullet acquires at its Exit from the Muzzel of the Cannon. And since the specific Gravity of Iron and Lead are (as the Quantities of Matter under equal Bulks) as 2 to 3 nearly;* therefore say (by 2097) as $\sqrt{3} : \sqrt{2}$ :: 1926,6″ : 1573″, the Feet *per* 1″ such a Bullet would move through if made with Lead.

3044. In these great Guns it will be necessary to understand what Space the Piece will recoil through in firing it, or rather what Space the Gun will move through Backwards, while the Ball is carried forwards to the Muzzel thereof. In order to this, we must consider, that it is one and the same Power (*viz.* the Elasticity of the Powder) that acts both on the Gun and Ball; and therefore in the Theorem Q V = G S M (1001) G is a given Quantity; and so in the present Case, we have Q V = S M, but Q = V M in every Case (970) therefore S = V^2; that is, *the Spaces described in a given Time, by given Forces, will be as the Squares of the Velocity.*

3045. Let S = (F B =) Space described by the Ball, and V its Velocity; and let s = Space through which the Gun recoils with the Velocity v; M = Quantity of Matter in the Gun, and m = the same in the Ball. Then we have S : s :: V^2 : v^2 :: $\frac{1}{m} : \frac{1}{M}$ (by 2097). Whence S : s :: M : m. Now the Weight of the Gun is about 47 *Cwt.* or 5264 *lb.* and that of the Ball 24 *lb.* also S = 98,37 (3042). Therefore we have 5264 *lb.* : 24 *lb.* :: 98,37

* The Reader is desired to correct an Error in the Table of specific Gravities (2054) where that of *Lead* is 10,131 instead of 11.131.

:: 98,37 : 0,448 = Space of the Recoil, which is not quite $\frac{1}{2}$ an Inch, when the Ball is just out of the Gun.

3046. Hence we see the usual Methods of constructing Platforms for Cannon with so much Strength and Firmness, and consequently with so great an Expence, are not at all necessary; since if it be but sufficiently steady at the Beginning of the Recoil, the remaining Part may be much slighter (as Mr. *Robins* observes) since its Unsteadiness or Shaking beyond the first $\frac{1}{2}$ Inch can have no Influence on the Ball, (which is then out of the Gun) nor any how alter the Direction of the Shot.

3047. Having thus finished a particular Application of these Experiments to elucidate the THEORY of GUNNERY according to the new Principles of Mr. *Robins*; we shall now give a general View of the many valuable Experiments he made, and digested into proper Tables, by which it will appear how surprizingly the Theory agrees with and is corroborated and confirmed by those Experiments. In these Trials it became necessary sometimes to change the Board of the Pendulum (when too much battered and over-charged with Bullets) but when its Weight varied any Thing considerable, Mr. *Robins* has taken Care to acquaint us with it. The following Account we shall give in his own Words.

3048. The first Table contains three Experiments only, made with a Barrel 45 Inches in Length, and the Board on the Pendulum was 4 *lb.* lighter than that described (3023).

N°.	Quantity of Powder. Dw.	Chord of ascending Arch measured on the Ribbon.	The same by the Theory.	Error of Theory.
1	12	18,7	19,0	+,3
2	12	19,6	19,0	—,6
3	6	13,6	13,4	—,2

3049. The next Experiments were made with the same Barrel, but the Board on the Pendulum was now of little more Weight than that in the Example of (3023).

N°.

Nº.	Length of the Cavity AF.	Quantity of Powder.	Chord of ascending Arch measured on the Ribbon.	The same by Theory.	Error of Theory.
	Inches.	Dw.	Inch.	Inch.	Inch.
4	2 $\frac{5}{8}$	6	11,9	12,1	+,2
5	2 $\frac{5}{8}$	6	12,2	12,1	—,1
6	1 $\frac{1}{4}$	6	13,2	13,6	+,4
7	1 $\frac{1}{4}$	6	13,9	13,6	—,3
8	2 $\frac{5}{8}$	12	16,7	17,2	+,5
9	2 $\frac{5}{8}$	12	17,5	17,2	—,3
10	2 $\frac{5}{8}$	12	16,9	16,8	—,1
11	2 $\frac{5}{8}$	12	17,0	16,8	—,2
12	2 $\frac{5}{8}$	6	11,7	11,5	—,2
13	2 $\frac{5}{8}$	6	11,1	11,5	+,4
14	2 $\frac{5}{8}$	12	16,7	16,3	—,4

The last five Numbers resulting from the Theory are corrected from the Quantity of Bullets lodged in the Board, which, as many other Experiments of a different Kind were tried in the Interval, amounted at last to above two Pounds; whence the Weight of the Pendulum being increased, its Vibration with the same Blow must be proportionably diminished.

3050. The next Experiments were made with a Barrel of the same Bore with the last, but only 12,375 Inches in Length: To distinguish them, we shall for the future denominate the first Barrel by the Letter A, and this short one by C. The Board on the Pendulum was at first rather lighter than in (3023).

Nº.	Barrel	Extent of the Cavity containing the Powder.	Quantity of Powder.	Chord of ascending Arch measured on the Ribbon.	The same by Theory.	Error of Theory.
		Inch.	Dw.	Inch.	Inch.	
15	C	2 $\frac{5}{8}$	12	12,7	12,8	+,1
16	C	2 $\frac{5}{8}$	12	12,6	12,8	+,2
17	C	2 $\frac{5}{8}$	12	12,4	12,8	+,4
18	A	2 $\frac{5}{8}$	12	17,0	17,3	+,3
19	A	2 $\frac{5}{8}$	12	17,2	17,2	,0
20	A	2 $\frac{5}{8}$	12	17,1	17,2	+,1
21	A	2 $\frac{5}{8}$	12	17,2	17,2	,0
22	A	2 $\frac{5}{8}$	6	12,4	12,2	—,2

3051. In ſome of the following Experiments a third Barrel was uſed of the ſame Bore with the other two, but 24,312 Inches in Length: This Barrel I denominate B; the Board fixed on the Pendulum was at firſt but little heavier than that in (3023); and when in the Courſe of the Experiments it is ſenſibly increaſed in the Weight, I diminiſh the Numbers ariſing from the Theory by a correſponding Part.

No.	Barrel	Extent of the Cavity containing the Powder. Inch.	Quantity of Powder. Dw.	Chord of aſcending Arch meaſured on the Ribbon. Inch.	The ſame by Theory. Inch.	Error of Theory.
23	A	2 $\frac{5}{8}$	12	17,1	17,2	+ ,1
24	A	2 $\frac{5}{8}$	9	15,2	15,0	— ,2
25	A	2 $\frac{5}{8}$	9	15,4	15,0	— ,4
26	C	2 $\frac{5}{8}$	12	11,5	12,8	+ 1,3
27	C	2 $\frac{5}{8}$	12	11,5	12,8	+ 1,3
28	C	2 $\frac{5}{8}$	6	8,7	9,	+ ,3
29	C	2 $\frac{5}{8}$	12	12,3	12,5	+ ,2
30	B	2 $\frac{5}{8}$	12	14,4	14,4	0,0
31	B	2 $\frac{5}{8}$	12	14,4	14,4	0,0
32	B	2 $\frac{5}{8}$	6	10,3	10,5	+ ,2
33	A	1 $\frac{3}{4}$	8	14,7	14,5	— ,2
34	A	4	12	15,7	15,3	— ,4

The Error in the 26th and 27th Experiments being much greater than what has occurred to me in any other Trials, I ſuſpect, that ſome Miſtake was made in the Weight of the Powder, or that the Barrel (which had indeed lain by in a moiſt Place) was very damp; which Circumſtance, I know by Experience, will conſiderably diminiſh the Action of the Powder.

3052. The following Experiments were made with a Pendulum much heavier, it weighing in the whole 97 *lb.* its Center of Gravity was 55,625 Inches diſtant from its Axis of Suſpenſion, and 200 of its ſmall Swings were performed in the Space of 255″ $\frac{1}{4}$, whence its Center of Oſcillation is 63,9 Inches diſtant from the Axis of Suſpenſion. Alſo ſometimes another Barrel was uſed 7,06 Inches in Length, and ,83 in Diameter, its Ball was exactly fitted to the Bore without any Windage, ſo that it went in with Difficulty, the Weight of this Ball was 33 $\frac{1}{2}$ *dw.* This Barrel we ſhall denominate D.

No.

No.	Barrel	Extent of the Cavity containing the Powder.	Quantity of Powder.	Chord of ascending Arch measured on the Ribbon.	The same by Theory.	Error of Theory.
		Inch.	Dw.	Inch.	Inch.	Inch.
35	A	2 $\frac{5}{8}$	12	9,2	9,2	,0
36	A	2 $\frac{5}{8}$	12	9,5	9,2	—,3
37	A	5 $\frac{1}{4}$	24	11,7	11,3	—,4
38	A	7 $\frac{7}{8}$	36	13,2	12,6	—,6
39	A	2 $\frac{5}{8}$	12	9,3	9,1	—,2
40	A	1 $\frac{3}{4}$	8	7,6	8,1	+,5
41	C	2 $\frac{5}{8}$	12	6,1	6,6	+,5
42	C	2 $\frac{5}{8}$	12	6,5	6,6	+,1
43	B	2 $\frac{5}{8}$	12	8,0	8,2	+,2
44	B	2 $\frac{5}{8}$	12	8,3	8,2	—,1
45	A	2 $\frac{5}{8}$	12	9,5	9,1	—,4
46	A	2 $\frac{5}{8}$	12	9,1	9,1	,0
47	A	2 $\frac{5}{8}$	6	7,2	6,5	—,7
48	A	2 $\frac{5}{8}$	6	6,7	6,5	—,2
49	C	2 $\frac{5}{8}$	12	6,8	6,7	—,1
50	C	2 $\frac{5}{8}$	12	7,5	6,7	—,8
51	C	2 $\frac{5}{8}$	6	4,7	4,8	+,1
52	C	2 $\frac{5}{8}$	6	5,0	4,8	—,2
53	D	2 $\frac{1}{4}$	12	7,0	7,2	+,2
54	D	2 $\frac{5}{8}$	12	7,1	6,8	—,3
55	D	2 $\frac{5}{8}$	6	4,7	4,8	+,1
56	D	2 $\frac{5}{8}$	6	4,8	4,8	,0
57	A	2 $\frac{1}{16}$	6	6,4	6,5	+,1
58	A	2 $\frac{1}{16}$	6	6,4	6,5	+,1
59	A	2 $\frac{1}{16}$	6	6,6	6,5	—,1
60	A	2 $\frac{1}{16}$	6	6,7	6,5	—,2
61	A	2 $\frac{1}{16}$	12	9,0	9,1	+,1

The Error in the 50th Experiment, the greatest in this Set, was doubtless owing to the Wind; for the 49, which was made immediately before it in the same Manner, and with the same Quantity of Powder, differs but little from the Theory. The Excess of the 38th Experiment above the Theory was in Part occasioned by the Impulse of the Flame on the Pendulum, which in this large Quantity of Powder was plainly to be discerned.

CHAP. IX.

The foregoing PRINCIPLES *applied to investigate the* VELOCITY *which the* FLAME *of* GUNPOWDER *acquires by expanding itself, supposing it be fired in a given* PIECE *of* ARTILLERY *without either Bullet or any other Body before it.*

3053. IN order to experiment the Velocity with which the Particles of Gunpowder expand themselves (in the Explosion) at the Muzzle of the Gun, Mr. *Robins* charged the Barrel of 45 Inches (Art. 3033) with 12 *pwt.* of Powder, and a small Wad of Tow only; and then placing the Muzzle 19 Inches from the Center of the Pendulum (mentioned Art. 3023) it was fired, and the Impulse of the Flame on the Pendulum made it ascend through an Arch, whose Chord was 13,7 Inches. Now since a Chord of $17\frac{1}{4}$ Inches agrees to the Velocity of $3\frac{1}{4}$ Feet *per* 1″, (see 3034) therefore say, as $17\frac{1}{4}$ Feet : $3\frac{1}{4}$ Feet :: 13,7 Feet : 2,6 Feet; that is, the Velocity of the Pendulum was at the Rate of 2,6 Feet *per* 1″.

3054. Now the Weight of the Powder and Wad was about 13 *pwt.* and that of the Pendulum 42 *lb.* $\frac{1}{2}$ *oz.* (when reduced to the Center 3024). Whence $m : M :: \frac{13}{320} : 42{,}031 :: 1 : 1034{,}6$; therefore the Velocity of the Powder $v = \frac{M + m}{m} \times V = 1035{,}6 \times 2{,}6 = 2692{,}56$; (see 1009 and 3034). Hence the Velocity of the Particles of Powder, (supposing the Whole of it (together with the Wad) impinged upon the Pendulum,) was at the Rate of $2692\frac{1}{2}$ Feet *per* 1″ of uniform Motion.

3055. And this is the least Velocity the Particles of Powder can be supposed to acquire in the Expansion. For first we may observe, that not more than $\frac{3}{10}$ of the Whole is converted into this elastic Fluid. Since 1 *oz.* = 437 *grs.* produced 460 Cubic Inches (2086) and each Cubic Inch weighs 0,293 = $\frac{3}{10}$ of a Grain (2082). Whence 460 × 0,393 = 134,78 Grains of

elastic

elastic Air. But $\frac{135}{437} = \frac{3}{10}$ of the whole Powder, nearly. Therefore the other $\frac{7}{10}$ must, in mixing with the elastic Part, greatly impede the Action, and retard the Motion or Velocity thereof in Explosion; especially if it be considered that this inert Part is in some Measure of an unctuous Nature, and will not be thrown out, but stick or adhere to the Inside of the Barrel an Impediment to the Rest.

3056. Again, some Part of the Flame must be lost in Expansion Sideways through 19 Inches of Air; for an elastic Fluid expands itself equally every Way, and consequently only a Part thereof can impinge upon the Pendulum. And even that Part will meet with a great Resistance from the Air, and so will have its Velocity diminished on that Account. The Quantity therefore of the Powder, and its Velocity, estimated this Way, is short of what it is in the Barrel, and at the Exit from the Muzzle thereof. Now to discover what that is very accurately, Mr. *Robins* contrived the Experiment in a different Manner, which I shall here explain, with a Variation of some Circumstance to render it easier to be understood and practised.*

3057. The Method of proceeding is this; let the Barrel be fixed to the Center of the Pendulum, and let the Weight of the Pendulum, Barrel and all, be 56 *lb.* (or 42 *lb.* reduced to the Center, as *per* Art. 596); in this Situation let it be charged with 12 *pwt.* of Powder only, put close together with the Rammer, and then upon discharging the Piece, the Pendulum will ascend through an Arch, whose Chord, at a Medium, will be $14\frac{1}{2}$ Inches. Now since in this Case the Powder (or rather its elastic Air) acts equally on its self and on the Pendulum, therefore the Quantity of Motion or Momentum of the Powder (*viz.* $q = v\,m$) will be equal to that of the Pendulum ($Q = V\,M$) that is, $V\,M = v\,m$ (970). Also, since to $14\frac{1}{2}$ Inch Chord there corresponds the Velocity of $2\frac{3}{4}$ Feet *per* 1″; therefore say, as $m = \frac{12}{320} : M = 42 :: V = 2\frac{3}{4} : V = 3082$, nearly; so that the Velocity is by this Means determined for the whole Mass of Powder to be at the Rate of 3082 Feet *per* Second.

* This Pendulum with the Barrel fixed upon it in the Manner represented by Fig. 2. of the preceding Plate, we apprehend is the most facile and simple of any.

3058. If now ſo light a Body as 1 *pwt.* of Tow be placed before the Powder contiguous to it, it will preſently acquire the Velocity with which the elaſtic Part of the Powder will expand itſelf when uncompreſſed, and by this Means we may be able to meaſure that Degree of Velocity pretty nearly. Thus if the Barrel be charged with 12 *pwt.* of Powder, and 1 of Wadding, and thoſe fired, the Pendulum will aſcend through an Arch, whoſe Chord is 17,3; from which, if we ſubduct 14,5 for the Powder, the Remainder 2,8 will be owing to the Wad; and therefore ſince 2,8 Inches of Chord gives the Velocity 0,53 of a Foot *per* 1″; therefore ſay, as $\frac{1}{320}$: 42 : : 0,53 : 7128 Feet, the Velocity of the Wad, or that which the ſwifteſt Part of the Flame moved with *per* 1″.

3059. In this Way alſo the Velocity of the Bullet may be determined to a greater Exactneſs than by the former Method where the Barrel was fired, and at a Diſtance from the Pendulum. Thus let the Barrel be charged with 12 *pwt.* of Powder, and Bullet $\frac{1}{12}$ *lb.* (as at 3033); then upon diſcharging it, the Pendulum will aſcend through an Arch, whoſe Chord will be 32,3 Inches. Now 14,5 of this Chord is owing to the Impulſe of the Powder (as above 3057); therefore 17,8 is occaſioned by the Motion of the Bullet, and is ſomewhat greater than that determined (in 3033). Now to a Chord of 17,8 Inches, there anſwers the Velocity 3,45 Feet (as *per* 3034); whence the Velocity of the Bullet $o = \frac{M+m}{m} \times V = 505{,}6 \times 3{,}45 = 1734$ Feet, nearly. Whence we find the Velocity of the Bullet, at the Muzzle of the Gun, is at the Rate of (at leaſt) 1700 Feet *per* 1″.

3060. From this Experiment it appears, that the Action of the Powder on the Gun is the ſame, whether it impels a Bullet before it, or whether it be fired alone; and it is therefore a convincing Proof, that the whole Quantity of Powder is fired in the latter Caſe, as well as in the former. In all the Experiments hitherto mentioned, the Bullet has been ſuppoſed contiguous to the Powder; and Mr. *Robins* found that it was laid at a ſmall Diſtance from it (as an Inch, or two, at moſt) the Theory will agree very nearly with the Experiments. But when the Bullet is

is laid at a considerable Distance from the Powder, as 12, 18, or 24 Inches, the Case will be very much altered, and the Bullet will be impelled or acted upon in a Manner very different from what it was before.

3061. For now we are to consider the Powder, by the Time it reaches the Bullet, as acting with two Forces, *viz.* one by *Pulsion* or *Percussion*, which it receives from the great Velocity with which it's parts expand, and strike against the Bullet, as shewn above (615), and the other Force is by Pressure; for when the inflamed Powder has expanded itself into all the Cavity behind the Bullet, it will, after the first Impact, continue to press on the Bullet with all the Force of its Elasticity, till the Bullet be out of the Barrel; and hence it will be easy to understand that the Velocity of the Bullet will be in this Case greater than if it were impelled by its elastic Pressure of the Powder only; and it is demonstrably so by Experiment.

3062. For Mr. ROBINS charged the Barrel of 45 Inches with 12 *pwt.* of Powder as usual, and then placed the Bullet at the Distance of $11\frac{1}{4}$ from the Breech (or $8\frac{1}{2}$ from the Powder) and upon discharging it against the Pendulum, he found that it had acquired a Velocity of about 1400 Feet *per* 1″. Whereas if it had been acted upon by the Pressure of the Flame only, it would not have acquired a Velocity of 1200 Feet *per* 1″, as may be easily made appear by the Theory, thus; let M be the Place of the Bullet (see Fig. to 2097) then will MN represent the Force of Pressure on the Ball the first Instant; but it is $AM = 11,25 : AF = 2\frac{5}{8} :: HF = 69800 : MN = 16131$, (2099). Then we find (by the Methods in 3006 and 3002) the Rectangle $MRPB$: Space $MNQB :: MR \times MB \times 0,43429 : AM \times MN \times L\,\frac{AB}{AM}$; which Quantities are all known; the Subduplicate of this Ratio is that of 1 to 86,338; now the Velocity acquired in falling through the Space $= MB$, is 13,51 Feet *per* 1″. Therefore say, as $1 : 86,338 :: 13,51 : 1166$ Feet, the Velocity acquired by the Force of Pressure only (by 2096).

3063. And to this the Experiment agrees; for the same Gentleman (with his usual Sagacity) in order to separate the two different Actions of Powder upon the Bullet, and to retain that only which arose from the continued Pressure of the Flame, contrived

trived the following Method. He no longer placed the Powder at the Breech from whence it would have full Scope for its Expansion, but scattered it as uniformly as might be through the whole Cavity of 11 $\frac{1}{4}$ Inches left behind the Bullet, conceiving that by this Means the progressive Motion or Velocity of the Flame in each Part would be prevented by the Expansion of the neighbouring Parts. And he found upon discharging the Barrel, the Velocity of the Ball was (instead of 1400 Feet *per* 1″) no more than 1100 Feet *per* 1″. Which was 66 Feet short of what it should be by the Theory.

3064. This Deficiency he supposes was owing to some intestine Motion of the Flame; for the Powder being kindled in a Space much larger that it could fill, must have produced many Reverberations, and Pulsations of the Flame; and from these internal Agitations of the Fluid, its Pressure on the containing Surface, he supposes, must be considerably diminished; and from hence he judges it necessary, to avoid any such Irregularity, to take particular Care to have the Powder confined closely in as small a Space as possible, even when the Bullet lies at some little Distance from it.

3065. From what has been said, it will be apparent, that when the Ball lies at a great Distance from the Charge, the Action of the Powder will be so greatly augmented in the Space behind, and will be so accumulated and condensed by the Velocity each Part has acquired by the Time it comes to the Ball, that if the Barrel be not of an extraordinary Firmness in that Part, it must by this reinforced Elasticity of the Power infallibly burst. And the Truth of this Reasoning he experienced in an exceeding good *Tower Musquet*, forged of very tough Iron; for charging it with 12 *pwt.* of Powder, and placing the Ball at 16 Inches from the Breech, on firing it, the Part of the Barrel just behind the Bullet was swoln out to double its Diameter, like a blown Bladder, and two large Pieces of two Inches long, were burst out of it.

3066. From what has been said, we see the Reason of all the extraordinary and enormous Effects of warlike Engines in which Gunpowder is used; as in *Grenades*, *Bombs*, *Petards*, *Mines*, *&c.* For as we have shewn the Force of this Powder upon every square Inch is 15000 *lb.* (2093), and that it expands itself with

a Velocity more than 3000 Feet *per* 1″ (3057) it is no Wonder it ſhould burſt the Shells of Bombs and Grenades with ſuch Force and Violence, and rend the Planks and Parts of Gates, Bridges, Walls, *&c.* in Petards, with ſuch ſudden inſuperable Power and Impetuoſity. For the Powder in the Petard weighing 5 *lb.* will have an Effort equal to that of a Cannon-Ball of 10 *lb.* moving with half the Velocity, *viz.* at the Rate of 1500 Feet *per* 1″. For the Momentum is the ſame in both Caſes (970).

CHAP. X.

The QUANTITY *of the* AIR'S RESISTANCE *to* PROJECTILES, *and* BULLETS *in particular, determined by* EXPERIMENTS *on the* Balliſtic PENDULUM.

3067. WE have demonſtrated the Principles on which the Computation of the Velocity and Reſiſtance of Bullets depends; and have confined the Theory by Experiments in reſpect of the former: We now proceed to do the ſame Thing for the latter, that is, we ſhall ſhew how the Quantity of the Air's Reſiſtance to Bullets is to be computed from the Theory joined with the Experiments made by Mr. ROBINS for that Purpoſe.

3068. We have already ſhewn, that when two Bodies impinge on, or ſtrike each other, the Magnitude of the Stroke is proportional to the Loſs of Motion in the percutient Body (1008). Now in the Caſe of a Shot made with a Gun, Bow, *&c.* the two impinging or percutient Bodies are the Bullet and the Air; which, indeed, are very different in their Natures; the leaden Ball being a continuous and unelaſtic Subſtance; but the Air a diſcontinued elaſtic Body. The former in Motion, the latter at Reſt. Yet this, notwithſtanding their Actions on each other, may be eaſily eſtimated both by Theory and Experiment as follows.

3069.

3069. Mr. Robins charged the Musket-barrel of 45 Inches with the Bullet and Powder as usual, and fired it against the Pendulum at the Distance of 25, 75, and 125 Feet from the Muzzle of the Piece at three several Times respectively; and he found that it impinged against the Pendulum in the first Case, with a Velocity of 1670 Feet *per* 1″; in the second Case, with a Velocity of 1550 Feet *per* 1″; and in the third Case, with a Velocity of 1425 Feet *per* 1″. Therefore in striking against 50 Feet of Air, it lost a Velocity of about 120 Feet *per* 1″. Now since the whole Motion of the Bullet in the first Case was $1670 \times \frac{1}{12}$ (by 970, 3003) and in the second Case it was $1550 \times \frac{1}{12}$; therefore the Difference $120 \times \frac{1}{12} = \frac{120}{12} = 10$ *lb.* will be the Loss of Motion in the Bullet which it sustained in passing through the 50 Feet of Air; but this Loss of Motion was the Effect of the equal Reaction or Resistance of the Air; consequently the Resistance of the Air to a Bullet moving with the mean Velocity of 1610 $\left(= \frac{1670 + 1550}{2}\right)$ Feet *per* 1″ is about 120 Times its Weight.

3070. To find the Time which was spent in passing through this 50 Feet of Air; say, as the mean Velocity 1610 Feet : 1″ = 60‴ :: 50 Feet : 1,87‴. And therefore to find the Resistance of the Air to the Bullet passing through the same Space with any given Velocity, and Time will be easy as follows. In a second Experiment made with all possible Care, the Mean of three Discharges against the Pendulum placed at 25 Feet Distance was the Velocity of 1690 Feet *per* 1″, and of 5 Shot against the Pendulum at the Distance of 175 Feet, the Mean was a Velocity of 1300 Feet *per* 1″. The Velocity lost in this Case in passing through 150 Feet, was that of 390 Feet *per* 1″, or 130 Feet *per* 1″ for 50 Feet of Air.

3071. Now the Mean Velocity in this Case was $\left(\frac{1690 + 1300}{2} = \right)$ 1490 Feet *per* 1″. And since the Resistance is at least as the Square of the Velocity (2068): Therefore say, as $1490^2 : 1610^2 :: 130$ Feet : 152 Feet, nearly; so that the Loss of Motion would be 152 Feet *per* 1″, with the Velocity 1610 Feet *per* 1″. But the Time spent in passing this 50 Feet of Air is about 2‴; and so this Loss of 152 Feet in

in $2'''$ will be but 142 in $1,87'''$, the Time of passing through 50 Feet with the Velocity of 1610 *per* $1''$. Consequently the Loss of Motion or (its Cause) the Resistance of the Air in this Case was $142 \times \frac{1}{12}$ *lb.* $= 11,83$ *lb.* or almost 12 *lb. viz.* 142 Times the Weight of the Bullet.

3072. Let us next see what Resistance a less Degree of Velocity met with from the following Experiment made by the same Gentleman with great Accuracy; he charged the same Gun with the same Bullets, but with a less Quantity of Powder; and the Mean of 5 Shot made against the Pendulum, at the Distance of 25 Feet, was a Velocity of 1180 Feet *per* $1''$; and then of 5 others against the Pendulum removed to the Distance of 250 Feet, the mean Velocity was that of 950 Feet in $1''$. Whence the Ball in passing through 225 Feet of Air lost a Velocity of 230 Feet *per* $1''$, or 51 Feet in striking against 50 Feet of Air. Now the mean Velocity is that of 1065 Feet *per* $1''$; whence the Time of describing that Space 225 Feet was about $14'''$; and that of describing 50 Feet, about $3,1'''$; consequently in $1,87'''$ the Loss of Motion will be about 31 Feet *per* $1''$; wherefore $31 \times \frac{1}{12} = 2$ *lb.* 10 *oz.* nearly, which is the Resistance of the Air to this Velocity.

3073. Having thus determined by Experiment what Resistance the Bullet meets from the Air with two different Degrees of great Velocity; let us next see what will result from a Computation made from the Theory established by Sir *I. Newton* for slow Motions, the Principles of which we have already explained in Chap. V. and shall now apply them. Since the Weight of a Globe of Air, equal to the Bullet, is inconsiderable in Comparison of the Weight of the Bullet itself, we have (2061) $A = B$. And consequently $\sqrt{\frac{S}{16,2}} = T$, (2066). Also because it is $D = \frac{3}{4}$, we have $\frac{4}{3} D = 1$; and so $S = \frac{D}{d}$; but Lead is 11,345 Times heavier than Water, and Water is 860 Times heavier than Air; whence $D : d :: 9756,7 : 1$. Consequently $S = 9756,7$ Inches $= 813$ Feet; whence

whence $\sqrt{\frac{S}{16,2}} = 7'' = T$, the Time of the Fall through S.

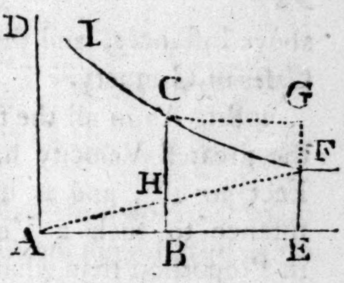
Then the Velocity $V = \frac{2\,S}{T} = 229,5$ Feet *per* 1″.

3074. Now this is the greatest Velocity the Bullet can acquire by falling in the Air (2061) and the Resistance it then meets with is equal to $B = A = \frac{1}{12}$ *lb.* = Weight of the Bullet; if we would have the Resistance, therefore, to any other Velocity, as that of 1600 Feet *per* 1″, we must say, as $229,5^2 : 1600^2 :: \frac{1}{12}$ *lb.* : 4 *lb.* nearly. But we have before found by undeniable Experiment, that a Velocity of about 1500 Feet *per* 1″ (3072) meets with a Resistance of 12 *lb.* nearly; therefore the Resistance to swift Motions is greater than that to slow Motions (which is as the Squares of the Velocity) in the Ratio of 12 to 4, or 3 to 1; according to what was delivered in the Theory (2672).

3075. If we enquire by the Theory what Resistance a Velocity of 950 Feet *per* 1″ will meet with, we shall find it to be about 1,427 *lb.* but by Experiment it was found to be in Reality 2 *lb.* 10 *oz.* which is nearly as 1 to 2. We shall see that this Resistance is very sensibly encreased, even in so small a Velocity as that of 400 Feet *per* 1″. For by Experiment, a Bullet discharged with that Velocity ranged but 319 Yards, or 957 Feet, on the Surface of Water. Whereas by the Theory for slow Motions, it should have ranged to the Distance of 1109 Feet, as I shall now demonstrate.

3076. The Resistance to a Velocity of 400 Feet by the Theory is 0,253 *lb.* (found as above); then say, as $\frac{1}{12} = 0,083$ *lb.* : 0,253 *lb.* :: 16,2 Feet : 49,4 Feet, the Space a Body would descend through in one Second if urged by a constant uniform Force equal to the Resistance 0,253 *lb.* (999). Whence $49,4 \times 2 = 98,8$ Feet is the uniform Velocity acquired in 1″. And since the Times are as the Velocities (1000) when the Force is given; therefore say, as 98,8 Feet : 400 Feet :: 1″ : 4″,049 = the Time in which the Velocity of 400 Feet *per* 1″ will be generated by the Force = 0,253 *lb.*

3077.



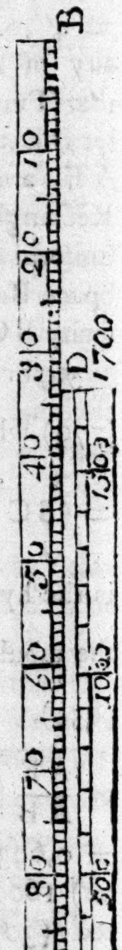
3077. Now let ICF be an Hyperbola, whose Asymptotes are AE, AD. Let AB be the Time just now found $= 4'',049$; and BC the given Velocity of 400 Feet *per* 1″, generated in that Time by the Force 0,253 *lb.* Also let BE $= 4''$, draw CG parallel to AE, and EG to BC, then will the Rectangle BCGE be as the Space described by the constant uniform Velocity BC in the Time BE; and the hyperbolic Space BCFE will be as the Space described with the same Velocity BC decreasing by the Resistance (2096).

3078. Now $AB \times BC = 1 =$ Power of the Hyperbola, (779) whence $BC = \frac{1}{AB}$; therefore the Rectangle BCGE $= (BC \times BE =)\, BE \times \frac{1}{AB} = \frac{BE}{AB}$; and the Area BCFE is the hyperbolic Logarithm of the Ratio $\frac{AE}{AB}$, which is equal to the tabular Logarithm of the same Ratio multiplied by 2,302585 (854). But $\frac{BE}{AB} = \frac{4}{4{,}049} = 1$, nearly; and the Logarithm of $\frac{AE}{AB}\left(\frac{2}{1}\right) = 0{,}301030$; therefore $0{,}301030 \times 2{,}302585 = 0{,}6931$, &c. Therefore say, as 1 : 0,6931 :: $(BC \times BE =)$ 1600 Feet : 1108,96 = 1109 Feet nearly, as before asserted (3075). The Resistance therefore by this Theory is much less than what it really is, in as much as it gives a Velocity of 152 Feet in 1109 more than the Truth, in this so slow a Motion.

3079. Where this Theory can take Place, we have FE = Velocity at the End of the Time BE; and is easily had thus, as AE (= 2) : AB (= 1) :: BC (= 400 Feet) : FE (= 200 Feet). Also, by drawing AF, it will cut BC in H; and BH will be as the Resistance at the End of the Time BE, in the Hypothesis of the Resistances being simply as the Squares of the Velocity, which is very erroneous as we have shewn in all the

above Inſtances, and therefore cannot be applied to any actua Caſes of Gunnery.

3080. As in all the foregoing Caſes of ſmall Shot, the greateſt Velocity has not exceeded that of 1700 Feet *per* 1″; and as it has been ſhewn that the Reſiſtance to ſuch a Velocity is three Times more in Proportion than what it is in ſlow Motions (3074) it will be ſtill neceſſary to ſhew what Proportion of this increaſed Reſiſtance belongs to all other leſſer Degrees of Velocity in order to compleat the Theory, and render it of general Uſe. For this Purpoſe let A B be a Right-line divided into 100 equal Parts, and let $CD = \frac{2}{3} AD$, be divided into 1700 equal Parts; then ſince DB : AB : 1 : 3 :: Reſiſtance to a Velocity of 1700 Feet *per* 1″: Reſiſtance given to the ſloweſt Motion; it is evident, that to a Velocity of any given Number of Feet in the Line C D, there will correſpond a Number in the Line A B, which in reſpect of the whole Line, will ſhew what Ratio of the encreaſed Reſiſtance belongs to that given Velocity; thus for Inſtance, to the Velocity of 1500 Feet *per* 1″ correſponds the Ratio of $\frac{40}{100} = \frac{2}{5}$, nearly; to the Velocity of 1000 Feet *per* 1″, the Reſiſtance is nearly in the Ratio of $\frac{60}{100}$ or $\frac{3}{5}$; and that of 500 Feet *per* 1″ has the Ratio $\frac{80}{100} = \frac{4}{5}$, nearly. And ſo for any other Velocity propoſed.



3081. Hence then if the Reſiſtance for any propoſed Degree of Velocity be calculated by the Theory, the Error of the Theory in ſwift Motions may be hereby corrected, and brought pretty near the Truth. Thus, ſuppoſe I find by the Theory that to a Velocity of 1000 Feet *per* 1″, there is a Reſiſtance of 1,5 *lb.* to a Ball of $\frac{1}{12}$ *lb. wt.* if this be encreaſed in the Ratio of $\frac{3}{5}$, by ſaying, as 3 : 5 :: 1,5 *lb.* : 2,5 *lb.* this 4th Number 2,5 = 2 *lb.* 8 *oz.* will be nearly the ſame as was found by Experiment (3072) to be the true Quantity of Reſiſtance. So that by this Contrivance the Theory for ſlow Motion may ſtill be applied to very good Purpoſe.

3082. We have ſhewn that when a Cannon Ball of 24 *lb.* is impelled by a full Charge of Powder, it acquires a Velocity of 1900 Feet *per* 1″; but if it were impelled with only a common Charge (*viz.* 12 *lb.* of Powder $= \frac{1}{2}$ the Weight of the Ball) its Velocity will then be about 1700 Feet *per* 1″. With this Velocity we have ſeen a Bullet of $\frac{3}{4}$ of Inch Diameter will meet with a Reſiſtance of 10 *lb.* (3096). And ſince the Reſiſtance is (*cæteris paribus*) in the duplicate Ratio of the Diameter, or as the Surfaces; and the Square of 5,325 = Diameter of the Cannon Ball, is equal to about 50 Times the Square of $\frac{3}{4}$, the Diameter of the Bullet; therefore the ſaid Cannon Ball will meet with about 50 Times the Reſiſtance of the Bullet, or 500 *lb. wt.* or about 20 Times its own Weight.

3083. That the Reſiſtance is in the duplicate Ratio of the Diameter in Globes or Balls of different Size, is deducible from (2033) where it was ſhewn to be as $\frac{1}{4}ap$, which is half the Area of a great Circle (830) and $\frac{1}{8}$ of the Superficies of the Sphere (839); therefore the Reſiſtance is in the Ratio of the Superficies of Globes, or in the duplicate Ratio of their Diameters (842).

CHAP. XI.

The THEORY *of* RESISTANCE, VELOCITIES, TIMES *and* SPACES *deſcribed by* PROJECTILES *in their perpendicular* ASCENT *and* DESCENT *in* reſiſting Mediums.

3084. BEFORE we can proceed further, it will be neceſſary to raiſe Theorems for aſcertaining the *Spaces*, *Times*, and *Velocities* in the *perpendicular Aſcent and Deſcent of Projectiles*, in a Medium reſiſting in any multiplied Ratio of the Velocity. In order to this, let s = Space, t = Time, v = Velocity, r = Reſiſtance. Then (by 971) we have $vt = s$; and ſince this holds in every Caſe of t and s, therefore $v\dot{t} = \dot{s}$,

3085.

3085. The Resistance being the Reaction of the Medium, will be proportional to the Decrement of Motion it causes in the moving Body; that is, $r = -\dot{q} = -\dot{v}\,m$ (970); and when m is given, it is $r \doteq -\dot{v}$, or the Resistance is as the Decrement of Velocity. The Resistance (r) will also be inversely as the Time (t) in which it produces a given Effect $(-\dot{v})$, therefore $r \doteq \frac{1}{\dot{t}}$; wherefore in general $r = \frac{-\dot{v}}{\dot{t}}$, or $r\,\dot{t} = -\dot{v}$. Hence $\dot{t} = \frac{-\dot{v}}{r} = \frac{\dot{s}}{v}$ (3084) therefore $r\,\dot{s} = -v\,\dot{v}$.

3086. If the Body ascends, it will be retarded by two Forces, the Resistance (r) of the Medium, and the centripetal Force, or Gravity (g); and this Retardation will produce a Decrement of Velocity $(-\dot{v})$ which will be as the Moment of Time $(\dot{t})$, and the Sum of the retarding Forces $r + g$, (as is evident from the Nature of the Thing); therefore in general $r\,\dot{t} + g\,\dot{t} = -\dot{v}$. And for a descending Body, if the Resistance be less than Gravity, it will be $g\,\dot{t} - r\,\dot{t} = \dot{v}$. But if Gravity be less than the Resistance, then $r\,\dot{t} - g\,\dot{t} = -\dot{v}$.

3087. We have shewn that $v\,\dot{t} = \dot{s}$, whence $v = \frac{\dot{s}}{\dot{t}}$; therefore for the ascending Body, we have $\overline{r\,\dot{t} + g\,\dot{t}} \times \frac{\dot{s}}{\dot{t}} = -\dot{v} \times v = r\,\dot{s} + g\,\dot{s}$. But for a descending Body, if Gravity be greater than the Resistance, the Theorem will be $g\,\dot{s} - r\,\dot{s} = v\,\dot{v}$; $r\,\dot{s} - g\,\dot{s} = -v\,\dot{v}$, when the Resistance exceeds Gravity.

3088. If the Body ascends in an unresisting Medium (or in Vacuo) then $r = 0$; and $g\,\dot{t} = -\dot{v}$, and $g\,\dot{s} = -v\,\dot{v}$; but if the Body descends, it will be $g\,\dot{t} = \dot{v}$, and $g\,\dot{s} = v\,\dot{v}$. When the Resistance becomes equal to Gravity, then $r = g$, and $g\,\dot{t} - r\,\dot{t} = \dot{v} = 0$; that is, the Velocity will then become constant or uniform. And since till then the Velocity is continually increasing (for there will be $\dot{v} = g - r$, in any given Time,) it is evident, the Velocity is in that Case a *Maximum*, as we have before shewn (2063).

3089. Suppose the Resistance (r) be as any multiplied Ratio of the Velocity $\left(\frac{v^n}{a^{n-1}}\right)$, that is, let $r = \frac{v^n}{a^{n-1}}$, where (a) is

is a given Quantity. Then for a descending Body it will be $g\dot{s} - \frac{v^n \dot{s}}{a^{n-1}} = v\dot{v}$ (644) and so $\dot{s} = \frac{a^{n-1} v\dot{v}}{g a^{n-1} - v^n}$. Also, because $\dot{t} = \frac{\dot{s}}{v}$, we have $\dot{t} = \frac{a^{n-1}\dot{v}}{g a^{n-1} - v^n}$. And for the ascending Body, $\dot{s} = \frac{-a^{n-1} v\dot{v}}{g a^{n-1} + v^n}$, and $\dot{t} = \frac{-a^{n-1}\dot{v}}{g a^{n-1} + v^n}$.

3090. If we suppose the Resistance to be as the Square of the Velocity, then $n = 2$, and $r = \frac{v^2}{a}$ (3089) let V = greatest Velocity acquired in the Descent, and because in that Case $r = g$ (3087) and v becomes V, we have $g = \frac{V^2}{a}$, and so $ag = V^2$. Let z = Space which a Body falls through in Vacuo, to acquire the greatest Velocity V; then $g\dot{z} = V\dot{V}$ (3088), and (taking the Fluents) $gz = \frac{1}{2}V^2$; therefore $2gz = V^2 = ag$. Whence $a = 2z$.

3091. Therefore in the descending Body, it will be $\dot{s} = \frac{av\dot{v}}{ag - vv} = \frac{2zv\dot{v}}{VV - vv}$; let $V^2 - v^2 = x^2$, and taking the Fluxions, we shall have $v\dot{v} = -x\dot{x}$; and so $\dot{s} = \frac{-2zx\dot{x}}{xx} = \frac{-2z\dot{x}}{x}$. And taking the Fluents, $s = Q - 2zLx$. (Here Q is some constant Quantity, and $L.x$ is the Fluent of $\frac{\dot{x}}{x}$ by (849). But $Q - 2L.x \times z = Q - zL.x^2 = Q - zL.\overline{V^2 - v^2}$. Now when $s = 0$, and then $v = c$ = the Celerity with which the Body begins to descend, we shall have Q in that Case $= zL.\overline{V^2 - c^2}$. And consequently $s = zL.\frac{\overline{V^2 - c^2}}{\overline{V^2 - v^2}}$.

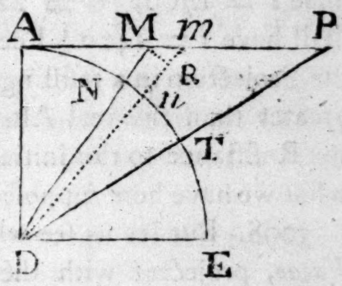
3092. Let $L.d = 1$; and it will be $sL.d = zL\frac{V^2 - c^2}{V^2 - v^2}$, and $\frac{s}{z}L.d = L.d^{\frac{s}{z}} = L\frac{V^2 - c^2}{V^2 - v^2}$; and therefore $d^{\frac{s}{z}} = \frac{V^2 - c^2}{V^2 - v^2}$; whence

whence we get $v^2 = \dfrac{V^2 \, d\overset{s}{\bar{z}} + c^2 - V^2}{d\overset{s}{\bar{z}}}$. And for the Time ($t$) of the Descent, we have $\dot{t} = \dfrac{a\dot{v}}{ag - v^2}$ (3089) $= \dfrac{2z\dot{v}}{V^2 - v^2}$ $= \dfrac{\frac{z}{V}\dot{v}}{V + v} + \dfrac{\frac{z}{V}\dot{v}}{V - v}$ (as will appear by reducing the two last Fractions to a common Denominator.) Therefore taking the Fluents, we have $t = Q + \frac{z}{V} L . \overline{V + v} - \frac{z}{V} L . \overline{V - v} =$ $Q + \frac{z}{V} L \dfrac{V + v}{V - v}$; let $t = o$, then $v = c$; and we shall have $Q = - \frac{z}{V} L \dfrac{V + c}{V - c}$. Consequently, we have $t = \frac{z}{V} L . \dfrac{\overline{V + v} \times \overline{V - c}}{\overline{V + c} \times \overline{V - v}}$.

3093. If the Body falls from a State of Rest, then the inceptive Velocity $c = o$; and the above Equations will become for the Space $s = z L \dfrac{V^2}{V^2 - v^2}$; for the Velocity at the End of the Fall $v^2 = \dfrac{V^2 d\overset{s}{\bar{z}} - V^2}{d\overset{s}{\bar{z}}}$; and for the Time of the Fall $t = \frac{z}{V} L \dfrac{V + v}{V - v}$. If we put $d\overset{s}{\bar{z}} = m$, then $v = V \sqrt{1 - \frac{1}{m}}$.

3094. In the same Manner, we find the Theorems for an ascending Body are $s = z L \dfrac{V^2 + c^2}{V^2 + v^2}$; $v^2 = \dfrac{V^2 + c^2 - V \, d\overset{s}{\bar{z}}}{d\overset{s}{\bar{z}}}$; and $\dot{t} = \dfrac{a\dot{v}}{ag + vv} = - \dfrac{2z\dot{v}}{VV + vv}$ (3089, 3090). Now this fluxionary Equation is analogous to that which is found for the Fluxion of an Arch of a Circle, by Means of the Radius and Tangent of that Arch; and therefore the Fluent of the Time may

may be found in the Measure of the Arch of a Circle, as follows.



3095. On the Center D, with the Radius $AD = V$, describe the Quadrant ATE; let the initial Velocity c be expounded by the given Tangent AP, the residual Velocity v by a Part of that Tangent AM; and draw Dm infinitely near to DM; and then $\dot v = Mm$; draw DP; and from the Point M let fall the Perpendicular MR upon Dm; then are the Triangles DNn, and DMR similar; and so DM : DN (or DA) :: MR : Nn. Also the Triangles mRM and MAD are similar; and so DM : DA :: mM : MR. The respective Terms of these two Analogies being multiplied together, give $DM^2 : DA^2 :: Mm : Nn$; that is, $VV + vv : VV :: \dot v : Nn = \frac{VV\dot v}{VV + vv} =$ the Fluxion of the Arch AN.

3096. If this last Equation be multiplied in each Part by $\frac{2z}{VV}$, it will become $\frac{2z \times Nn}{VV} = \frac{2z\dot v}{VV + vv}$; therefore $\dot t = -\frac{2z \times Nn}{VV}$ (3094); and taking the Fluents, it is $t = Q - \frac{2z \times AN}{VV}$. Let $t = o$, then will $AM = AP$, and $AN = AT$, and in that Case, therefore, $Q = \frac{2z \times AT}{VV}$. Therefore it will be $t = \frac{2z \times \overline{AT - AN}}{VV} = \frac{2z \times TN}{VV} = \frac{TN}{g}$; because $2zg = V^2$, by (3090).

3097. To apply these Theorems to the Motion of a Bullet shot from a Gun in a perpendicular Direction upwards, on Supposition that the Resistance is always in the *duplicate Ratio* of the Velocity. Let us take the Example of (3059) where the Bullet is projected with a Velocity $c = AP = 1700$ Feet *per* 1″. To determine the Height or Space of that Projection, we have $s =$

$z\,L\,\frac{V^2 + c^2}{V^2}$ (by 3094) becauſe in this Caſe $v = o$. Now ſince $c = 1700$, $V = 229{,}5$ (3073); and $z = 813$ (*ib.*) we ſhall have $s = 1420\frac{1}{2}$ Feet, for the perpendicular Altitude of the Projection in a reſiſting Medium. Now this is conſiderably greater than the real Altitude, becauſe we have ſhewn (2072) the Reſiſtance to the initial Velocities is three Times more than what we have here ſuppoſed.

3098. But let us ſee what Height the Bullet will aſcend to *in Vacuo*, projected with the ſame Velocity of 1700 Feet *per* 1″. This is found by the Theorem $\frac{m^2}{4n} = a$ (1154) where $m = 1700$, and $n = 16{,}11$, and $t = 1''$. Whence $m^2 = 2890000$, and $4n = 64{,}44$; and ſo $a = 45675$ Feet, which is about 32 Times higher than before; and very likely 40 Times higher than it does really aſcend in Air with the given Velocity.

3099. The Time in which this perpendicular Altitude *in Vacuo* is deſcribed, is found by Theorem (1152) $\frac{stm}{n} = T$, for in this Caſe $s = r = 1$, and $t = 1''$; therefore the Time of Aſcent and Deſcent is $\frac{m}{2n} = \frac{1700}{32{,}22} = 53''$ nearly.

3100. Now the Time in which the Bullet aſcends perpendicularly to the Height of $1420\frac{1}{2}$ Feet in the Air is equal to $\frac{TA}{g}$ (by 3096), becauſe in this Caſe T N becomes T A (ſee Fig. to 3095). To find the Arch A T we have the Radius $AD = V = 229{,}5$, and $AP = c = 1700$; wherefore ſay,

As A D	229,5 —	2.360783
Is to A P	1700 —	3.230447
So is Radius	90° 00′ —	10.000000
To the Tangent of the Angle P D A =	82° 18′ —	10.869664

3101. The Diameter is $2\,AD = 459$; therefore ſay, As $1 : 3{,}14159 :: 459 : 1442$, the Circumference of the Circle. Then ſay, As the Circumference 360° is to the Arch 82° : 18′ ſo is 1442 to the Length of the Arch $AT = 329{,}8$. Now

Gra-

Gravity produces an uniform Velocity $g = 32,22$ Feet *per* 1″. Whence $\frac{329,8}{32,22}$ $\left(= \frac{AT}{g}\right) = 10''$ nearly, the Time of the Ascent in the Medium of Air, resisting according to the duplicate Ratio of the Velocity. The Time therefore in this Case is not more than $\frac{1}{5}$ of that which is spent in the Ascent *in Vacuo* (3099).

3102. The Time of the Descent through the Air from the said Altitude of $1420\frac{1}{2}$ Feet, is $\frac{z}{V} L \frac{V+v}{V-v} = t$ (by 3093). But in order to determine this, we must first find the Velocity (v) acquired in the Descent from the Theorem $v = V\sqrt{1 - \frac{1}{m}}$ (in Art. 3093) where $V = 229,5$; $d = 10$, (because its Logarithm $= 1$ (by 3092), $s = 1420,5$; $z = 813$; and $m = d^{\frac{s}{z}}$ $= 55,86$. Whence we find the Velocity $v = 227,44$ Feet *per* 1″. Therefore $V + v = 56,94$; and $V - v = 2,06$; and $L \frac{V+v}{V-v} = 2,345992$; also $\frac{z}{V} = 3,543$; therefore $t =$ 8,″31. Whence it appears that the Times of the Ascent and Descent are in the Ratio of 10 to 8,3; or nearly as 5 to 4, and both together make but 18″,3 which is 35″ less than the Time *in Vacuo* (3099).

3103. Hence it appears they are widely mistaken who assert, that if an heavy Body be projected upwards with a Velocity greater than that which can be acquired in falling, the Time of the Descent will be greater than that of the Ascent; since it is demonstrated, that a longer Time is required to destroy a great Velocity in ascending, than in generating a much smaller one in descending through the same Space.

3104. From this Theory also it is manifest, that if a Ball or Shot be projected downward with a Velocity equal to the greatest that can be acquired in falling, the Motion will be uniform; but if it be projected with a Velocity greater than that, the Motion will be retarded; if with a less Velocity, it will be accelerated, but never will become equal to that greatest Velocity, all which is evident from the Theorem in Art. 3092.

3105. Hence alſo we ſee the Reaſon why the Force acquired by a Bullet in falling is ſo much leſs than that with which it is projected; for ſince the Maſs of Matter continues the ſame, the Forces will be as the Velocities, that is, in the foregoing Example (Art. 3097, 3102). The Force of the Bullet, when projected, from the Muzzle of the Gun, is to the Force it acquires in falling, as 1700 to 227,5; or nearly as 8 to 1.

3106. Indeed, with reſpect to very light Bodies, the Caſe may be a little different; and the Times of Deſcent of an Arrow, a Ball of Wood, *&c.* may be greater than that of the Aſcent, ſince they are reſiſted by the Air in a much greater Proportion to their Quantity of Matter, on Account of their larger Quantity of Surface; what this may be, as nothing of Conſequence depends upon it, I have not here calculated; but it may be eaſily done by any one who underſtands the foregoing Theory, and has Curioſity and Leiſure for ſuch Amuſements.

3107. In all that has been hitherto ſaid of the perpendicular Projection in a reſiſting Medium, we may obſerve what an egregious Difference there is between ſuch a Shot and one made in *Vacuo* in regard of the *Height*, the *Time*, and the *Velocity* and *Force* of the PROJECTILE; and how widely the latter is different from the Truth.

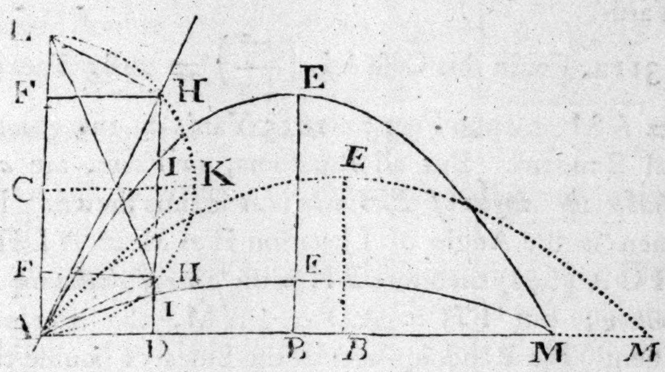
CHAP. XII.

The common practical RULES *of* GUNNERY, *derived from the* parabolical HYPOTHESIS, *compared with* EXPERIMENTS, *and thereby ſhewn to be extremely* fallacious, *and of no Uſe in* PRACTICE.

3108. WE next proceed to ſhew how very fallacious all the Rules and Caſes of *practical* GUNNERY are, as they are derived from the Principles of the *parabolical Hypotheſis*, which we have explained (in 1141, *&c.*) and ſhall here apply, in order to confute them by Experiments. After this we ſhall conſider the Nature of the Motion of a Body projected obliquely in a re-

resisting Medium, and the Form of the Path it describes, and shew it to be no PARABOLA (as is always supposed) nor any other *regular* or *geometrical Curve*.

3109. It was shewn (1155) that the greatest Amplitude or Random of a Shot (in Vacuo) was that which was made upon an Elevation of 45°. And it appears, that AD is equal to half AL, and therefore since AD $= \frac{1}{4}$ AM, we



shall have AM $=$ 2AL $=$ the *greatest Random* possible. But we have shewn (in our common Example (3098) that AL $=$ 45675 Feet; and therefore 2AL $=$ 91350 Feet $=$ AM, the Random on 45°. Now this is about 17,3 Miles; but all experienced Gunners, and practical Writers assure us that this Range is actually not quite $\frac{1}{2}$ a Mile; and so *Mersennus* found the horizontal Range of an Arquebuse of 4 Feet to be less than 800 Yards, which is not quite $\frac{1}{2}$ a Mile. The Range therefore by this false Theory is about 35 Times greater than the Truth.

3110. The Case is the same with respect to large Shot as in small ones; for St. *Remy* (as quoted by Mr. *Robins*) tells of some Experiments made by Mr. *Du Metz*, in which the Range at 45°, of a Piece 10 Feet in Length, carrying a Ball of 24*lb.* and charged with 16*lb.* of Powder, was 2250 *French* Fathom, which is not quite 3 Miles. But such a Ball was projected with a Velocity of about 1900 Feet *per* 1″, (Art. 3043.) and therefore *per* Theorem (1154.) $\frac{m^2}{4n} =$ AL $=$ 56021 Feet; and so 2AL $=$ AM $=$ 112042 Feet $=$ 21 Miles, which is more than 7 Times the real Random, in the resisting Medium of Air.

3111. The same Difference between this Hypothesis, and the Truth, will be found in Proportion to obtain in smaller Degrees of Velocity, and on any other Elevations, as is evident from Mr.

Mr. *Robins*'s Experiments compared with this erroneous Theory. For a leaden Bullet, $\frac{3}{4}$ of an Inch Diameter, discharged with a Velocity of 400 Feet *per* 1″, and in an Angle of 19° : 05′, of Elevation ranged on the horizontal Plane no more than 448 Yards, whereas by the Theory it ought to have ranged 1023 Yards.

3112. For in this Case A L $\left(\frac{m^2}{4n}\right) = 2483$ Feet; and 2 A L $=$ A M $= 4966$ Feet $= 1655$ Yards $=$ the greatest horizontal Random. But all Randoms, *in Vacuo*, are *as the Sines of double the Angles of Elevation*; as is thus shewn. Draw C H,* then is the Angle of Elevation H A M $=$ A L H (665) $= \frac{1}{2}$ H C A (642) therefore F H is the *Sine of double the Angle of Elevation*; but F H $=$ A D $= \frac{1}{4}$ A M, the Random. Consequently the Randoms are as the Sines of double the Angles of Elevation. Therefore say,

As the Sine of double 45° = 90°	——	10.
To the Sine of double 19° : 5′ = 38° : 10′	——	9.790954
So is the greatest Random 1655 Yards	——	3.218798
To the Random on 19° : 5′ = 1023 Yards	——	3.009752

So that the Range made in the resisting Air is but about $\frac{3}{7}$ of what it would be in the *parabolic Hypothesis*.

3113. Again, a Ball was discharged with the same Velocity as in the last Experiment, but on an Elevation of 9° : 45′; and its Range on the Horizon was at a Medium 990 Feet, or 330 Yards. Now if this were to be deduced directly from the Theorem (in 1152) $\frac{c\,s\,m^2}{n} =$ A M, we should find A M $= 1655$ Feet, or 552 Yards nearly, as in the following Operation.

Add the Logarithms of	Sine of 9° 45′	——	9.228784
	Cosine, 80° 15′	——	9.993681
	And of $m^2 = 160000$	——	5.204120
From that Sum	—— —— ——		4.426585
Subduct the Value of $n = 16{,}11$	—— ——		1.207095
The Remainder is the Range A M $= 1665{,}6$	——		3.219490

3114.

* The Reader is desired to draw this Line in the preceding Figure, as it is there (by Accident) omitted.

3114. But if this Random were to be computed from the Random on an Elevation of 19° : 30′ found in (Art. 3112.) it would come out a very different Number from either of the foregoing; for say

As the Sine of double 19° : 5′ = 38° : 10′	—	9.790954
Is to the Sine of double 9° : 45′ = 19 : 30	—	9.523495
So is the Random on 19° : 5′ = 448	—	2.651278
To the Random required —— 242	—	2.383819

Now this is as much too little as the other was to big, and proves the Theory false on every Account.

3115. Again, for greater Variety, a Ball was fired at an Elevation of 8°, and with a Velocity of 700 Feet in 1″, and the horizontal Range at a Medium was 690 Yards; but if this Range be computed from the common Theory, we shall have

$$2\,AL\,\frac{m^2}{4\,n} = AM = 15208 \text{ Feet, and then by (Art. 1154.)}$$

we shall find the Random at an Elevation of 8° to be 4192 Feet, or 1400 Yards nearly, which is more than double the real Distance.

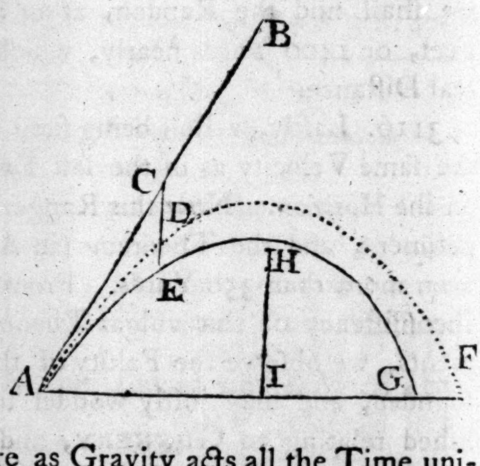
3116. Lastly, a Ball being fired at an Elevation of 4° with the same Velocity as in the last Experiment ranged 600 Yards on the Horizon. Now this Range if deduced from the last Experiment, and the Theorem (in Art. 1154.) should not have been more than 350 Yards. From all, which Instances of the Inconsistency of this vulgar Theory, with Facts and Experiments, we observe the Falsity of the Hypothesis on which it is founded, and may justly wonder to see every Day Books published relating to GUNNERY, and the *Doctrine* of Projectiles, with as strong a Presumption of these false Principles, as though Philosophy were not in the least understood, and the Nature of these Things had never been enquired into by any one.

CHAP.

CHAP. XIII.

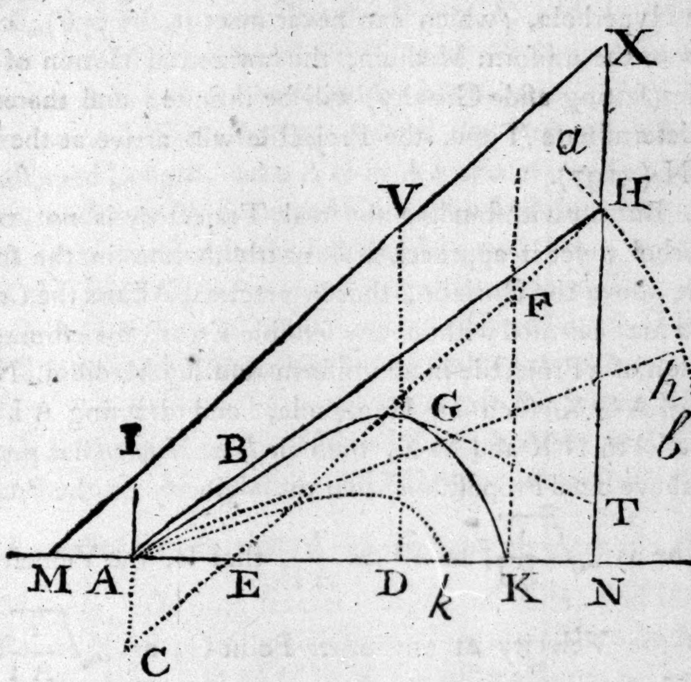
Sir ISAAC NEWTON'*s Method of investigating the* PATH *of a* PROJECTILE *in a* resisting *Medium illustrated, applied to practical* GUNNERY, *and exemplified by* EXPERIMENTS.

3117. BUT now that the ENGINEER may be assured from a genuine *Physico-mathematical* THEORY, that the Path which a Projectile describes, when thrown from the Gun in any oblique Direction, even when the Resistance of the Medium is no more than proportional to the *Square of the Velocity*, cannot possibly be a PARABOLA, nor any regular *geometrical Curve*, he is to consider in the first Place, that since the Body is prevented going on in the Right-lined Direction A B by the perpendicular Action of Gravity by which it is brought down from any Point C to the Point D in the Curve of the Parabola A D F in Vacuo, so if we suppose the Projection to be made in a resisting Medium, the Resistance will lessen the Velocity of the Ball, and prolong the Time of its arriving to the Distance of D, and therefore as Gravity acts all the Time uniformly, it will produce in the Projectile a greater perpendicular Descent from the Point C, and carry it from C to some Point E below D; and since this is the Case every where, it follows that the Curve A E G described by the Projectile, cannot be the *Parabola* A D F, but some other Curve contained within it.



3118. Again, since the Velocity of the Projectile in the descending Part of the Curve H G will be always much less than that in the ascending Part A H, as we have above demonstrated (3102).

(3102) therefore Gravity will in this Part carry the Ball through much greater perpendicular Spaces (in the ſame Time) from the parabolic Curve; and ſo the Curve in the Part HG will have a leſs Degree of Curvature, and therefore the Point G, where it falls on the Horizon, will be much nearer to the Line HI of the greateſt perpendicular Altitude, than the Point A from whence it was projected, different from what happens in the *Parabola*.



3119. And Sir ISAAC NEWTON has ſhewn, that the Curve AHG approaches nearer to the Form of an *Hyperbola* than that of a *Parabola*; for an *Hyperbola* will be truly deſcribed in a reſiſting Medium, whoſe Denſity is every where inverſely as the Tangent to the ſaid Curve, as he has proved in Prop. X. Lib. II. of the *Principia*. And therefore if AGK be the hyperbolic Trajectory of the Ball, projected from the Point A in the Direction AH, (in a Medium of a variable Denſity) whoſe Aſymptotes are MX, XN, cutting the horizontal Line MN in the Points M and N; and of which the latter XN is perpendicular thereto; then ſince in this Caſe the Denſity in any Points A and G is inverſely as the Tangents AH and

GT, it is manifeſt if the Medium be now ſuppoſed of an uniform Denſity, ſuch as is equal to the mean Denſity of the Medium in Queſtion, then in ſuch an one, on Account of the greater Denſity at A, the Velocity of the Projectile will be more diminiſhed than in the other Caſe, and conſequently the Trajectory will be continued within the Hyperbola AGK (by 3117) in the aſcending Part AG. But in the deſcending Part GK it will approach nearer to the Aſymptote XN than the Hyperbola, (which can never meet it, by 776); conſequently in the uniform Medium, the horizontal Motion of the Projectile (ſetting aſide Gravity) will be infinite; and therefore after a determinate Time, the Projectile will arrive at the ſaid Line XN (3097).

3120. But notwithſtanding the real Trajectory is not truly an Hyperbola, yet it approaches ſo nearly thereto in the ſmall Part of it above the Horizon, that in practical Affairs the *Conic-Hyperbola* may be uſed without any ſenſible Error, for eſtimating the Motion of a Projectile in an uniform reſiſting Medium. Now ſuppoſing AGK ſuch an Hyperbola, and drawing AI and HC parallel to NX and MX, then Sir *Iſaac Newton* has proved (in the above cited Propoſition) that the Velocity of the Projectile will be as $\sqrt{\frac{1}{AI}}$, or $V^2 = \frac{1}{AI}$, that is, the Velocity at A is to the Velocity at any other Point G, as $\sqrt{\frac{1}{AI}}$ to $\sqrt{\frac{1}{GV}}$. Alſo, that the Reſiſtance of the Medium is to the Force of Gravity (in any Point A) as AH to $\frac{4}{3}$ AI. Moreover, from the Nature of the Curve, $AM = KN$; and $AI = AC$; and by Conſtruction $IC = (2\,AI =)\,HX$.

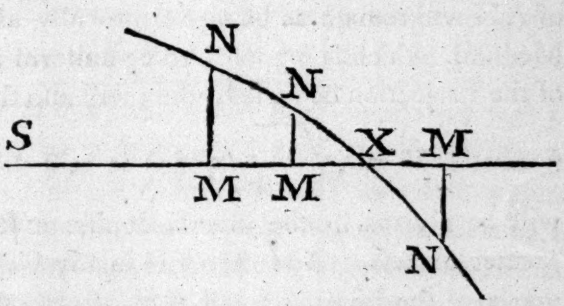
3121. Theſe Things premiſed, it is evident that if the Lines AI and AH be given in Magnitude and Poſition, the Hyperbola AGK may be deſcribed, becauſe in this Caſe we have given $HX = 2\,AI$, and therefore the Center of the Hyperbola X. Hence alſo the Aſymptote XN, and ſince the Point I is given, we have alſo the other Aſymptote XM cutting the horizontal Line in M. Hence, laſtly, we have given the Point K, by

taking

taking K N = A M (776 to 780). It remains now to ſhew how the Lines A I and A H may be determined in the preſent Caſe of the Problem by Experiments, and thence an expeditious Method of deſcribing the Hyperbola A G K, or Trajectory of the Projectile; and thence a Solution of all the Caſes of the Problem.

3122. In order to this, let two equal Balls be diſcharged with the ſame Velocity on two different Angles of Elevation H A K, and *h* A *k*; and let the Points K and *k* where they fall on the Horizon be obſerved, and the Proportion of their Diſtances A K and A *k* be found by Meaſuration. And let A K : A *k* :: *d* : *e*. Then having erected the Perpendicular A I of any aſſumed Length, take A H or A *h* alſo of any Length whatſoever, and then by Scale and Compaſſes find the Lengths A K, and A *k*, as directed in the laſt Article; and if they are found to be in the ſame Ratio as *d* to *e* (by Experiment) then was the Length A H rightly aſſumed, and the Hyperbola A G K every Way ſimilar to that deſcribed by the Ball in the Air.

3123. But if not, take in the indefinite Right-line S M, the Length S M equal to the aſſumed Length A H; and erect the perpendicular M N equal to the Difference of the Ratios of A K to A *k*, and of *d* to *e*; that is, let $MN = \frac{AK}{Ak} - \frac{d}{e}$. And in like Manner, aſſuming ſeveral Lengths of A H, find ſeveral other Points N, thro' all which draw a regular Curve N N X N, cutting the Line S M M M in X. Laſtly aſſume A H = S X, and thence again find the Lengths A K and A *k*; theſe ſhall have the ſame Ratio with *d* and *e*, or the ſame Lengths found by Experiment; and A I, and this laſt found A H, ſhall be ſimilar



to, or have the ſame Proportion with thoſe which belong to the Hyperbola deſcribed by the Ball in the Air. For ſince $\frac{AK}{Ak} - \frac{d}{e}$

$\frac{d}{e} =$ MN, where SM = SX there MN = o, and consequently there $\frac{AK}{Ak} = \frac{d}{e}$; and so AK : A$k$ :: d : e.

3124. To illustrate this by an Example; let it be required to find AI and AH for the two Shots made upon the two Elevations of 19° 5′ and 9° 45′ (in Art. 3112, 3113) where the horizontal Ranges were 448 Yards = d, and 330 = e. Therefore $\frac{d}{e} = \frac{448}{300} = 1,36$. Then assuming AI = 10, and AH = 70, you find AK = 45, and Ak = 37,8, and so $\frac{AK}{Ak} = \frac{45}{37,8} = 1,19$ (by 3122). And then $\frac{AK}{Ak} = \frac{d}{e} = 1,36 - 1,19 = 0,17 =$ MN. In like Manner, if you assume other Lengths AH = 60, 50, 45, 40, respectively you will find other Values of MN = 1,23. 1,28. 1,39. 1,50. Then drawing a Curve through the extreme Points of all the Ordinates MN, and it will intersect the Line SM in X, so as to give the Abscissa SX = $46\frac{1}{2}$ = AH sought; and in this Case AK will be to Ak as 448 to 330; or measured upon the Scale by which AI and AH were laid down, they will be AK = 25,7 and Ak = 18,8.

3125. Having thus obtained the Proportion or Values of the Lines AI and AH in this one Case, they remain unalterably the same for all other Angles of Elevation, for Discharges of the same Velocity. And indeed in all Cases whatsoever, the Value of AH will remain as being reciprocally as the Density of the Medium, which is supposed to be uniform; but if the Velocity of the Projection be varied, then will also the Value of AI. For since it is $V = \sqrt{\frac{1}{AI}}$ (3120) it will be $AI = \frac{1}{V^2}$, or AI will be always in the inverse duplicate Ratio of the Velocity greater or less. And therefore in any Case of a given Velocity you have the Lines AI and AH, and so the Trajectory may be described for any Angle of Elevation HAN, according to this Theory.

3126.

3126. Hence likewiſe the Ratio of the Reſiſtance (R) in the Point A to Gravity (G) is given. For R : G :: A H : $\frac{4}{3}$ A I (3120) :: 4,65 : $\frac{4}{3}$ = 1,333 (3124). Therefore if the Weight of the Bullet were $\frac{1}{12}$ *lb.* then $\frac{4}{3}$: 4,65 :: $\frac{1}{12}$: $\frac{13,95}{48}$ = 0,288 *lb.* the ſame nearly as found by the common Method to the ſame Velocity (in 3076).

3127. Having now found the Values of A I and A H for any propoſed Velocity of Projection, we may proceed to find the Random A K generally for any given Velocity and Angle of Elevation H A N; in order to this let A H = a, A I = b, and then H X = 2 A I = $2b$. Alſo let A K = x, A N = z, and N H = y; and we ſhall have $z - x$ = K N = A M = A E, and A C = A I = b. Alſo E N = A K = x. Then by the ſimilar Triangles E A C, E N H, we have this Analogy A E ($z - x$) : E N (x) :: A C (b) : H N (y). And thence compounding, $z : x :: b + y : y$; whence $x = \frac{zy}{b+y}$, and $z = \frac{bx + xy}{y}$; and $zz = xx \times \frac{\overline{b+y}^2}{yy}$. But becauſe of the right Angle A N H, it is $aa - yy = zz = xx \times \frac{\overline{b+y}^2}{yy}$. And ſo $\frac{a^2y^2 - y^4}{\overline{b+y}^2} = xx$; and therefore x = A K = $\sqrt{\frac{a^2y^2 - y^4}{\overline{b+y}^2}}$.

3128. Hence alſo if the Amplitude A K be given, then may the Sine H N of the Elevation neceſſary for ſtriking any Object K at the given Diſtance A K be found by the ſame Theorem. For we ſhall thereby get $b^2x^2 = \overline{a^2 - x^2} \times y^2 - y^4 - 2bx^2y$; from which Equation by a few eaſy Trials, the Value of y = N H may be found. Or in a Method eaſy for Practice thus. On the Center A with the Radius A H deſcribe a Circle, as (ab); and on the given Point K erect the Perpendicular K F; then apply a Ruler to the Point C in ſuch Manner that the Part F H intercepted between K F, and the Circle ab may be equal to C E. Then through the Point H, draw the Right-line A H, and it ſhall be the Direction or Elevation required. For ſuppoſing the Angle given, we have always C E = F H, becauſe (by

(by Construction 3120) the Angles ACE and HEN are similar, and therefore CE : AE :: EH : EN :: FH : KN; hence since AE ($=$ MA) $=$ KN, we have CE $=$ FH. And the Point H must be in the Circle $a\,b$, because of the given Length AH.

3129. By putting the above Equation (3127) into Fluxions, and making the Fluxion of $x =$ AK equal to nothing, we shall get an Equation for the Sine NH of the Angle or Elevation which of all others will produce the greatest horizontal Random AK; for then we shall get $2a^2y\dot{y} - 4y^3\dot{y} = 2bx^2\dot{y} + 2x^2y\dot{y}$; and dividing by $2\dot{y}$, we have $a^2y - 2y^3 = bx^2 + x^2y = x^2 \times \overline{b+y}$. But $xx = \frac{a^2y^2 - y^4}{\overline{b+y}^2}$; wherefore $a^2y - 2y^3 = \overline{b+y} \times \frac{a^2y^2 - y^4}{\overline{b+y}^2} = \frac{a^2y^2 - y^4}{b+y}$. Whence by Reduction, and by dividing by y, we get $aab = 2byy + y^3$.

3130. Here it is plain the Quantity $2byy$ is less than aab, and therefore yy less than $\frac{1}{2}aa$, or HN^2 less than $\frac{1}{2}$ AH^2; and consequently the Angle HAN is less than half a Right-angle, which it would be in an unresisting Medium, (see 1155) and in that Case, we have $yy = \frac{1}{2}aa$, and $y = a\sqrt{\frac{1}{2}}$, which Value of y, if it be substituted in the above Equation, in order to reduce it, it becomes $aab = 2byy + yya\sqrt{\frac{1}{2}}$; whence $yy = \frac{aab}{2b + a\sqrt{\frac{1}{2}}}$ and so $y = \sqrt{\frac{aab}{2b + a\sqrt{\frac{1}{2}}}}$.

3131. Let us now see what the Quantity of this Angle is for the Values of AI and AH as determined above (3124) where, if we put AI $= b = 1$, then AH $= a = 4{,}65$, and the Equation is $aa = 2yy + y^3 = 21{,}6225$. Now $y = \sqrt{\frac{21{,}6225}{2 + 4{,}65\sqrt{\frac{1}{2}}}} = \sqrt{\frac{21{,}6225}{5{,}255}} = 2{,}03$. But it is plain this Value of y is too small, for it makes $2yy + y^3$ little more than 16, whereas it must be $= 21{,}6255$; this was occasioned by making $y = a\sqrt{\frac{1}{2}} = 3{,}25$ according to the customary Method, which makes one of the Roots too large, and consequently the whole Quantity $2yy + y^3$ too small. The best Way, then, to approximate to the Value of y is (*Tentando*) by Trial, for by this Means we shall easily find that $y = 2{,}26$ nearly. Therefore say, as AH $= 4{,}65$: HN $= 2{,}26$ :: Radius : Sine of 29° nearly; whereas *in Vacuo* we have shewn this Angle is precisely 45 Degrees (1155).

3132. If it be required to find what this *Maximum* of Amplitude is for the above found Values of A I and A H, you have it determined by the Theorem in (3127) (and also for any other Degree of Velocity, as shewn in 3125). So according to the present Case (3124) we shall find the greatest Value of $AK = x = \sqrt{\frac{a^2 y^2 - y^4}{b - y^2}} = 2{,}83$ nearly; and therefore say, as $AK = 2{,}57$ (by Experiment) : $AK = 2{,}83$ (a *Maximum*) :: 448 Yards : 490 Yards nearly, the greatest Amplitude.

3133. If the greatest Ordinate or Height of the Projection be required, let the Line D N be taken in the horizontal Line A N, a Mean proportional between A M and A N, and through the Point D, draw the Ordinate G D, and it will be the *Maximum* required. This Ordinate G D is equal to the Difference between the vertical Line N X, and a 4th Proportional to the Lines DN, AN, and $2AI = IC$; or $GD = NX - \frac{2AI \times AN}{DN}$. The Demonstration hereof may be easily deduced from the abovementioned Proposition of Sir *Isaac's Principia* (3119). But it is too prolix and intricate to be here inserted.

3134. To exemplify this in the last Case of the greatest Amplitude A K; we have $AK = x = 2{,}83$, and $y = HN = 2{,}26$ (3131). Also we have $z = AN = \frac{bx + xy}{y} = 4{,}1$ nearly (3127); then $AN - AK = KN = 1{,}27 = AM$. Therefore $DN = \sqrt{AN \times AM} = 2{,}28$. Now $NX = NH + HX = 2{,}26 + 2 = 4{,}26$; and $\frac{2AI \times AN}{DN} = \frac{2 \times 4{,}1}{2{,}28} = 3{,}6$. Therefore $GD = 4{,}26 - 3{,}6 = 0{,}66$; which gives about 110 Yards for the greatest Height of the Projection. It is observable also, that $DK = (DN - KN =)\ 1{,}01$ and $AD = (AK - DK =)\ 1{,}82$; and so the Point D is nearly $\frac{2}{3}$ of the whole Random from the Point A; whereas *in Vacuo* it is always the Middle Point between A and K.

3135. From a bare Inspection of the Trajectory A B G K, it is evident, the Path of the Projectile will nearly coincide with the Tangent A H for a considerable Distance A B, which in a Shot of 24 *lb.* will be about 500 Yards (as the Angle there will not exceed $\frac{1}{2}$ a Degree;) and hence we see the Reason why Mr. *Anderson*, the most eminent of all practical Authors on Gunnery, found himself obliged to suppose that the Track of Shells and Cannon Balls was much less incurvated,

vated, than what it ought to be on the *parabolical Hypothesis*; and in order to reconcile this Circumstance with the said Theory, he imagined that every Shot was impelled to a certain Distance A B in a *strait Line*, or that in such a Distance it was not affected by the Power of Gravity. But so strange and absurd a Supposition, as the Suspension of the Power of Gravity for a Moment, plainly evinced how ignorant he was of the great Diminution of the Velocity of the Shot in its Flight from the Resistance of the Air. The Notion, therefore, of a *Point-blank Shot*, as they call the Distance A B, is entirely groundless, and owing to the vulgar Error of the Air's Resistance being inconsiderable.

3136. We have now delivered all we can think necessary relating the Theory and Practice of Gunnery; and it is with some Regret we are obliged to conclude, that from a true and genuine Theory, the *practical Part* of this most necessary Science appears very perplext and difficult; it is for this Reason we desist from pursuing it any further. The celebrated Mr. *J. Bernoulli* has given a most exquisite Theory of the Path of a Projectile in a resisting Medium, but as the Practice resulting thence depends upon the Quadrature of mechanical Curves, it can by no Means be rendered useful.

3137. The late learned Mr. *Simpson* has also given us an admirable Theory, but the practical Rules that might be from thence deduced would prove so difficult, that few of our ENGINEERS, I fear, will care to be at the Trouble either to understand, or reduce them to use. Mr. *Muller* has supplied us with *converging Series* for the various Cases of Projectiles in a *resisting Medium*; but then he has given no Demonstration of them, and they are upon Supposition that the Resistance is as the *Square of the Velocity* only, which is far from the Truth, as we have shewn; besides *infinite Series* seem but little adapted to Cases of *practicable Gunnery*.

3138. In the last Place, Mr. ROBINS himself, in a posthumous Tract, has left us some of the best Methods of reducing the Genuine (though complicated) Theory of military Projectiles to practice, but even these will be attended with Trouble and Difficulty enough; and besides, being published without any Investigation, or Rationale, which he proposed doing if he had lived, they will be found as little satisfactory as ready in Practice. And I think we may conclude, Sir *Isaac Newton* would not have given us the above Method for deducing the *practical Rules of Gunnery* if he had known of any other that was better.

INSTITUTIONES HOROLOGICÆ, OR, A PHYSICO-MATHEMATICAL THEORY OF CLOCK-WORK.

CHAP. I.

Of the Nature *and* Design *of* CLOCK-WORK *in general*; *and the* PRINCIPLES *on which it depends.*

3139. A THEORY of CLOCK-WORK, I am inclined to think will be looked upon as a Novelty by the *English Mechanic*; as I have seen nothing of that Kind published in our own Language. Indeed I have met with but one Treatise on the Subject of Clock-work in *English*, *viz.* The *artificial Clock-maker*; but this is altogether *practical* and refers in every Particular, almost, for the *Rationale*, to a *Treatise* in *Latin*, entituled *Horologium Oscillatorium*, by Mr. HUGENS.

3140. But this Treatise of Mr. HUGENS is upon the Theory of one Sort of Clock only, *viz.* that which moves with a *Weight*, and is regulated by a *single Pendulum*. This great Author being the first that applied a PENDULUM to a Clock for this Purpose; though *Pendulums* were long before in Use as CHRONOMETERS, to measure Time by their equable Vibrations in astronomical Observations, and on many other Occasions. But more of this

hereafter, when we come to treat directly of Nature, Use, and various Forms of the Pendulum.

3141. The Design of Clock-work is twofold, *viz.* (1.) To measure Time exactly, or to divide a given Portion of Time into very small equal Parts; for Instance, to divide the Time of a Day into *Hours*, *Minutes*, *Seconds*, &c. (2.) To produce Motions, or periodical Revolutions similar to any given ones, as those of the heavenly Bodies, together with their Phases, Aspects, Positions, &c. at a determinate Time. But Clock-work of this Sort is usually called a *Planetarium*, *Orrery*, &c.

3142. A Clock is the principal Machine, or Capital of all mechanical Contrivances, for *measuring* Time. Its Principle of Motion is derived from a two-fold Power or Force, *viz.* that of a *Weight*, or a *Spring*. For either of these Forces are sufficient to actuate, or put into Motion, the System of Wheels and Pinions which compose the intermediate Parts or Body of the Machine; the Indexes fixed on the Axles of the Wheel point out the proper Divisions of the integral Portion of Time on appropriated Circles upon the Face of the Clock, and a Pendulum or Balance is added, to regulate the Motion communicated to the Machine, or render it uniform.

3143. The two physical Principles of all *Automata*, *viz.* the Weight and the Spring are now to be considered. With Respect to the first, as its Force is derived from the Power of Gravity only, and this Power being always the same in a given Quantity of Matter (968) it follows that the Force of a given Weight is a *constant Quantity*, or always remains the same in the same Medium, and therefore in such Case, this becomes absolutely a uniform Power or Principle of Motion; such as is necessary in perfect Clock-work.

3144. The Elasticity of a well-tempered Steel *Spring* is a fit Power or Principle of Motion in a Clock, for when it is bent or coiled to any given Degree, the Intensity of its Force continues the same; but as it is coiled more or less about its Axis, the Force becomes increased or diminished; and since the Action of a Spring in Clock-work is by unbending itself by Means of its renitent Force, the Force it exerts on the Wheels would gradually decrease; and therefore if it was at first sufficient to keep the Clock in Motion, it could not long continue so, but

the *Pendulum* would at Length ceaſe to move, and the Clock ſtand ſtill.

3145. When a Spring, therefore, is applied to Clock-work, it becomes neceſſary to contrive a Method by which its variable Force may be rendered equable or uniformly the ſame on the Wheels of the Clock from its firſt or greateſt to its laſt or leaſt Degree of intenſive Force. And this is done by giving a requiſite Form to the Barrel on the Axis of the firſt Wheel, which it is connected with by a proper Chain, or String.

3146. The Force, therefore, of the Weight or Spring is the *Primum Mobile*, or firſt Mover, in Clock-work; and it may be proper here to obſerve that the Force they communicate to the Machine is continually diminiſhed by the Wheel-work, till at Length upon the ſerrated Teeth of the Crown-wheel it is but juſt ſufficient to keep the *Pendulum* in Motion; that is, the Force is there required to be equal to the Reſiſtance the Pendulum meets with from the Air and the Axis of its Motion, for then its Motion will be continued in the Arch of Vibration propoſed.

3147. But if the Force of the ſaid Crown-wheel on the Pallets of the Verge of the Pendulum be ſuperior to the Reſiſtance, it will only cauſe the Pendulum to vibrate farther, or in a ſomewhat larger Arch than the given one. But if the Force be leſs than the Reſiſtance of the Pendulum, then the Arches of Vibration will decreaſe gradually to nothing, or the Pendulum and Clock will ceaſe to move.

3148. The more perfectly the component Parts of the moving Syſtem are wrought and finiſhed, the leſs Force of a Weight or Spring will be required to keep it going. But when the Work is coarſe, and clogged with Duſt, inſpiſſated Oil, *&c.* the Force of the Weight or Spring will be ſo much diminiſhed thro' the Train of the Work, that unleſs it be in Proportion encreaſed it will not keep the Clock in Motion. But whatever be the Condition of the Clock, while it does move, it will meaſure Time equally; for on Suppoſition the Pendulum continues of the ſame Length, it makes all its Vibrations, in larger or ſmaller Arches, in the ſame Time, as we have formerly ſhewn (1122, 1126); and, therefore, if once a Clock be ſet right, or

has its Pendulum duly adjusted, it must (as *Hugens* says,) always *measure Time truly, or not measure it at all.*

3149. Upon the Whole we may conclude, that since a *Weight* is, in its own Nature, a constant Principle and must necessarily act with an uniform Tenour; and on the other Hand, the Action or Force of a *Spring* is in itself alway variable, and can only be rendered of an equable Tenour by an Artifice, it must follow that the former is more eligible in Clock-Movements than the latter; and that a Spring is preferable to a Weight only on a single Account, *viz.* of its bringing the Clock into a more compendious or portable Form than can be admitted where a Weight is made use of for giving it Motion.

CHAP. II.

The NATURE, FORM, *and* ACTION *of the* FUSEE, *explained from* Mechanical *and* Mathematical PRINCIPLES.

3150. WE are now to explain the Theory of that Invention by which the SPRING is made to act with an equable Force on the System of Wheel-work by the Mediation of a Part called the FUSEE, which for that Purpose is required to have a peculiar Form; every one knows how a Weight acts upon the Cylinder, and thereby communicates an equal Force and Movement to the Machine. But the Manner in which the Spring and Fusee do the same Thing conjointly, is not so obvious, but yet will be easy to conceive by attending to the following Particulars.

3151. The Chain being fixed at one End to the Fusee, and at the other to the Barrel, when the Machine is winding up, the Fusee is turned round, and of Course the Barrel; on the Inside of the Barrel is fixed one End of the Spring, the other End being fixed to an immoveable Axis in the Center. As the Barrel moves round, it coils the Spring several Times about the Axis, thereby increasing its elastic Force to a proper Degree; all

all this while the Chain is drawn off the Barrel upon the Fusee and then when the Instrument is wound up, the Spring by its elastic Force, endeavouring constantly to unbend itself, acts upon the Barrel, by carrying it round; by which Means the Chain is drawn off from the Fusee, and thus turns it about, and consequently the whole Machinery is put in Motion.

3152. Now as the Spring unbends by Degrees, its elastic Force, by which it affects the Fusee, will gradually decrease; and therefore unless there were some mechanical Contrivance in the Figure of the Superficies of the Fusee to cause, that as the Spring is weaker, the Chain shall be removed farther from the Centre of the Fusee, so that what is lost in the Spring's Elasticity is gained in the Length of the Lever; I say, unless it were for this Contrivance, the Spring's Force would always be unequal upon the Machine, and so would produce an unequable Motion of the Parts thereof.

3153. The Figure of the Curve, which shall form the Superficies of the Fusee by a Revolution about its Axis, may be investigated as follows. (Fig. 1.) Let B C D be the Curve, A L the Axis of the Fusee produced; let D be the Point where the End of the Chain is fixed on the Fusee when the Watch is down, or the Spring uncoiled, and B the Point where it touches it when the Spring or Machine is wound up. From the Points B and D let fall the Perpendiculars to the Axis B A and D H; in which produced, let there be taken A E and H I proportional to the Force or Strength of the Spring, when the Chain is at B and D. Through E, I, draw the Right-line E I K intersecting the Axis somewhere in K; and from any Point C in the Curve, draw C F perpendicular to the Axis in G; then will F G be as the Strength of the Spring when the Chain is at G.

3154. Now since the Force acting on the first Wheel ought always to be uniformly the same; and this Force being always as the Strength of the Spring expressed by F G, and the Distance at which the Chain acts from the Axis of the Fusee conjointly: Therefore the Force at any Point C will be as the Rectangle $FG \times GC$, and since this is a given Quantity it may be made $FG \times GC = ab$, and so we have $FG = \frac{ab}{GC}$.

3155.

3155. Therefore to determine the Equation of the Curve BCD, let KH $= a$, HI $= b$, HG $= x$, and GC $= y$. Then because of the similar Triangle HKI and GKF, we have HK : HI : : GK : FG $= \frac{ab}{y}$; that is, $a : b :: a + x : \frac{ab}{y}$; whence we have $aa = ay + xy$, which is the Equation of the Curve, and shews it to be that of the *Hyperbola* with Respect to the Space between the Curve and its Asymptotes, as is evident from (779).

3156. Hence when $x = o$, then $a = y$, or HK = HD; also when the Point G arrives at A, then $y =$ AB. And because EA × AB = IH × HD, we have EA : IH : : HD : AB : : $a : y$: : so is the greatest Force of the Spring to its least Force, on the Fusee (3154).

3157. Because the Ordinates HD, AB, *&c.* are at Right-angles to AL, (3153) the *Curve* BCD is that called an *equilateral Hyperbola* (see 780). By the Revolution of which about its Axis or common Asymptote AK the true Form of the Solid or *Fusee* is generated, as in Fig. 2.

3158. In that Figure ADE, FGH, and IKL, MNO, are opposite equilateral Hyperbolas described about the Asymptotes PQ, RS, intersecting at Right-angles in the common Center C. Put CT (= TD) = 1, then CD $= \sqrt{2} =$ Radius of the Circle DKGN touching the four equal Hyperbolas in their respective Vertices. And CF (= CY $= \sqrt{CQ^2 + QY^2}$) = 2, is the Distance of the Focus of each Hyperbola from the Center. Lastly, the Parameter $ab =$ KN, the Diameter of the Circle. All which is evident from what we have heretofore demonstrated of the Properties of the Hyperbola in general (765, *&c.*)

3159. It is therefore demonstrated that the Section of any given Fusee BDKX through its Axis ZT is determined by two equal Arches BD and KX of two equal and adjacent Hyperbolas beginning from their Vertices D and K.

3160. It is also evident, that since TD and ZB do represent the Force of the Spring, when it is wound up, and when quite down, therefore in every Fusee truly made the *Diameters*

DK

DK *and* BX *of the greatest and least Ends thereof must be exactly proportionate to the greatest and least Force of the Spring.*

3161. Also it follows, that when the Proportion of the greatest and least Force of the Spring is known, or the Ratio of TD to ZB is given, then also the Length of the Fusee TZ is a given Quantity; or *there can be but one determinate Length for the Fusee, to answer to the two given Forces of the Spring.*

3162. Lastly, it is evident, that when the Length of the Fusee, and one of the Forces, TD or ZB are given, then the other Force is given or determined, and not to be assumed at Pleasure.

3163. Having thus determined the *geometrical Form of the Fusee,* we next proceed to illustrate the Theory of the several Cases by Examples. Therefore put $TD = a$, $ZB = y$, and $TZ = x$; and then the Equation $aa = ay + yx$ (3155) will appear in its usual Form. Whence (1.) If a and y are given, to find x; we have $\frac{aa}{y} - a = x$. (2.) When x is given, or $x = 1$, we have given the Ratio of a to y; for then $aa = ay + y$, consequently $a : y :: a + 1 : a$. (3.) When a and x are given, then $\frac{a^2}{a + y} = y$. (4.) Lastly, when x and y are given; we have $a^2 - ay = xy$, and (compleating the Square) $a = \sqrt{xy + \frac{1}{4}y^2} + y$.

3164. The proper Numbers for expressing the Forces (a) and (y) of the Spring will be in *Ounces* and *Drams* Averdupois Wt. which *Ounces* may be made Tenths of an Inch in the Measures of the Fusee TD, TZ, and ZB. These Forces are thus determined. In Fig. 3. let ABCD be the Barrel containing the Spring, and let FBX be the Position of the Chain or Cord upon the Barrel and Fusee when the Spring is wound up; then suppose the Chain disengaged from the Fusee, and carried under the Barrel in the Direction EH to the Pulley at H, over which it is to be hung with such a Weight W appended, as will just counteract or balance the Force of the Spring coiled up. After the same Manner, if CDK be the Chain when the Clock is down, then if this be taken from the Fusee, and passed under the Barrel to the Pulley K, and a Weight L hung on to the End, such

as ſhall juſt keep the Barrel in the ſame Poſition; then this Weight L will be equal to the Force of the Spring ſo far uncoiled. Therefore $TD = a : ZB = y :: W : L$.

3165. Case I.

Suppoſe the Weight W be 63 Ounces, and the Weight L be 21; then $a = 63$ and $y = 21$; or becauſe $63 : 21 :: 3 : 1$, we have $a = 3$, and $y = 1$; and then we find $x = \frac{aa}{y} - a = 6 = 2a$, that is when the Forces are as 3 to 1, the Height or Length of the Fuſee TZ is equal to the Diameter of its Baſe or End DK. If $W : L :: 2 : 1 :: TD : ZB$; then $x = a$, or $TZ = TD$. When $W : L :: 3 : 2$; then $x = \frac{1}{2}a$; and univerſally if $W : L :: m : n :: a : y :: a + x : a$, then it will be $\frac{m-n}{n} a = x$.

3166. Case II.

Given the Length of the Fuſee $TZ = 6$, to determine the Ratio of the Forces of the Spring, or the Weights W, L, which will give the Diameters DK, and XB of the Ends of the Fuſee. Since $a^2 = ay + 6y$ (3163) we have $a : y :: a + 6 : a$; then by aſſuming the Value of a you have that of y. Thus ſuppoſe $a = 3$, then $a : y :: 3 + 6 : 3 :: 3 : 1$. And in this Caſe the Diameter $DK = 3XB$. And becauſe $a : y :: m : n$, (3165) therefore for any aſſumed Ratio, we have $\frac{n}{m-n} x = a$: thus if $m : n :: 2 : 1$, then $x = a$, or if $m : n :: 3 : 2$, then $2x = a$. Conſequently if $x = 6$, we have $a = 12$, and $y = 8$.

3167. Case III.

Given the Length of the Fuſee, and the greateſt Force of the Spring to find what the leaſt Force muſt be. Suppoſe $x = TZ = 6$, and $a = TD = 3$; then $y = \frac{a^2}{a + x} = \frac{9}{9} = 1$; ſo that if $W = 63$ Ounces, we have $L = 21$. If $x = a$, then $y = \frac{1}{2}a$, or $DK = 2XB$.

3168.

3168. CASE IV.

Given the Length of the Fusee, and the least Force of the Spring, to find the greatest. Let $x = 6$, and $y = 1$; then $\sqrt{xy + \frac{1}{4}y^2} + \frac{1}{2}y = \sqrt{6,25} + 0,5 = 3 = a$ (3163) so that if $y = 21$ Ounces, the greatest Force of the Spring will be $a = 63$. Therefore in every Case the Form and Dimensions of the Fusee are geometrically determined.

3169. It only remains now to shew the Method by which the Hyperbola A D E (Fig. 2.) is to be described *in Plano* in order to be made a *Gauge* for giving the true Form of the Fusee required in any particular Case. Thus suppose it be found by Experiment (3164) that the greatest and least Force of the Spring to be used is 63 and 21 Ounces. Then having drawn two Lines P Q and R S at Right-angles in C for Asymptotes, let the Angles P C S and R C Q be bisected by a Right-line G D continued each Way indefinitely.

3170. Then having determined the Diameter of the Base D K of the Fusee, take that Extent in your Compasses and set it off each Way from the Center C in the Line D G to F and C (in Fig. 4.) then will those two Points be the Focusses of two opposite Hyperbolas A D E and F G H (774).

3171. Having provided a Ruler A B C of the Form (in Fig. 4.) you fix one End of a String A B F on the End A, and the other End in the focal Point F of the intended Hyperbola. This Chord A B F must be just so much less in Length than the Ruler A B C, as is equal to the Diameter G D of the Circle; that is, A B C — G D = A B E. Then if with a steady Hand you move the Ruler about a Pin fixed in the opposite Focus C, and at the same Time keep the Chord nicely to the Edge of the Ruler, as at B, with a proper Pencil or drawing Point, that Point B will describe the required Hyperbola A D E (Fig. 2.)

3172. For the *Ellipsis* in the Figure to (768) becomes a *Circle* in the present Case, and the transverse Axis T V there becomes the Diameter D G here; but in every Case the Difference of two Lines C B, F B, drawn from any Point B in the Curve of the Hyperbola will be equal to the transverse Axis, as is shewn (769) and therefore universally C B — B F = G D; or the

Point B is constantly in, and therefore describes the *Hyperbolic Curve* required.

3173. When the Curve is thus drawn on Paper or Pastboard, it will be easy to transfer it to a Plate of Brass, or Steel; and thereby form a *Gauge*, for giving a true Figure to the Fusee proposed. In such Fusees of the larger Sort, where the Diameter of the Chord or Chain is large enough to be considered, it must be added to the Diameters DK and XB of the greatest and least Helix of the Fusee.

3174. In order to draw the Fusee by Scale and Compasses, it has been shewn that there is a stated Proportion between the Diameters, and Length of the Fusee; and therefore in whatever Numbers one is expressed, the other may be expressed in the same. Thus for Instance; if TD be to ZB, as 63 to 21, then, because, in this Case, $x = 2a$, or $TZ = 2\,TD$ (3165); therefore $TZ = 126$, of the same equal Parts. So that if $TZ = 1{,}26$ Inches, or 12,6 *Tenths* of an Inch, then $TD = 0{,}63$, or 6,3 *Tenths*; and $ZB = 0{,}21$, or 2,1 *Tenths* of an Inch, which are laid down from any decimal Scale, or other Scale of equal Parts.

3175. These Things are, I think, all that belong essentially to the *Theory of the Fusee* in Clock-work; and could the Artist in Practice execute this Part to the Perfection of Theory, it would then communicate a Motion to the Machine as equable as that produced by a *Weight* itself. And by this geometrical Construction of the Fusee, those which are usually made by Trial with the Lever, may be compared and corrected in regard to their Figure and Dimensions.

CHAP.

CHAP. III.

The Rationale *of* CALCULATION *in* CLOCK-WORK; *the ſame applied and illuſtrated in a* DESCRIPTION *of the* original AUTOMATON *invented by* HUGENIUS.

3176. THE Nature of the *Weight*, and the *Spring* with its *Fuſee*, being, as the Origin or Principles of Motion in this Kind of Machinery, explained; it remains to ſhew the Conſtruction and Diſpoſition of the Syſtem of Wheel-work to anſwer the general Purpoſe of a Clock. But as little of a mathematical Theory is here required, or employed, it will the ſooner be diſpatched.

3177. The Communication of Motion being by Wheels and Pinions, it is in the firſt Place neceſſary to take Care, that the Diameters of the Wheel and Pinion it drives, have the exact Proportion of their Numbers of Teeth reſpectively, that the Teeth of one may exactly correſpond to the Cavities or Interſtices of the other; thus if a Wheel of 80 Teeth be propoſed to drive a Pinion of 8 Teeth, then the Diameter of the Wheel muſt be to that of the Pinion exactly as 80 to 8, or as 10 to 1. If this be done throughout the Syſtem, the Movements will be every where natural and exact.

3178. In the next Place, from the Nature of the Lever and Axis in *Peritrochio*, it appears that the Power or Force on the Pinion is to that on the Circumference of the Wheel on the ſame Axis, as the Diameter of the Wheel is to that of the Pinion; conſequently by this Means, and by the Friction of the Parts, the Force at firſt impreſſed by the Weight or Spring is conſtantly diminiſhing through the whole Compages, till at laſt it is but very ſmall on the Pendulum, *viz.* juſt enough to continue it in Motion.

3179. The Revolutions of two immediate Axis in a given Time are inverſely as the Number of Teeth in the Wheel of one to that Number of Teeth in the Pinion of the other, which it drives, thus if the Number of Teeth in the Wheel and Pinion

be 80 and 8, then the Axis of the Pinion will be turned round 10 Times to 1 Turn of the Axis of the Wheel; therefore the *Quotient of the Wheel, divided by the Pinion it drives*, is the Ratio of Turns to Unity. Thus 8) 80 (10, as before.

3180. Hence if there be any Number of Wheels A, B, C, D, *&c.* acting upon so many contiguous Pinions a, b, c, d, *&c.* and let the Quotients of each Wheel, by the Pinion it drives, be m, e, f, g, *&c.* then $\frac{A}{a} = m$; or $a : A :: 1 : m$. Again, B is the Wheel on the same Axis with the Pinion a, and drives the Pinion b on the next Axle; then $\frac{B}{b} = e$, is the Number of Revolutions in this third Axis to one on the Axis of the Pinion b; and therefore $me = \frac{mB}{b} = n$ the Number of Turns in this third Axle to one of the first Axle, or that of the Wheel A, so that $b : B :: m : n$. Let the Wheel on the Axis of the Pinion b drive another Pinion c, then $\frac{C}{c} = f$ with Respect to the Turns of the preceding Axis which are expressed by n; therefore $\frac{nC}{c} = nf = o$; and so we have $c : C :: n : o$; and for the next Axle, we have $d : D :: o : p$, and so on for as many Axes as are required in the Train of the Work.

3181. Then placing these Analogies one under another, and multiplying all the Antecedents and Consequents together, we have $abcd : ABCD :: mno : mnop :: 1 : p$. Wherefore

$$\left\{\begin{array}{l} a : A :: 1 : m \\ b : B :: m : n \\ c : C :: n : o \\ d : D :: o : p \end{array}\right.$$

$\frac{ABCD}{abcd} = p$; whence we have this Rule. *Divide the Product of the Number of Teeth in the Wheels by the Product of the Teeth in the Pinions, the Quotient will be the Number of Turns of the Axle of the last Pinion* d *in one Turn of the first Axle of the Wheel* A.

3182. Since the Expressions $\frac{A}{a}$, $\frac{B}{b}$, $\frac{C}{c}$, *&c.* are but Ratios of the Numbers of the Teeth in the Wheels and Pinions, it is evident, any Numbers having the same Ratio will answer the same Purpose, or give the same Number of Revolutions to an Axis

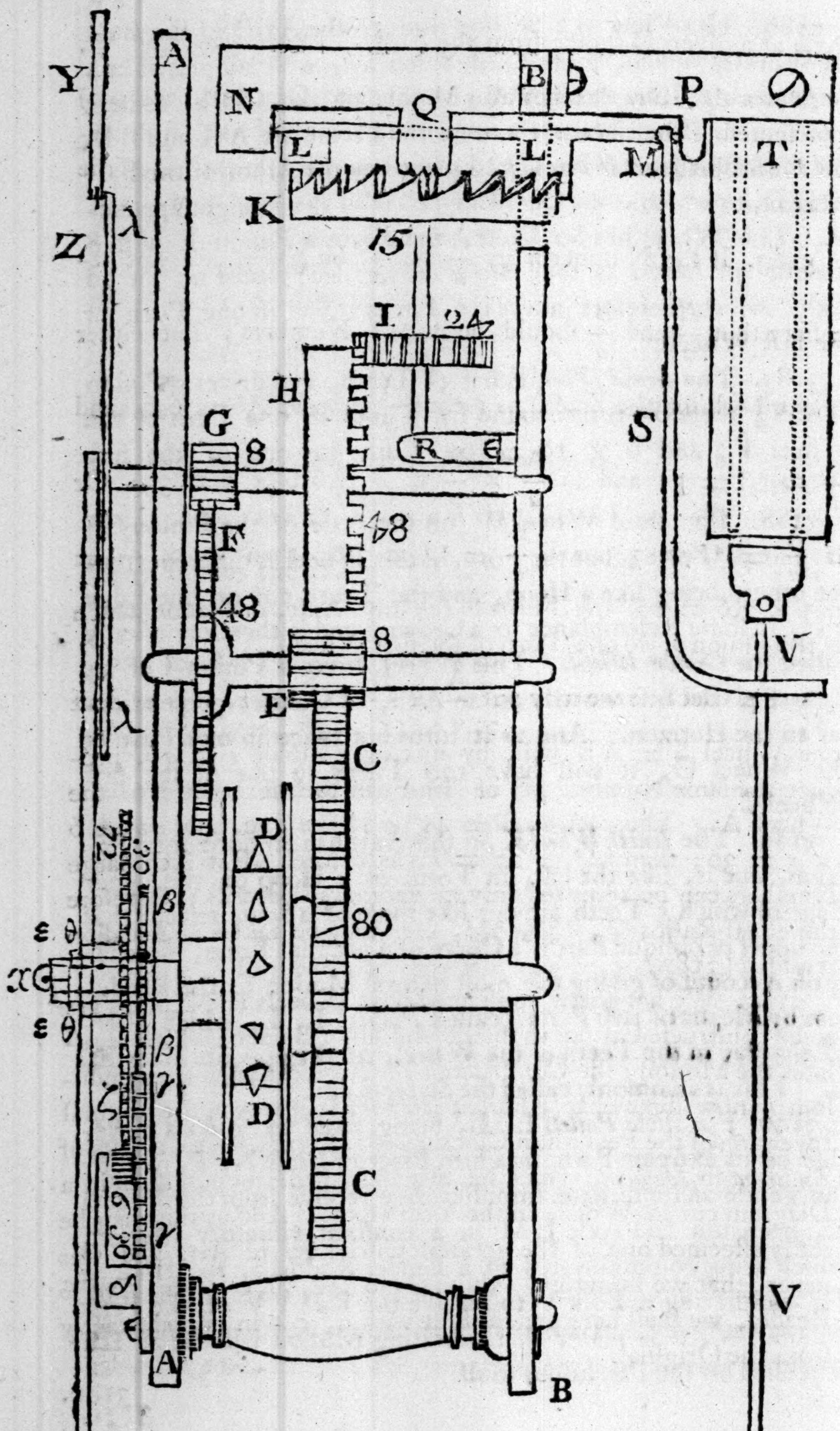
Axis at a given Diſtance from the Firſt. Thus $\frac{A}{a}$ may be $\frac{80}{8}$, or $\frac{60}{6}$, or $\frac{50}{5}$, &c. (as they all give the ſame Quotient, *viz.* 10) and therefore ſuch Numbers are to be choſen by the ſkillful Artiſt as will beſt ſuit the general Deſign and Circumſtances of the Machine.

3183. If the Number $n = \frac{A}{a} \times \frac{B}{b}$, be given, it is not neceſſary that $\frac{A}{a}$ and $\frac{B}{b}$ ſhould be whole Numbers; but either one or both may be Fractions; thus $\frac{A}{a}$ may be $\frac{33}{5} = 6,6$, and $\frac{B}{b} = \frac{40}{8} = 5$; and ſo $\frac{A}{a} \times \frac{B}{b} = n = 6,6 \times 5 = 33$. Or if $\frac{A}{a} = \frac{45}{6} = 7,5$, and $\frac{B}{b} = \frac{18}{5} = 3,6$; then $\frac{A}{a} \times \frac{B}{b} = 7,5 \times 3,6 = 27$, the Turns in the third Axle, or that on which is the Pinion b, to one Turn of the firſt Wheel.

3184. Becauſe we may put $\frac{A}{a} \times \frac{B}{b} = \frac{S}{s} = n$, it is evident one Wheel $S = AB$ will, by driving a Pinion $s = ab$, produce the ſame Number (n) of Turns in the next Axle to the Wheel A. Thus let $AB = 45 \times 18 = 810$, and $ab = 6 \times 5 = 30$; then $\frac{810}{30} = 27 = n$, as before. But ſuch large Numbers can be admitted only in very large Works; therefore the equal Ratios $\frac{81}{3}$, $\frac{162}{6}$, $\frac{243}{9}$, &c. may be taken as Occaſion requires (3182).

3185. We have now conſidered how Wheels and Pinions are to be conſtructed as far as the Pendulum or Balance which regulates the Motion; the Conſtruction for Application of the Pendulum is now ſomewhat different from what it was in the original Invention of the Pendulum Clock by Mr. *Chriſtian* HUGENIUS of *Zulichem* in *Holland*, which he firſt deſcribed and publiſhed in a Diagram cut in Wood, in the Year 1657. And as this may be juſtly eſteemed one of the greateſt Curioſities of Art, and was never (that we know of) exhibited to the View of an *Engliſh* Reader, we ſhall here preſent him with it, cut in Wood exactly from the Original.

3186.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3186. The View is a Section through the Axles of the several Wheels, which, in this first *Automaton*, were all placed in a perpendicular Line through the Middle of the Clock, as here represented. The parallel Plates of the Frame are A A and B B, and the first or *great Wheel* C C is placed on the Arbor of the Barrel D D, to which is the Chord applied with the Weight appended. This Wheel has 80 Teeth, and drives a Pinion E with 8 Teeth, (or *Leaves* as they are usually called) placed on the 2d Axle, which therefore moves ten Times round in one Turn of the first Wheel.

3187. The *second Wheel* F has 48 Teeth, and drives a Pinion G of 8; therefore turns round six Times in one Turn of the Wheel F; and $6 \times 10 = 60$ Turns in one of the first Wheel.

3188. The third Wheel H (on the Axle of the Pinion G) has also 48 Teeth; but the Form of this Wheel is different from the other, being like a Hoop, and the Teeth cut on one Edge gives it some Resemblance to a Crown, and is therefore usually called the *Crown Wheel*. This Wheel drives a Pinion I of 24 Leaves placed horizontally on an Axis, of Course, perpendicular to the Horizon. And as it turns but twice in one Turn of the Wheel G, it will have 120 Turns to one of the first Wheel C.

3189. The *fourth Wheel* K on this Axis has an horizontal Position, and is, like the last, in Form of a Hoop on the upper Edge of which 5 Teeth are cut like those of a *Saw*, except that the upper or oblique Part is a Curve of a peculiar Form, necessary on Account of giving the most natural Motion to the Pendulum by Means of two Parts (called *Pallets*) L, L, which alternately play in the Teeth of the Wheel, at the opposite Sides K, L. This is commonly called the *Swing-wheel*.

3190. For these *Pallets* L, L, being fixed to an Axis moveable on its extreme Parts in a firm Piece of Brass N, P, will by the gentle and alternate Impulses they receive from the Wheel K, keep the said Axle L M in a constant vibratory Motion, which being communicated to a slender Rod M S, fixed to it at M, and having a Fork V to receive the Rod V V of the Pendulum T V, it is plain, that vibrating Motion will be at last impressed on the Pendulum itself.

3191. But this Motion thus communicated to the *Pendulum*, must be such as nicely quadrates or coincides with the proper Motion of the Pendulum itself, which depends on its Length. Now as the Wheel K has 15 Teeth, and each Tooth strikes each Pallet 15 Times in one Turn, both the Pallets together must receive 30 Impulses, and consequently the Pendulum must have 30 Swings, or Vibrations in one Turn of the Wheel K; and therefore it must vibrate $30 \times 120 = 3600$ Times in one Turn of the great Wheel C.

3192. Now 3600 is the Number of Seconds in one Hour; therefore a Pendulum vibrating Seconds, whose Length is 39,2 Inches (1125) was applied to the Clock; and then the Axis of the first or great Wheel C turned round once in an Hour.

3193. As the Axis of this Wheel passed through the Plate A A, then an Index put on the End $\epsilon\epsilon$ was carried round once in an Hour, and over a large Circle (on the Face of the Clock) divided into 60 equal Parts, thereby indicating the *Minutes* at its extreme Part.

3194. On the same Axis (near the Plate A A) was fixed a Wheel $\beta\beta$ of 30 Teeth which drove another Wheel $\gamma\gamma$ of the same Number of Teeth, and a Pinion of 6 Leaves. This Pinion drove the Wheel ζ of 72 Teeth; and therefore it moved but once round in 12 Turns of the Pinion or Axle of the Wheel C. Consequently an Index placed on the Socket $\theta\theta$ of the Wheel ζ (moveable about the common Axis) will shew the Hours as it passes over a Circle divided into 12 equal Parts, on the Face of the Clock Y Y.

3195. Lastly; the Axis of the Crown-wheel H has 60 Turns to one of the great Wheel C, (3187) that is, it has 60 Turns in an Hour, or it turns once round in a *Minute*. The Axis of this Wheel, therefore, passing through the Plate A A had a Plate $\lambda\lambda$ fixed upon its End with a Circle divided into 60 equal Parts denoted by their proper Numbers, which shewed the Seconds of Time as they passed by a Hole z made in the exterior Plate or Face of the Clock Y Y.

3196. This original Pendulum-clock, constructed with such Simplicity as to have no more than *four Wheels and Pinions* in the Body of the Clock, shewed Hours, Minutes, and Seconds of Time, with all the Exactness and Equability that a mechanical

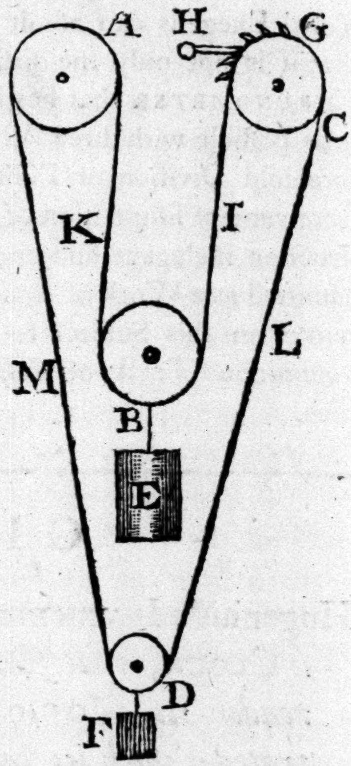
nichal Exegesis can admit of. And I think we may conclude that it is not only the first but the most perfect Pattern of a CHRONOMETER that has been or can be proposed; for though it be possible with three Wheels and Pinions only to produce this threefold Division of Time; yet the great Disproportion, and inconvenient Disposition of the Parts, would render such a Construction inelegant and immechanical, and therefore not to be admitted as a Work of Art. They who would see a great deal wrote on this Subject to very little Purpose, may consult the voluminous Tracts of *Casp. Scottus*, and others of the like Genius.

CHAP. IV.

Hugenius's INVENTION *for applying a* WEIGHT *to a* CLOCK, *that shall act upon it incessantly, and render its* MOTION *constant, explained and illustrated; with its Improvement in* ROYAL PENDULUMS.

3197. THE celebrated Author of the foregoing Invention, so useful in public and private Life, did afterwards improve it by another, which no less demonstrated his singular Genius and Sagacity for *automatical Machinery*. This second Invention was to render the Motion of the Clock, or rather of the Pendulum, more equal than before; but to understand the Reason of it, the Manner in which he suspended the Weight for keeping the Clock in an unintermitting Motion, is first to be explained.

3198. It is easy to understand that a Weight suspended upon a Clock in the Manner it is upon a Jack, will keep the Clock going till it wants winding up; but during the Time of winding it up, the Action of the Weight is taken off from the great Wheel of the Clock, as it is all that while disengaged from the Barrel on which the Weight hangs; and therefore the Clock not being impelled by the Weight, will, during that Time, stand still, and consequently so much Time will be lost.



3199. Our Author therefore tells us, that he invented the Method which follows, of a *perpetual* or *endless Line* for that Purpose, which being of a proper Length, the two Ends were nicely spliced and connected together; it was then applied in the Clock as represented in the Figure adjoined, by the Letters ABCDA. Where A represents the sulcated Wheel DD (in the Figure of the Clock) with several *Spiculæ* or small pointed Pins fixed in the Surface of the evacuated Part, as is there shewn.

3200. The Pulley C is fixed to a *Rachet-wheel* G, but both moveable on a fixed Axis, from the Left-hand towards the Right only; for all Motion the contrary Way is prevented by the Trigger or Catch at H falling constantly into the serrated Teeth of the Wheel G. The Surface also of this Pulley C is spiculated, like that of the Wheel DD on the Arbor of the first Wheel of the Clock.

3201. The endless Line or Cord, being put over the Barrel A, and Pulleys B and C, will, by Reason of the *Spiculæ* or pointed Pins, hang firmly on A and C, suspending the large Weight E affixed to the Pulley B. Now it is evident from the Nature of the Pulley (1050) that the Weight E equally stretches the Parts of the Chord I and K; and therefore acts with an equal Force, *viz. half its Weight*, on the Wheel A, and Pulley C.

3202. But since this Force upon the Pulley C tends to move it from Right to Left, and all Motion that Way is stopped (3200) it follows, that the Line I is to be considered as fixed all the Time the Clock is going; but the Parts K and M, by Vertue of the Weight E, will be constantly moving over the Surface of the Wheel A, and thereby communicating Motion to it, and, of Course, to all the Machinery of the Clock.

3203.

3203. As to the Pulley D, and the ſmall Weight F, it is plain they are deſigned only to give free Motion to the Parts of the Chord L, M; and keep them in a perpendicular Poſition.

3204. Thus it appears, that as long as the Weight E can deſcend, ſo long the Clock will keep going; and therefore as many Times as the Circumference of the Wheel or Barrel A is contained in the Diſtance through which the Weight can deſcend, ſo many Hours will the Clock go without drawing up, becauſe the ſaid Wheel A goes once round in an Hour (3192). In this original Clock, the Wheel D D is one Inch Diameter, or bout 3 Inches in Circumference, and went 30 Hours without drawing up; therefore 90 Inches, or 7½ Feet was the perpendicular Deſcent of the Weight, or Height of the Clock.

3205. When the Weight was down, it was eaſily drawn up again by applying the Hand to the Part of the Line at L, and drawing downwards it will move the Pulley C, till the Weight E aſcends to the Top, during which Aſcent it conſtantly acts or gravitates on the Wheel A, by Means of the Rope K, with the ſame Force as when at Reſt, and therefore the Clock goes conſtantly by this Contrivance without ever looſing a Moment of Time, if it be not neglected.

3206. *Hugenius* informs us that in thoſe of his Clocks which were eſteemed the beſt, the Weight E was ſix Pounds; the Power therefore by which the Clock was animated, was that of 3 Pounds. The Weight of the Pendulum was alſo 3 Pounds; and of the Length, to ſwing Seconds.

3207. By an Addition of Pulleys, it would be eaſy to make the Weight deſcend more ſlowly, and, of Courſe, to make the Clock go longer before it wants to be drawn up. In the Modern Conſtruction of Clocks, a different Method of applying the Weight is uſed, and it is wound up upon a Barrel with a Handle or Winch; by this Means, the uſual Time the Clock will go, is 8 Days in thoſe called *Royal Pendulums*, in which the Swing-wheel K has a Poſition (not parallel, as here, but) perpendicular to the Horizon, or the ſame with all the other Wheels. Every Thing of this Sort is evident by Inſpection of any 8 Day Clock, and needs no other Inſtruction.

3208. I shall only here observe, that the Form of the Teeth in this Wheel should be exactly circular on that Side or Part by which acts on the *Pads* or two Arms of the *Clock* of the Pendulum; in the extreme Parts of those Arms there should be a small cylindric Pin fixed, always touching and moving in the circular Surface of the Teeth of the Wheel; the Radius of the Circle for the Teeth is the nearest Distance of the Pin in the End of the Pad to the Axis or Arbor of the Pendulum on which it is fixed.

3209. Moreover it is farther necessary, that the Teeth on this Wheel be of such a Number and Length, that the Pins upon the Ends of the Pads, may very nicely *take and escape* them alternately, or without the least Intermission of Action. Such a Construction is both natural and mathematical, but it is not in Use, that I have seen; I know of no Method that can be substituted that is easier, or that will produce a more steady or equable Motion of the *Second-hand* fixed on the End of this Arbor, or (in the Clock-maker's Language) so accurately cause the Pendulum to *beat dead Seconds*.

CHAP. V.

The Rationale *of* HUGENIUS's Invention *for* continuing and regulating the MOTION *of the* PENDULUM *by a* single Wheel *only*; *thereby rendering it most* EQUABLE *for* USE *at* SEA.

3210. HUGENIUS having considered (and probably found by Experience) that the Action of a Weight being propagated through a Multiplicity of Wheels and Pinions to the *Pendulum*, could not be so constant, and agitate the Pendulum with a Force so equable, as it would do if the Number of Wheels were less; and of Course concluded, that if the Weight could be applied to the Arbor of *that Wheel only* which actuated the Pen-

Pendulum, it would then communicate a Motion to it that would be the moſt ſimple and conſtant that the Nature of Things admits of in this Kind of Machinery.

3211. At Length he invented the Method of ſuch an Application, which was as follows. An *endleſs Line* with its proper Weights and Pulleys, (every Way ſimilar to that deſcribed in Fig. to Art. 3199, only as much leſs in Proportion as the Force of the Weight was leſs,) was applied to the Arbor of the Swing-wheel in a ſmall pinnulated Grove of a Barrel, (which we now ſuppoſe to be A) on one Part, and to a Ratchet-wheel C on the other, ſupporting the ſmall Wheel E, with the other much ſmaller at F.

3212. It is manifeſt that the Weight E depending on the Line I B A, conſidered as fixed on the Pulley C, does now ſolely act on the Arbor of the Swing-wheel by the Part A B, and therefore the Motion communicated to the Pendulum, muſt be the moſt Uniform poſſible, as we cannot conceive any Thing incident to the Weight E that can make any Alteration in its gravitating Force, and as there can be no Cauſe for any Irregularity in the Motion of the Pendulum.

3213. As the Body of the Machinery in the Clock is in this Conſtruction diſingaged from the Swing-wheel and Pendulum, ſo none of its Anomalies can affect either; the proper Office of all this Part being now only to raiſe up the Weight E as often as it is down, which the Author tells us, was every *Half-minute*, or 30 Seconds. For ſince in every Revolution of the Wheel A, the Weight E muſt deſcend through a Space equal to the Circumference of that Wheel; therefore it muſt want very frequently to be drawn up, which could not be done but by Means of the Machinery of the Clock, which by its Nature becomes a conſtant Agent for that Purpoſe.

3214. Where it not for the Frequency of drawing the Weight up, the Pendulum, with one Wheel to continue its Motion, would be the moſt perfect *Chronometer* that could be made; but as it is thus connected with the Clock-part, it becomes juſt as uſeful as if it wanted no Attendance to draw it up at all.

3215. By this Invention, therefore the Clock with all its Wheels, is, in Effect, reduced to *one ſingle Wheel*; and the Mo-

Motion just as equable and perfect as if it consisted of no more; and the Author assures us, that the Equability of Motion in Clocks of this Sort was apparently greater than in Clocks of the common Construction.

3216. As a Proof of this, he mentions a very extraordinary Circumstance, which he discovered by Accident; it was this; as two of those Clocks stood together on a Shelf, he observed the contrary Vibrations of each of them did so exactly agree and coincide, that they never in the least receded from it. And if this Concordance of their Motions was on Purpose disturbed, yet in a very little Time it would of it's one Accord return, and continue, in such perfect Coincidence, that only *one Sound* could be heard of both their Vibrations.

3217. Having wondered some Time at so unusual a Phænomenon, he resolves to enquire into its Cause with the utmost Diligence, and at Length found that it proceeded from a Motion of the Shelf or Plank on which they stood, though in itself altogether insensible. For the Reciprocations of the Pendulums communicated some Degree of Motion to the Clocks, firmly placed as they were; and this Motion being impressed on the Board, did necessarily cause that, if the *Pendulums* vibrated otherwise than by contrary Strokes very exactly, they must at Length do so, and then the Motion of the Shelf would entirely cease.

3218. For whatever Motion be communicated to the Shelf by the contrary Motion or Vibrations of the Pendulums as those Motions are impressed in contrary Directions, they will, if equal, destroy each other ( ) and if they are not impressed equally, and at the same Instant on the Board, yet as the Difference equally affects the Board and the Clock, it will, by this Alternation, constantly tend to an Equibrium both in the Clocks and the Shelf, till at Length that being affected, the Pendulums must move by contrary Strokes very nicely.

3219. But our Author concludes, that notwithstanding the Fact is evident, yet a Cause of such a tender and delicate Nature could never have Efficacy enough to produce such an Effect, unless the Motions of the Clocks had been previously, and by other Means rendered most equable and consentient between them-

themselves; and which therefore is by this Experiment fully ascertained and demonstrated.

3220. It is above 100 Years since this Discovery was made and published to the World, and was intended, by the Inventor, to render his Clock more immediately useful at Sea; for this Purpose he contrived it in a particular Form with a triangular Pendulum, and a Construction which gave it always an upright and steady Position on Ship-board; and thus many of them were made and used at Sea for the better discovering of the Longitude; and with considerable Success, as we find by an Account thereof (in several Voyages) by the Author in his *Horolog. Oscillator.* Pag. 17, *&c.* where also you have Iconisms of the Pendulum, the Clock, and its Frame in which it was suspended in the Ship.

3221. But after all, we find the Use of this most promising Invention discontinued at Sea, which would never have been, were it possible for any Thing of this Nature to succeed there; and if this be the Fate of a Clock actuated and regulated by the constant Power of a Weight, it may tend to moderate our Expectations of the Longitude by Means of *Watches*, wherein the same Artifice has been attempted by Means of *Springs*. But how much Inferior all *Automata* by Springs are to those which move by Weights, common Experience can sufficient testify, and will be farther evident by considering their Nature and Construction, explained in the next Chapters.

3222. I shall conclude this with a Query to the ingenious Artificer, *viz.* whether it is not practicable to move a Pendulum by a *single Wheel*, and a Weight that shall not need winding up at all, by the Addition of another equal Pendulum to the Clock? —— I see nothing to contradict it in the Theory; if this were done, the utmost Equability of Motion from Clock-work would thence be derived to the Pendulum, and such a *double Pendulum* Clock would become the most perfect *Automaton* in Nature.

CHAP.

CHAP. VI.

Concerning the Invention *of* portable AUTOMATA, *or* WATCHES; *of the* BALANCE, *and* REGULATOR; *and the* THEORY *of* Isochronal VIBRATIONS *in* SPRINGS.

3223. IN the Invention and Application of Pendulums to *Horological Automata*, Mr. *Hugens* stands unrivaled; but with regard to the Invention of *Clocks in Miniature*, or of a portable and pocket Form, which we generally call WATCHES, there is (at least, there has been) great Dispute. Of these Watches there is not the least Mention or Intimation in the *Horologium Oscillatorium* by *Hugenius*, though printed in 1673, from whence it seems probable, that at that Time he knew of no such Thing.

3224. It is also certain, that many Years before, our celebrated Countryman, Dr. HOOKE, exhibited several of those Watches as his own Contrivance and Invention, and a Patent was offered him for the same in 1663, but not liking the Conditions, he refused it. Dr. *Derham* (in his *Artif. Clock-maker*) tells us, he saw a Watch presented to King *Charles* II. on which was this Inscription, *viz. Robert Hook, inven.* 1658. *T. Tompion, fecit,* 1675. Mr. *Ward*, in the Life of Dr. *Hooke*, says, he saw the same Inscription on a round Brass-plate, which formerly had been a Cover to the Balance of one of Mr. *Hooke*'s Watches; and he farther adds, that Mr. *G. Graham*, informed him, he heard Mr. *Tompion* say, he was employed three Months that Year by Mr. *Hooke*, in making some Parts of those Watches before he let him know for what Use they were designed; and that Mr. *Tompion* use to say, he thought Mr. *Hooke* was the Inventor of them.

3225. From hence it appears, that these Watches were invented by Dr. *Hooke*, within one Year after the Clock itself, which (as before observed, 3185) was not made public till the Year 1657. Notwithstanding this, in the Year 1674, Mr. *Hu-*

Hugens published a Pocket Watch of his own Invention, as he asserts, but it was in several Respects different from Dr. *Hooke*'s. And as it is inconsistent with the great Character of *Hugenius*, for Learning and Probity, to suppose him capable of *Plagiarism*, therefore it must be allowed, that the Reduction of Clock-work, to the Size of a *Pocket-watch*, was separately the Invention of each of these ingenious *Competitors*.

3226. This Invention consists in applying a *Balance-wheel*, to beat the Time instead of a *Pendulum*; for though they are called *Pendulum-watches*, it is only because of the *alternative Motion* of the Balance being somewhat like that of a *Pendulum*.

3227. This *Balance* has *its Motion* regulated by a *Spiral-spring* properly applied, an Idea of the Balance and Spring is easy to be formed from the View of any Watch-work. As also the Manner in which they are actuated by the Machinery of the Watch; for this being the same as in a Spring-clock, only *in small*, there is nothing new till we come to the Regulation by a *Wheel and Spring* instead of a *Pendulum*.

3228. From barely considering the Size of a Watch, it is plain no Pendulum can be admitted into its Construction; for in the first Place, a Pendulum cannot be made of so short a Length as is requisite, with any Degree of Exactness; and secondly, the Pendulum always must have a perpendicular Position, which cannot be allowed in the Use of a Watch, which is incident to all Kinds of Positions.

3229. As the Balance-wheel has in itself no Principle of, or Disposition to Motion, but is actuated solely by the Machinery of the Watch, it is to be considered alone as no other Thing than a Check upon the Motion of that System, or a Reservoir in which it is ultimate received and absorbed, like the *Fly of a Jack* in a great Measure, only the Fly has a circular Motion, whereas that of the Balance is an Oscillatory one.

3230. The Motion of the *Balance* and the *Fly* is therefore equally subject to the Inequalities of the Motion of the Wheel-work of the Watch; but this can be regulated in the Vibrations of the Balance by Means of a Spring, whereas a *circular Motion* of a Wheel admits of no such Regulation or Correc-

 tion,

tion, and therefore can have no Place in this Sort of Mechanism.

3231. Hence then it is evident, that the whole Artifice of a Watch consists in giving Motion by the first Spring and Fusee to the Balance, or oscillating Wheel, as nearly equable as possible; and then to correct the Irregularities of those Oscillations, and render them isochonal as far as may be, by a fine Spring properly applied to the said Balance, though not by its *vibrating Property*, as many have supposed.

3232. But as this Doctrine of the *isochronal Vibrations* of a Spring is a curious Point, I shall be a little more particular and explicit in deriving it from its first Principles, and then illustrating it by a Figure. Therefore let G be any Force (whether *Gravity*, *Elasticity*, *&c.*) which constantly acts on a Body and produces an accelerated Velocity; and let S, T, V, be the Space, Time, and Velocity of such Motion; then we have shewn (991) that it is $S : TV$, universally; and also, that it is $GT : V$ (*ib.*) therefore we have $S \times V : GT^2 V$, or $S : G \times T^2$; and consequently we have $S : G :: T^2 : 1$; therefore when the Ratio of S to G is given, the Time T will be a given Quantity or always the same. So that if G be the Force of Elasticity, and S the Space through which it vibrates by that Force, then, however those Quantities vary in Magnitude if they keep the same Proportion, the Vibrations of the Spring will all be isochronal, or performed in equal Times.

3233. For a farther Illustration of the Similarity of Motion in a Pendulum and Spring; let A *a* (Fig. 1.) be a Spring, or a strait elastic Wire fixed at the End A, and at the other End suppose it drawn out of its natural or perpendicular Situation, by a Line passing over a Pulley B, with a Scale C at the End, and Weights put into it for trying the Experiments. Then admit we put a Dram into the Scale C which draws the Wire or Spring from A *a* to the Site A *b*; and then we add another Dram, and it draws it into the Position A *c*; and a third Dram being put into the Scale, draws the Wire into the Situation A *d*; and so on, as long as the Wire or Spring can retain its rectilineal Form; then will the Spaces *a* A *b*, *a* A *c*, *a* A *d*, be as 1, 2, 3; that is, as the Weights applied which retain the Spring in those Positions respectively; therefore the elastic Force of the Spring at

at *a*, *b*, *c*, being alſo as thoſe Weights, and conſequently as the Spaces deſcribed by the Wire, *viz.* the ſmall right-lined Triangles *a* A *b*, *a* A *c*, *a* A *d*, it follows, that the Wire will deſcribe each of them in the ſame Time; and therefore all its Vibrations, as long as it continues rectilineal, will be performed in equal Time.

3234. But if upon adding 3 Drams more, we obſerve the Wire drawn into the Poſition A *e*, and there appears incurvated, it is evident then, becauſe the Space *a* A *e* is not equal to twice the Space *a* A *d*, the Time in which it moves from *e* to *a* cannot be equal to the Time in which it moves from *d* to *a*. If 3 more Drams be added, ſuppoſe all the Nine draw the Spring to the Poſition A *f*, which is there more curved than before; and therefore *a* A *f* is ſtill more deficient from 3 Times the Space *a* A *d*, and conſequently the Time of deſcribing it will differ ſtill more from the Time of deſcribing the Space *d* A *a*. Upon the Whole, then, it appears, *that after the ſtrait Spring begins to be bent into a Curve, the Times of Oſcillation are no longer iſochronal.*

3235. In like Manner, with regard to a Pendulum *Aa* (Fig. 2.) we have formerly demonſtrated (1105) that the Times of Deſcent through any Number of Chords (*ad*, *ae*, *af*,) of a Circle are all equal. Therefore if we take very ſmall Arches *ab*, *ac*, *ad*, they will, as to Senſe, coincide with their Chords, and ſo of Courſe the Time of deſcribing each of them will be equal; but if the Arch be ſo large that it can no longer be eſteemed rectilineal, as *ae*, or *af*, then the Times of deſcribing them ceaſe to be iſochronal. So that very ſmall Diſtances only on each Side the Perpendicular *a* A, both in the Spring and Pendulum, will admit of iſochronal Vibrations.

3236. But what has been hitherto ſaid of the Spring and its Vibrations has been with a View rather to ſhew its Nature and Congruity with the Pendulum, than its Application to Uſe in automatical Machinery on that Account; for though the Pendulum by Means of its oſcillatory Property, becomes ſuch an excellent Regulator in all Kinds of Time-pieces, yet the Spring poſſeſſed of the ſame Property, will anſwer no ſuch Purpoſe on that Account; becauſe the Time of a Vibration is ſo exceeding ſmall. Now the Vibrations of a Spring (ſuch as we have conſider-

ed) ought to coincide with thoſe of the Balance, if its Regulation were to be produced by them; but the Vibrations of the Balance, or *Beats* of a Watch, are not more than about 16000 in an Hour, that is, about 4 ½ *per* Second; whereas a ſtrait Spring, of the Length adapted to a Watch, will vibrate many hundred Times in a Second; and, indeed, a Spring of any Size, or Kind, can have no Uſe in that Reſpect, all its Effect, as a *Regulator*, being derived from its elaſtic re-active Force, by Way of *conſtant Preſſure* (and not *Oſcillation*) which therefore muſt be more particularly explained, and is the Subſtance of the following Chapter.

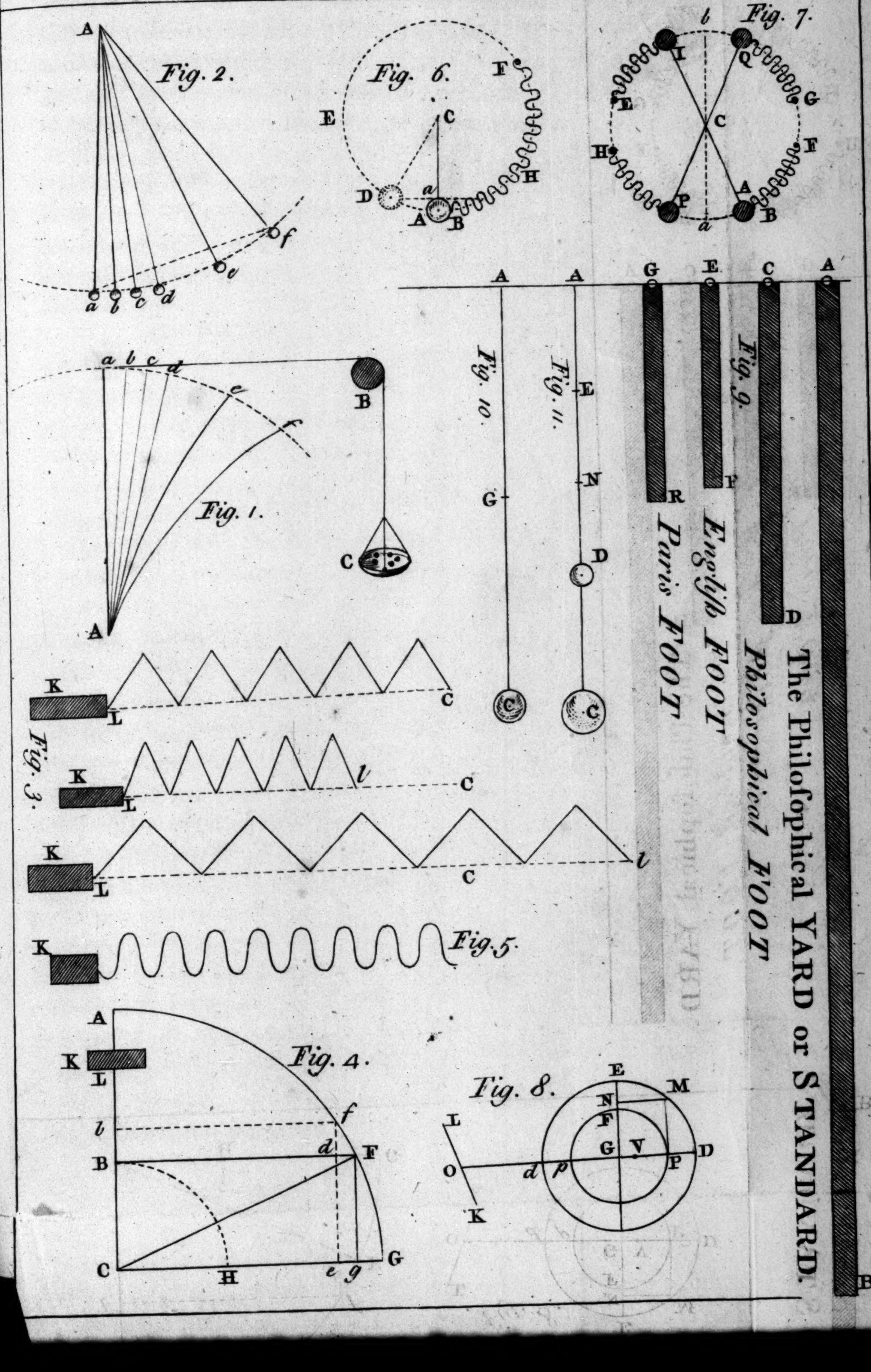
CHAP. VII.

The THEORY *of* SPRINGS *conſidered as* REGULATORS *of the* BALANCE *of* CLOCKS *and* WATCHES.

3237. AS a ſtrait Spring, in Length, cannot be more than about ⅔ of the Diameter of the Watch-plate, it will be found too ſhort to admit of a proper Tenour of Action as a Regulator of Motion in ſuch ſmall Balances; for the Action of a Spring ought to be very free and eaſy, but yet at the ſame Time ſtrong enough to govern the Motion of the Balance, and controul the Irregularities of its Vibrations derived from the Syſtem of Machinery. Therefore a Spring of a Spiral, undulated, or ſome other Form, wherein there is a conſiderable or ſufficient Length in a conciſe Space for anſwering its Purpoſe in a WATCH, muſt be choſen.

3238. As the Action of the Spring under all thoſe different Forms will be the ſame, I ſhall chuſe that of (Fig. 3.) to explain it by. Thus let C L be the Spring in its natural Situation, or ſuch as it has when left to its itſelf; at the End L ſuppoſe it to reſt againſt, or be fixed to, an immoveable Support K, then by its elaſtic Force it will make Reſiſtance to any Power or Weight by which it is preſſed inwards, or drawn outwards. Thus let

The THEORY of SPRINGS & PENDULUMS.



p be a Force applied which bends it through any Space or Length $Cl = s$; then from the Nature of the Spring (3233) $2p$ will bend it through a Space equal to $2s$, and $3p$ will bend it thro' $3s$; and so on. So that s will be every where proportional to p.

3239. Then if P denote the Power by which the Spring is wholly compressed, and S = Space of total Compression; then it will be $S : s :: P : p$; and the same Analogy will result from a partial and total Extension. This Principle is not only deduced from a physical Theory as before-mentioned, but is confirmed most accurately by Experiments on Springs on every Form.

3240. And here I may observe, that useful Instrument called the *Spring* (or *Cylindric*) BALANCE for weighing Bodies, is nothing more than the Practice of this Principle; for the Number of pound Weights in the Body appended is indicated by the Number of *equal Divisions* on the Bar, which is the Space thro' which the Spring is compressed by the Weight of the Body, as each single Division corresponds to that of a pound Weight.

3241. This is the Effect of a Spring in regard to *Pressure*; we shall next institute a Comparison between this *elastic Force* and a *percussive Force*, or that of a *striking* Body moving with a certain Degree of Velocity. For if a Body whose Weight or Mass of Matter is M, and moving with an uniform Velocity V, strike upon the End C of the Spring, it will bend the same through a Part or the whole Space S; and the Quantity of Motion by which it affects the Spring at first will be MV (970); but by the Reaction of the Spring, the Velocity V will be gradually diminished; so that when the Spring is bent through any Space s, the original Velocity V will be reduced to v, and the Quantity of Motion or Momentum of the Body will there be only Mv, so that the Loss of Motion in bending the Spring through the Space s, will be $MV - Mv$.

3242. Let A = Space through which a Body descends *in Vacuo* in one Second, by the Power of Gravity; and C = Celerity acquired in that Descent. Then we have $C^2 : A :: V^2 : a =$ Space of the Body M must descend thro' by Gravity to acquire the Velocity V. (991) Also $\sqrt{A} : 1'' :: \sqrt{a} : T =$ Time of the Descent through a.

3243.

3243. Suppose, then, by the Stroke of the Body M moving with the Velocity V, the Spring be bent through a Space s in the Time t. In this Case we have $S : s :: P : p = \frac{sP}{S} =$ to the *elastic Force* of the Spring at l, or at the End of the Space s.

3244. Now let the momentary Decrement of Velocity in the percutient Body be $-\dot{v}$, then will the instantaneous or fluxionary Momentum produced by the elastic Force of the Spring be $-M\dot{v}$ in the Moment of Time $\dot{t}$; and we have shewn (1600) that in Case of Forces $\left(M \text{ and } \frac{sP}{S}\right)$ acting uniformly, the Quantities of Motion (MV, $-M\dot{v}$,) generated are proportional to the generating Forces, and the Times (T, $\dot{t}$,) conjointly; and therefore $MV : -M\dot{v} :: M \times T : \frac{sP}{S} \times \dot{t}$; which gives $-\dot{v} = \frac{VPs\dot{t}}{MST}$.

3245. Again, in the same Case of Forces acting uniformly, the Spaces are as the Velocities and Times conjointly (971) therefore $2a : VT :: \dot{s} : v\dot{t}$; whence $\dot{t} = \frac{TV\dot{s}}{2av}$.

3246. Therefore $-\dot{v} = \frac{VPs}{MST} \times \frac{TV\dot{s}}{2av}$; which gives $2v\dot{v} = -\frac{V^2Ps\dot{s}}{MSa}$; the Fluents of which are v^2 and $-\frac{V^2Ps^2}{2MSa}$; but when the former of these was V^2, the latter was $= 0$, because of $s = 0$; therefore $v^2 = V^2 - \frac{V^2Ps^2}{2MSa}$.

3247. To reduce this Equation to more simple Terms by a Construction, let $\frac{2aMS}{P} = R^2$, then will $v^2 = V^2 - \frac{V^2s^2}{R^2}$, or $v^2 = V^2 \times \frac{R^2 - s^2}{R^2}$. Now if $R =$ Radius CG of a Circle, (Fig. 4.) and $s = CB = gF$, the right Sine of the Arch GF; then by the Property of the Circle, it is $R^2 - s^2 = BF^2$; therefore $v^2 = V^2 \times \frac{BF^2}{R^2}$; and so $v = V \times \frac{BF}{R}$, that is, *the Ve-*

Velocity v is to the original Velocity V, *as the Cosine* BF *to the Radius* CG.

3248. Because $\dot{t} = \frac{TV\dot{s}}{2av}$, and $v = V \times \frac{\sqrt{R^2 - s^2}}{R}$, therefore $\dot{t} = \frac{TV\dot{s}}{2a} \times \frac{R}{V \times \sqrt{R^2 - s^2}} = \frac{T}{2a} \times \frac{R\dot{s}}{\sqrt{R^2 - s^2}}$. But drawing bf indefinitely near to BF, and fd perpendicular to the same, we have from the similar Triangles BCF and dfF, as $BF : df :: CF : fF = \frac{CF \times df}{BF} = \frac{R\dot{s}}{\sqrt{R^2 - s^2}}$. Therefore $\dot{t} = \frac{T}{2a} \times \dot{fF}$; and the Fluents of this Equation are $t = \frac{T}{2a} \times GF$; which gives this Analogy $t : T :: GF : 2a$.

3249. Thus the *Times*, *Velocities*, and *Spaces*, respecting the Motion of Springs, are easily assignable by a *trigonometrical Calculus* in the Parts of a Circle. But in most practical Cases, particularly *Watch-work*, the Space through which the Spring is bent by the Stroke is but a Part of the whole Length, and therefore the Quantity of Motion MV, or Force of the Stroke being consumed in bending the Spring, through the Space s in the Time t, the Velocity will there cease, or $v = 0$; in which Case the Time may be expressed independent of any particular Arch of a Circle, as follows.

3250. It is evident from the Theorem in (2247) that $R^2 = s^2$, when $v = 0$; therefore s being now equal to Radius CH = CB (Fig. 4.) and also the Sine of the Arch GF, that Arch now becomes the Quadrant of a Circle HB. Then if m = Periphery of a Circle whose Diameter is 1, we have $1 : m :: 2s : 2sm$, therefore $\frac{2sm}{4}$ = Quadrant HB. Wherefore in this Case, $t = \frac{T}{2a} \times GF = \frac{T}{2a} \times \frac{2sm}{4}$. But $\frac{T}{2a} = \frac{T}{\sqrt{a} \times 2\sqrt{a}}$; also $1'' : T :: \sqrt{A} : \sqrt{a}$; therefore $\frac{T}{\sqrt{a}} = \frac{1''}{\sqrt{A}}$; whence $t =$

$\frac{1}{\sqrt{A}} \times \frac{2sm}{4 \times 2\sqrt{a}}$. But it being $s = R = \sqrt{\frac{2SMa}{P}}$ (3247) we have $\frac{s}{\sqrt{a}} = \sqrt{\frac{2MS}{P}}$. Consequently $t = \frac{m}{2} \times \sqrt{\frac{MS}{2PA}}$ $= \frac{m}{2} \times \sqrt{\frac{Ms}{2pA}}$ in Seconds of Time.

3251. Because $v^2 = V^2 \times \frac{BF^2}{R^2}$ (3247) we have $V^2 - v^2 = V^2 - V^2 \times \frac{BF^2}{R^2} = V^2 \times \frac{R^2 - BF^2}{R^2} = V^2 \times \frac{CB^2}{R^2} = \frac{V^2 s^2}{R^2}$; then because $R^2 = \frac{2MSa}{P}$ (2247) we have $V^2 - v^2 = V^2 s^2 \times \frac{P}{2MSa}$.

3252. But $\frac{V^2}{a} = \frac{C^2}{A}$ (2242) therefore $V^2 - v^2 = \frac{C^2 P s^2}{2MSA}$. And therefore $MV^2 - Mv^2 = \frac{C^2 P s^2}{2SA} = \frac{C^2 ps}{2A}$. This Theorem may be of Use in that Part of *Mechanics* concerned in the Doctrine of the *Vires Vivæ & Mortuæ*.

3253. Hence when $v = 0$, we have the initial Velocity $V = Cs \sqrt{\frac{P}{2MSA}} = C \sqrt{\frac{ps}{2MA}}$. And because $t = \frac{m}{2} \times \sqrt{\frac{MS}{2PA}}$; therefore the Product of the Velocity and Time will be $Vt = \frac{mCs}{4A} \times \sqrt{\frac{MSP}{MSP}} = \frac{mCs}{4A}$.

3254. And because $\frac{mC}{4A}$ is a constant Quantity, we have $Vt : s$; or the *Space through which the Spring is bent always as the Velocity and Time conjointly*, the same as in the Case of Bodies descending by *Gravity* (971).

3255. Wherefore in the same, or different Springs, *the Spaces through which they are bent in a given Time will be as the Velocities*; and with a given Velocity, *they will be as the Times employed in bending them.*

3256.

3256. Because in the Expression of the Time, $t = \frac{m}{2} \sqrt{\frac{MS}{2PA}}$, the Quantities m and $2A$ are constant; therefore it will always be $t : \sqrt{\frac{MS}{P}}$; or $t^2 P : MS$. But we had also $t : \frac{s}{V}$ (2254) therefore $t : \frac{MS}{tP} : \frac{s}{V}$. Consequently $MSV : tsP$. Hence we obtain a Comparison between these six principal Quantities in a *Spring*, and a Body bending it with a *percussive Force*.

3257. Thus if the Length S and Force P of a Spring be given we have $MV : ts$; but in this Case $t^2 : M$ (2256); therefore $tV : s$, or *the Space is as the Rectangle of the Time and Velocity*, as in *Falling-bodies* (991).

3258. Likewise in the same Spring, since $t^2 : M$, therefore when M, or the Body is given, *the Time* (t) *of bending the Spring will be the same*, whatever be the Degree of Velocity V, or the Space s through which it is bent.

3259. But in a given Spring S, P, and Body M, since t will be constant, we have $V : s$; that is, the *Space through which the Spring is bent, will always be proportional to the Velocity of the Body*; which is another Case similar to descending Bodies urged by different accelerating Forces.

3260. It was proper on some Accounts to give the foregoing Theorems expressing the *Velocity*, *Time*, *&c.* in the Form they there have; but in the particular Case of the Space described, or Descent in *one Second*, we have $2A = C$; those Theorems, therefore, will be expressed more simply for Calculation, thus (putting $m = 3{,}1416$) $t = \frac{m}{2} \times \sqrt{\frac{MS}{2AP}} = 1{,}57 \sqrt{\frac{MS}{CP}} = 1{,}57 \sqrt{\frac{Ms}{Cp}}$; and $V = C \times \sqrt{\frac{ps}{2AM}} = C \times \sqrt{\frac{ps}{CM}} = \sqrt{\frac{Cps}{M}}$; and $MV^2 = Cps$; and lastly, $Vt = \frac{mCs}{4A} = 1{,}57s$;

$= 1,57\, s$; or $t = 1,57\, \frac{s}{V}$, in Seconds. And in all these Cases, $C = 32\frac{1}{6}$ Feet, or 386 Inches.

CHAP. VIII.

The foregoing THEORY *of* SPRINGS *farther considered, and exemplified by* CALCULATIONS *in their various* APPLICATION *to* WATCH-WORK.

3261. IT has been already shewn, that an *Automaton* put into Motion by a Weight, and having that Motion regulated by a Pendulum, is the most perfect that can be in Nature; and that in the next Degree of Perfection is the *Spring-clock*; but when we consider the Structure of a WATCH, left wholly to the Power and Action of Springs, we shall find it naturally destitute of any absolute Principle of equable Motion.

3262. The Action of a *Weight* is constant, and the Oscillations of a Pendulum are equable, from an *innate Principle*; but we cannot consider the Action of a Spring either as a *First-mover* or *Regulator* of Motion, as constant and equable in its artificial Application in Watches; this we have also shewn in regard to the First, in the *Theory* of the *Spring* and its *Fusee*; and in respect to the Second, the Office of the Spring as a *Regulator*, is not derived from the *Natural*, but what we may properly call, the *artificial Vibrations* thereof. The natural Vibrations of all Springs are in themselves as perfectly isochronal as those of Pendulums (3232); but the Case is quite otherwise with respect to the *artificial Vibrations*, whose Equability of Motion, being the Result of the compound Action of two Strings together, cannot be supposed so perfectly constant, and regular, as no *mechanical Combination of Causes* can act with the Simplicity and Uniformity of *Nature itself*.

3263.

3263. If the Watch be required to beat *Quarter-seconds*, or to oscillate 14400 Times in an Hour, these Oscillations by a Pendulum are rendered equable without any Trouble; but to effect this by a Spring, or Number of Springs, will require much more mechanical Skill and Contrivance, and at last be attended with some Degree of Inconstancy or Irregularity.

3264. Indeed, the fine Spring connected with the Balance is to be considered as somewhat more than a *Regulator*; for it does, as it were, *form* or *modulate* those Oscillations, as well as *regulate* them; without such a Spring the Balance would oscillate, it is true, but since the Motion produced by the Action of the Crown-wheel, on one Pallet, must be in a Moment stopped, destroyed, and generated anew in a contrary Direction by its Action on the other Pallet, and this in so quick and constant an Alternation, the Effect would be too violent and shocking to be sufferable, and too irregular to be of any Use.

3265. But by the Application and Co-operation of the Spring, the said contrary Motions are gradually generated and destroyed; and the Times of the Oscillations of the Balance-wheel rendered as equable and uniform as the nicest Mechanism can admit of. To effect this, many different Methods have been invented, principally by Dr. *Hooke*, Mr. *Hugens*, Mr. *Leibnitz*, *&c.* Some of these were by single Balances, others by double Ones; some had only one regulating Spring; others had two, or more. The Attempt to regulate a Watch by a *Loadstone* instead of a *Spring*, is not worth mentioning.

3266. There were two Ways in which the double Balance was applied; in one of them each single Balance had a Verge with one Pallet only, placed on each Side of the *Crown-wheel* diametrically opposite to each other, for that Wheel had the same Position with the *contrate Wheel*, or as it now has in the present *horizontal Watches*. On the Verge of each Balance was fixed a small Wheel; these Wheels were proportioned to the Diameter of the Crown-wheel; having the same Number of fine Teeth, they played in each other, and so gave an equal Motion to the larger Balance-wheels just above them.

3267. The other Way, with two Balances, had the two small Wheels, by which they moved each other, and more-

over to each Balance-verge, there was added a ſpiral Spring as a *Regulator*. In this Method, one Balance only had a Verge on which both the Pallets were, and it was moved by the Crown-wheel, placed in the ſame perpendicular Poſition it now has in common Watches. With reſpect to theſe two Inventions, Dr. *Derham* ſuppoſes the firſt was never proſecuted ſo far as it deſerves; and the Second has this Excellency, that no Jerk or the moſt confuſed Shake can in the leaſt alter its Vibrations, And that the Reaſon why this Method of conſtructing Watches came into Diſuſe, he judges was the great Trouble and vaſt Niceneſs required in it. If this was the Caſe, it reflects no great Honour on the Reputation of Artiſts in this Way.

3268. The Spring, as a *Regulator*, is applied in Watches, with one End fixed to the Verge of the Balance, and being coiled ſeveral Times round, to give it a ſufficient Length, it has its other End faſtened to a Part towards the Extremity of the Watch-plate, which has lateral Teeth, and is moved by a ſmall Pinion on an Axis in the Center of a ſmall ſilvered Plate on one Side of the Cock or Balance, divided into 30 ſmall equal Parts. This Plate is alſo fixed on the ſaid Axis, and by the Key is moved againſt an Index to the Right or Left, thereby increaſing or remitting the Force of the Spring when the Watch goes too ſlow or too faſt; and this Apparatus for correcting the Regulator, and thereby the Time of the Watch, we find by common Experience, is but too frequently neceſſary in the moſt excellent Pieces of this Sort of Machinery.

3269. Mr. *De la* HIRE, in a *Memoir of the Academy of Sciences*, objects very much to the common ſpiral Form of the Regulator, and eſpecially to the fixing one End thereof to, or near the Verge or Axis of the Balance. He thinks, if the End of the common ſpiral Regulator were faſtened to a Part of the Radius of the Balance inſtead of the Axis, it would have more Force to govern the unequal Movements thereof, and at the ſame Time be leſs ſubject to its Irregularities; and farther adds, that the Spring bent into an *undulatory* or *wave-like Form* (as in Fig. 5.) is much better than the Spiral, in that a greater Length may be had in a leſs Space, and the Spring thereby ſuſtain itſelf with greater Eaſe, and act without that Compound and diſtorted

Mo-

Motion which the Spires of the common Regulator are ſubject to, in their lengthening and ſhortening. And what he has aſſerted, he tells us, he has fully evinced by Experiments with Watches of this new Conſtruction.

3270. But I have conſtantly obſerved, that theſe Inventions, Alterations, and Innovations concerning the Balance and its Regulator, have all proceeded (*rudi Minerva*) from the natural Force of mechanical Genius, with very little, if any, *Rationale* from Principles of a *philoſophical* or *mathematical* THEORY of the *Nature and Action* of SPRINGS; which certainly muſt reflect very great Light on a Subject that has always been looked upon as overwhelmed in Difficulty and Obſcurity. For, (if I judge right in theſe Matters) there ſeems to be a much better and more natural Method of governing and regulating Watch-work pointed out by the foregoing Theory, than that of the Balance-wheel and its uſual Regulator.

3271. For by this Theory, it is evident, that with Reſpect to the *Power* of the Spring, the *Momentum* of a *Body* which ſtrikes it, the *Space* through which it is bent, and the *Time* of bending it, if any of theſe Quantities are known or given, the reſt may be found; becauſe their Relations are all determined by the Theorems $t = 1{,}57\sqrt{\frac{Ms}{Cp}}$, and $V = \sqrt{\frac{Cps}{M}}$ (2260) as alſo the real Quantities of each reſpectively.

3272. Therefore ſuppoſe the Train of a Watch be 14400, or 4 Beats *per* Second; then let A be a Body ſuſpended by a Thread A C from the Center C, (Fig. 6.) and being raiſed to the Point D let it deſcend through the Arch D A and ſtrike the Spring B F with the Velocity acquired in that Deſcent; and then A *a* is the verſed Sine, or perpendicular Deſcent of the Body to acquire the ſame Velocity. Laſtly, the Momentum of the Body A will by its Impulſe on the Spring bend it through a certain Space B H. Now for the given *Length* and *Strength* of the Spring, and the given Space and Time of bending it, the other Quantities may be found by Calculation, as follows.

3273. Conſidering this Spring B F as a *Regulator*, it muſt be very fine and tender; and therefore we will ſuppoſe it ſuch that a ſingle *Scruple* only, or 20 Grains, ſhall be juſt equal to its whole

whole Force, or be able to keep it bent through its *whole Length*, which may be one Inch, and let the Time of bending it through half its Length be $\frac{1}{4}$ of a Second; then there is given B F $=$ S $= 1$, P $= 20$, $s = 0,5 =$ A H, $p = 10$, and $t = 0'',25$; to find M, the Weight of the Body A; V, the uniform Velocity of its Motion; and $a =$ A a, the Height it must descend perpendicularly to acquire that Velocity.

3274. To find the *Weight of the Body* A, we have from the Theorem (3260) $t = 1,57 \sqrt{\frac{MS}{386\,p}}$, this Equation $386\,p\,t^2 = 2,4649\,M\,s$; and therefore this Analogy, $p : M :: s : 156,6\,t^2$; which will give $M = \frac{156,6\,t^2\,p}{s} =$ 196 Grains, in the present Case; so that the striking Body A is near 20 Times the Weight of p.

3275. The uniform Velocity of its Motion is $V = \sqrt{\frac{C\,p\,s}{M}} =$ 3,138 Inches *per* Second, this is equal to the Circumference of a Circle A E F whose Diameter is just *one Inch*.

3276. Lastly, we find $A\,a = a = \frac{A\,V^2}{C^2}$ (3242) $= \frac{V^2}{4\,A} = \frac{V^2}{772} = 0,0127$, or about $\frac{13}{1000}$ of an Inch; and therefore the Arch A D so exceeding small as to be altogether inconsiderable.

3277. If therefore the Pallets of the Verge be so proportioned to the Teeth of the Crown-wheel as to move the Body A thro' $\frac{3}{4}$ of an Inch each Stroke without the Spring; it will, when the Spring is added, be moved through only half an Inch, or bend the Spring from B to H in $\frac{1}{4}$ of a Second as required.

3278. If the Mass of Matter in the Body or Globe A were disposed into the Form of a Circle A E F, or so as to make the Perimeter of a fine Wheel, it would then become the Balance-wheel of a common Watch; and being connected with the End of the Spring B F, that Spring would also become the Regulator; and the Watch thus constructed, would beat Quarter-seconds.

3279. Hence in Watches of a large Size, and eſpecially Table-clocks, where more Springs than one may conveniently be applied, this Regulation of Time might be moſt commodiouſly performed by a ſingle Balance-lever with two Springs, or a double One with four; the Reaſon of which will appear from (Fig. 7.) where AI and PQ are two Levers, whoſe Weights A, I, P, Q, are equal to each other, and to the Weight A in Fig. 6th. Alſo the Springs BF, IE, PH, GQ, are all ſeverally equal to BH in that Figure. Theſe Levers croſs each other in the common Center of Gravity C, where they are fixed at Right-angles to the Verge.

3280. As the Weights and Springs are increaſed in the Balance, it is neceſſary the Force of the Spring at the Fuſee ſhould be encreaſed in Proportion. And it is further to be obſerved, that in the quieſcent State of the Levers, the four Springs are in a State of Compreſſion; the two Springs fixed at F and H being compreſſed on each Side from (*a*) and the other two from the Point (*b*). If *a*B be one Half of the Arch through which the Levers oſcillate, then it is plain, that in each Oſcillation, while two Springs are compreſſed by one Lever, the other two Antagoniſt Springs are relaxing; and therefore as one Pair retards, the other equally accelerates the Motion of the Levers; and ſo no Inequality of Motion can ariſe from the joint Action of Springs; but on the contrary, as perfect a Correction of the Irregularities of the Watch-work is obtained, as can be produced by the Agency of *Springs*.

CHAP.

CHAP. IX.

The METHOD *of finding the* CENTER *of* OSCILLATION *in all Kinds of* PENDULUMS, *deduced from a* NEW THEORY. *Also the* NATURE *of a* universal MEASURE of LENGTH, *or* philosophical FOOT, *explained and exemplified.*

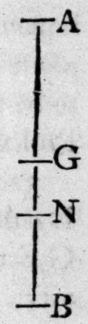
3281. WE now quit the Subject of *Watch-work*, where nothing valuable with regard to the Equation and Regulation of Time can be hoped for, or expected; and return again to the farther Consideration of those *Automata* which are regulated more by *Nature* than *Art*, *viz.* by the Power of Gravity governing the Oscillations of *Pendulums*. But here the Artist must follow pretty closely the Dictates of the omniscient Mechanic, and work, if he proposes to merit Applause, by the Rules of divine Geometry.

3282. Now as the PENDULUM is the Principle of Truth and Perfection in Clock-work, all Circumstances relative to it should be considered with the greatest Attention, and principally that which concerns the *Center of Oscillation*; and here will arise the following Questions, *viz.* what this *Center* of *Oscillation* is in the BALL or BOB of a common Pendulum? What is the Distance thereof from the Point of Suspension? And how that Distance is to be preserved unaltered?

3283. I believe very few Mechanics in this Way know so little of Art or Nature, as to suppose that the *Center of the Bob*, is the *Center of Oscillation*. But fewer still know where it is, or how to find it in the Pendulum. Yet the Knowledge thereof is of the last Consequence in very large Balls, and some Mr. GRAHAM had which exceeded 60 Pounds; also in short Pendulums in Table Clocks, this Center of Oscillation should be nicely ascertained. Mr. *Hugens* lays the greatest Stress on this Point; and all his Folio-treatise is wrote professedly on this important Subject.

3284.

3284. But the Method he purſued to diſcover it, will give us too much Trouble and Fatigue; a nearer and eaſier Way has, ſince his Time, been found out, and which we ſhall now elucidate in that Caſe firſt, where the Weight of the Ball or Bob is ſo great, that the Weight of the Rod by which it is ſuſpended is inconſiderable in Compariſon of it. In order to this, a Retroſpect to a few Things already demonſtrated in theſe Inſtitutions will be neceſſary, and tend greatly to ſhorten the Operation.



3285. Let A B be a Line oſcillating about a Point or Center A; the Diſtance of the Center of Gravity A G $= g$, and of the Center of Oſcillation A N $= n$; $\dot{x} =$ a ſmall Particle or Weight; $x =$ its Diſtance from A; and let the Sum of all the Particles or Weight of the whole Line be S. Then $\dot{x}x$ is the *Moment*, and $\dot{x}xx$ the *Force* of the Weight $\dot{x}$; and the Sum of all the $\dot{x}x^2$ is $\frac{x^3}{3}$, and when $x = S$, we have $\frac{S^3}{3}$ for the Force F of the whole Line. All which is evident from (1086 to 1098).

3286. But $\frac{S^3}{3} = S \times \frac{1}{2}S \times \frac{2}{3}S = Sgn = F$, agreeable to what was ſhewn in (1095). Alſo when $x = \frac{1}{2}S$, then the Sum of all the $\dot{x}x^2$ will be $\frac{x^3}{3} = \frac{S^3}{3 \times 8} = \frac{S^3}{24}$; and twice that Sum will be $\frac{2}{24}S^3 = \frac{1}{12}S^3$; which therefore will be as the *Sum of the Products of all the Particles* ($\dot{x}$) *each multiplied by the Square of its Diſtance from the Center of Gravity* G.

3287. Therefore put $\frac{1}{12}S^3 = q$; and we ſhall have $Sgn = q + z$; and $Sgn - q = z = \frac{1}{4}S^3 = S \times \frac{S^2}{4} = Sg^2$. Therefore $Sgn - Sg^2 = q = Sg \times \overline{n - g}$. And conſequently $\frac{q}{Sg} = n - g = GN$; which gives this general Rule for finding the Diſtance of the Center of Oſcillation N from the Center of Gravity G, *viz. divide the Sum of the Products of the Particles or Weights, ſeverally multiplied by the Squares of their Diſtances from the Center of Gravity, by the Solidity or Maſs multiplied by the Diſtance of the Center of Gravity from the Axis of Oſcillation, and the*

Quotient will be the Distance of the Center of Oscillation from the Center of Gravity.

3288. If we put GN $= d$; then becaufe $q = Sgd$ it will be $\frac{q}{d} = Sg = \frac{F}{n}$ (2286) therefore $qn = Fd$, and we have $F : q :: n : d ::$ AN : GN. The reafoning is the fame in regard to Planes and Solids, and confequently the Rule is the fame in all; but I have chofen the eafieft and moft fimple Method by which it may be demonftrated. As a Proof of its Univerfality I fhall apply it to find the Center of Ofcillation in a Globe, and fhew it to be the fame as *Hugenius* found it with fo much Labour and Prolixity.

3289. Let D E d be the Section of a Globe through its Axis; D d the Diameter perpendicular to the Axis K L of Ofcillation; G E the Radius at Right-angles to D d; G the Center of Gravity, and V the Center of Ofcillation in the Line of Sufpenfion O G; and let P F p be any concentric Circle. Let the Ordinate PM $= y$, GP $= x$, and $n : 1 ::$ Circumference of a Circle : to Radius; and draw N M parallel G D. Then fuppofe a cylindric Surface to ftand on the Circumference P F p perpendicular to the Plane D E d, and terminated by the Surface of the Sphere, then becaufe $b : 1 :: p : x$, we have $nx = p$ = Circumference P N p P, and therefore $2ynx =$ to the faid cylindric Surface (832).

3290. Now GP $= x$, is the Diftance of the Particles in each Section of this Surface (perpendicular to the Axis of Ofcillation) from the Center of Gravity of the Section; therefore $2ny x \times \dot{x}^2 = 2nyx\dot{x}$, is the Fluxion of the Surface or Weight, which multiplied by (the Square of the Diftance from the Center of Gravity) $GP^2 = x^2$, gives the Fluxion $2nyx^3\dot{x}$, whofe Fluent is $\frac{1}{2}nyx^4$.

3291. The Sine and Co-Sine of an Arch are inverfely as their Fluxions; for (in Fig. 4.) the Triangles fdF and F Cg are fimilar, therefore $fd : d$F :: Cg : gF; that is, $\dot{x} : \dot{y} :: y : x$; whence we have $\dot{x} = \frac{y\dot{y}}{x}$; and putting the Radius of the Globe GD $= a$, we have $x = \sqrt{a^2 - y^2}$; therefore $2nyx^3\dot{x} = 2ny \times \frac{y\dot{y}}{\sqrt{a^2 - y^2}} \times x^3 = 2ny^2\dot{y} \times \overline{a^2 - y^2} = 2na^2y^2\dot{y} - 2ny^4\dot{y}$,

whofe

whose Fluent is $\frac{2}{3} n a^2 y^3 - \frac{2}{5} n y^5$; and when $y = a$, this Fluent is $\frac{4}{15} n a^5$, analogous to the Quantity (q) in the general Expression (2287).

3292. Again, the Solidity of the Sphere is $\frac{1}{6} p d^2$ (836) but $p = n a$, and $d = 2a$, therefore $S = \frac{1}{6} \times n a \times 4 a^2 = \frac{2}{3} n a^3$, and putting $OG = z$, we have $\frac{2}{3} n a^3 \times z$ analogous to Sg in (2287); therefore $\frac{2}{3} n a^3 z)\ \frac{4}{15} n a^5\ (= \frac{2}{5} \frac{a^2}{z} = GV$, the Distance of the Center of Oscillation from the Center of the Sphere, as required. And is the very same with that found by *Hugenius*, *Bernouilli*, *Varignion*, *Mac Laurin*, *Simpson*, *Emerson*, *&c.* by tedious and obscure Methods.

3293. If $Od = d$, then $OG = d + a = z$; and then also $\frac{2a^2}{5z} = \frac{2a^2}{5d + 5a}$; and when $d = 0$, or the Sphere oscillates about a Point d in its Surface, we have $dV = \frac{2a^2}{5a} = \frac{2}{5} a$, or $\frac{7}{10}$ of the Diameter dD, and not $\frac{6}{10}$, as it has been determined by Mr. *Carrè*, *Hayes*, *Stone*, and others.

3294. Put $\frac{2a^2}{5z} = (GV =)\ v$, then $2a^2 = 5zv$, and $a^2 = \frac{5}{2} vz$; therefore $z : a :: a : \frac{5}{2} v$. Now the Length of a Pendulum vibrating Seconds is 392 *Tenths* of an Inch. Then suppose the Diameter of the Globe or Ball = 4 Inches, we have $a = 20$, and $z + v = OV = 392$. And then $392 + v : 20 :: 20 : 1$, very nearly; so $\frac{5}{2} v = 1$, or $5v = 2$, and $v = \frac{2}{5}$ of the Tenth of an Inch, or $\frac{1}{25}$ of an Inch.

3295. In a whole Day, or 24 Hours, there are $(24 \times 60 =)$ $1440'$; and if we put $OV = 1440$, it will be easy to find the Length of a Pendulum that shall gain just one Minute *per* Day. For since the Times of a Vibration in the two Pendulums will be inversely as the Number of Vibrations performed in a given Time (or *one Day*,) those Times will be as the Numbers $1440'$ and $1439'$. Again, the Lengths of Pendulums are as the Squares of the Times of their Vibrations, or as 1440^2 to 1439^2; but $1440^2 : 1439^2 :: 1440 : 1438$, very nearly; then $1440 : 1438 :: 392 : 391{,}46$, then $392 - 391{,}46 = 0{,}54$, or a little more than $\frac{1}{20}$ of an Inch; therefore $\frac{1}{20} : \frac{1}{25} :: 25 : 20 ::$

60″ : 48″. Whence it appears, that the Diſtance G V, ſmall as it is, in a *Second Pendulum*, is enough to make the Clock err 48 Seconds *per* Day, if not regarded.

3296. A Cubic Inch of Braſs weighs 4,4 Oz. Troy, and the Cube being to its inſcribed Sphere nearly as 2 to 1 (846). A Sphere or ſolid Braſs Globe of one Inch Diameter, will weigh 2,2 Oz. Troy; therefore as $1^3 : 4^3 :: 1 : 64 :: 2,2 : 141$ Oz. or 11 $\frac{3}{4}$ *lb.* Troy, (841) the Weight of the Ball of the abovementioned Pendulum. But if it were 5 Times heavier, the Quantity G V would ſtill be greater in the Ratio of the Squares of the Diameters of the Globes; and hence it appears, of how great Conſequence it is, in the Science of meaſuring Time, to be able to aſcertain this Point or *Center of Oſcillation in ſimple Pendulums.*

3297. Before we quit this Theory, it will be very material to obſerve, that the Doctrine of a *perpetual and univerſal* MEASURE is founded in it; and that what is called a phyſical or *philoſophical* YARD, is nothing more than the Length of a ſimple Pendulum vibrating in a *Second* of Time. That is, the *univerſal* YARD is equal to 392 *Engliſh Lines*, or *Tenths* of an Inch. And the *univerſal* FOOT is a third Part of that Number, or 130 $\frac{2}{3}$ of ſuch Lines. But this Notion of a HORARY YARD, and FOOT, and the Manner of aſcertaining it by the Pendulum of a Clock well adjuſted to equal Time by the Revolution of the Stars, as deſcribed by HUGENIUS, is to tedious here to inſiſt on.

3298. I ſhall offer the following Method as the moſt conciſe and eaſy for this Purpoſe. Let any Meaſure propoſed be conſidered as a uniform ſolid Body, whoſe Length is the *general Standard.* If ſuch a Meaſure or Solid were ſuſpended on an Axis to vibrate freely on one End (paſſing through the central Line of the Axis) it has been ſhewn, that the Center of Oſcillation in ſuch a Body will be juſt $\frac{2}{3}$ of its Length from the Axis of Suſpenſion (1097). And therefore any ſuch Body equal in Length to $392 + 196 = 588$ *Tenths* of an Inch, will vibrate exactly in *one Second of Time*; and is of Courſe, the *univerſal* or *philoſophical* YARD.

3299. This YARD muſt be divided into 10000 equal Parts, in which the Length of other Standard Meaſures are to be ex-

preſſed

pressed by comparing the Times of their Vibrations with that of the universal Yard. For this Purpose, let its Length be $L = 10000$, and $N = 3600$, the Number of Vibrations in an Hour; and let l and n denote the same Quantities in any other Measure proposed; then we have $T : t :: \sqrt{L} : \sqrt{l} :: n : N$, the Times of a single Vibration T, t, being inversely as the Numbers N and n performed in a given Time. Whence this Theorem $\frac{\sqrt{L} \times N}{n} = \sqrt{l}$.

3300. Then as the *English* Standard Foot is $= 120$, say, as $588 : 10000 :: 120 : 2040$; so that the *English* Foot is 204 of the same Parts of which the *universal* YARD contains 1000. Now this is discoverable from its Number of Vibrations in an Hour, which suppose were found, by a well adjusted Clock to be 7969; for by the Theorem $\frac{\sqrt{10000} \times 3600}{7969} = \sqrt{l}$, whence $l = 2040$, as before.

3301. After the same Manner the *Paris Royal* Foot would be found 2179,6 Parts of the *universal* YARD; and therefore as $2040 : 2179{,}6 :: 1000 : 1068 :: 12$ Inches : 12,8 Inches; or more accurately the *Paris Foot* is $12 \frac{816}{1000}$ Inches. The *Philosophical*, *English*, and *Paris* Feet are therefore as 333,3, 204, 218; and after this Manner, by counting the Vibrations made in a Minute or an Hour by any other Standard Measure, may its Length be ascertained, and Ratio expressed, in Parts of the *universal* YARD.

3302. And as a farther Illustration of this Matter, I have represented these Measures separately in the Copper-plate (Fig. 9.) where A B is the *philosophical* YARD; C D the *philosophical* FOOT; E F the *English Foot*; and G H the *Parisian Foot*; all in their due Proportion of Length. From what has been said, it is evident of how much Importance the Theory of a simple Pendulum is, not only as the most perfect CHRONOMETER, but as a STANDARD for Measures of LENGTH of every Kind, and to all Ages.

CHAP.

CHAP. X.

The Theory *of a* Compound Pendulum *explained, and its* Center *of* Oscillation *investigated.*

3303. THE Consideration of a *Compound Pendulum* comes next in Course, nor is the Theory thereof less curious or necessary than that of a simple One, since in the common Construction of Clocks, the Weight of the Rod or Wire is too considerable to be neglected; and then the Rod and Ball together are to be esteemed as a Pendulum consisting of two different Weights, whose compound Center of Gravity, and Center of Oscillation are to be determined.

3304. Again, after all the Efforts of human Art or Skill, the *Pendulum* can never be perfectly equable in its Motion, but will, from the very Condition of Nature itself, be liable to have the Times of its Vibration altered, by an insensible Acceleration or Retardation from the variable State of natural Causes. Therefore, for common Use, Hugenius recommended a moveable Ball, or Weight for correcting the Irregularities of Vibration, as by its different Position on the Rod, it will cause a *small Alteration of the Distance of the Center of Oscillation*, by which Means the Clock may be adjusted to true Time.

3305. In a Pendulum of this Sort there are three different Weights to be considered, *viz.* that of the large *Ball or Bob*, that of the *Rod*, and that of the *Corrector*, moveable upon it. The Center of Oscillation of all which must be found, which will not be difficult after we have found that for two Weights. Therefore let A C (Fig. 10.) be a Pendulum, Bisect its Length in G; and put $AG = g$, and $AC = a$. Also call the Weight of the Bob (c) and that of the Rod or Wire (b).

3306.

3306. Now the *Momentum* and *Force* of each Part are separately to be estimated. The *Momentum* of the Rod is universally $\frac{1}{2}ab$ (1072); the *Moment* of the Ball C is (ac), being as the Weight multiplied by its Distance (1089). The *Force* of the Rod is $b \times \frac{1}{2}a \times \frac{2}{3}a$ (1095) $= \frac{1}{3}ba^2$; and the *Force* of the Ball C is ca^2 (1089); therefore the *Sum of the Forces* divided by the Sum of the Moments, gives the Distance (n) of the Center of Oscillation, that is, $\frac{\frac{1}{3}ba^2 + ca^2}{\frac{1}{2}ba + ca} = n = \frac{\frac{1}{3}ba + ca}{\frac{1}{2}b + c}$ as in (1094). And from hence the Length of the Pendulum $AC = a = \frac{\frac{1}{2}bn + nc}{\frac{1}{3}b + c}$; which is therefore determined for a *Second-pendulum*, by putting $n = 39{,}2$ Inches; or for *Half-seconds*, if $n = 9{,}8$ Inches. *Note*, in all these Cases, the Length A C is the Distance between the central Line in the Axis of Suspension, and the Center C of the Bob.

3307. Now let AC (Fig. 11.) be the Pendulum with the Addition of the *Corrector* D, or small moveable Weight $= d$. And let its Distance be $AD = f$. Then will its *Momentum* be fd, and its *Force* dff; and now the Sum of the Forces of each Part divided by the Sum of the Moments will be $\frac{\frac{1}{3}a^2b + ca^2 + df^2}{\frac{1}{2}ab + ac + df}$ *the Distance of the Center of Oscillation or Length of an isochronal Pendulum*, which for the Future we will call p.

3308. Then if it be required to find (f) or the Position of the Corrector D, we have $\frac{\frac{1}{3}a^2b + ca^2 + df^2}{\frac{1}{2}ab + ac + df} = p$, which gives this Equation $f^2 = fp + \frac{\frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}$, then by compleating the Square, and extracting the Root, we have

$$f = \frac{1}{2}p \pm \sqrt{\frac{\frac{1}{4}p^2 + \frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}}.$$

3309. Hence it is obvious, there will always be two real Roots or Values of (f) while $\frac{1}{2}abp + cap$ is less than $\frac{1}{3}aab + aac$, or while the Length p is less than $\frac{\frac{1}{3}a^2b + a^2c}{\frac{1}{2}ab + ca}$, which

is

is the Length of the isochronal Pendulum (π) consisting of the Rod AC, and the Weight C only (3306).

3310. When, therefore, we would accelerate the Motion of the Pendulum by the Application of the Weight D, we have the Choice of two Places for it between A and C, *viz.* D or E, which Places are equally distant from N, the Point which Bisects the isochronal Pendulum p. For $f = \frac{1}{2}p +$ or $-$ the radical Quantity as above (3308).

3311. Therefore, when $f = \frac{1}{2}p = AN$, it will accelerate the Motion of the Pendulum the most of all; in which Case we have $\sqrt{\dfrac{\frac{1}{4}p^2 + \frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}} = 0$, which will give the Equation $\frac{1}{4}p^2 + \dfrac{\frac{1}{2}abp + cap}{d} = \dfrac{\frac{1}{3}a^2b + a^2c}{d}$; from which, by Reduction, compleating the Square, &c. we shall find $p = \dfrac{a}{2d}\sqrt{\frac{4}{3}bd + 4cd + bb + 4bc + 4cc} - \dfrac{ab - 2ac}{2d}$; and from hence the Value of $f = \frac{1}{2}p$, is known when its Effect is a *Maximum*.

3312. For accommodating this Theorem to practice it will be most convenient to assume $b = d = 1$; then we shall have the Equation f or $AD = f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{1}{2}ap + acp - \frac{1}{3}a^2 - a^2c}$; and when $f = \frac{1}{2}p$; then $p = \frac{a}{2}\sqrt{2\frac{1}{3} + 8c + 4cc} - a \times \overline{1 - 2c}$.

3313. To give an Example of the Use of these Theorems; let the Pendulum of the Clock be required to beat *Seconds* precisely, and suppose the Weight C $= 50$*lb.* and $b = d = 1$*lb.* Then we have $p = n = 1440$ (2295), $c = 50$, and the Theorem in (2306) will become $\dfrac{\frac{1}{2}n + nc}{\frac{1}{3} + c} = a = 1444{,}8$. And by substituting these Values of a, b, and d, in the Equation for (f), we get $f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + 72962\,p - 105061210}$.

3314. Therefore when the Accelleration is greatest of all, we have $f = \frac{1}{2}p$; and $\frac{1}{4}p^2 + 72962\,p = 105061210$, or $p^2 + 291848\,p = 420244840$; whence we find $p = 1436$, nearly; but

but 1440 — 1436 = 4′, or the greateſt Acceleration is 4 Minutes *per* Day, by placing the Weight D at N, or making $f = AN = 718 = \frac{1}{2}p$, when a *Minimum*.

3315. Since by ſhortening the iſochronal Pendulum (p) only 4 Parts of the 1440, the Length of $f = AD$, or the Space through which the Weight D moved upwards is 718; it is evident that a ſmall Alteration in the Length of A C, will allow of a very large *Scale of Correction* on the Rod of the Pendulum both in regard to *retarding* as well as *accelerating* its Motion.

3316. For, to accelerate the Motion of the Pendulum one Minute *per* Day, the Center of Oſcillation moves through but 2 Parts of the 1440, or the Length of the Pendulum is then $p = 1438$ as we have ſhewed (3295). Therefore putting $n = 1438$, we ſhall have by Theorem (3306) $\frac{\frac{1}{2}bn + nc}{\frac{1}{3}b + c} = a = 1441,8$ which is juſt 3 leſs than 1444,8, the Length for ſwinging in a Second preciſely (3313). Hence the Motion of the Center of the Weight C is to that of the Center of Oſcillation as 3 to 2. And it is univerſally $a : n :: \frac{1}{2}b + c : \frac{1}{3}b + c$; or, in the preſent Caſe, $a : n :: 50,5 : 50,3 :: 1444,8 : 1438 :: 392 : 391,46$, as we have before obſerved (3295).

3317. From all which it appears, that the Center of Oſcillation is but 5 Parts of 1440 diſtant from C the Center of the Ball, and alſo, that when the Corrector D has its Poſition, ſuch as to produce no Acceleration, it muſt be either upon the Center of Oſcillation, or at the Axis of Suſpenſion A; or $f = \frac{1}{2}p = \frac{1}{2}a = \frac{1}{2}AC$, very nearly. But in the common Conſtruction of the Pendulum this Diſpoſition of the *Corrector* D can have no Place, becauſe it cannot be brought nearer to the Center of the Weight C than the Length of its Semidiameter, which, in a Ball or Globe of 12*lb.* only, is 2 Inches (3296); but in a Globe of 50*lb.* it is more than 3 Inches. Now this is more than the Diſtance to which the Weight D muſt be removed to accelerate

 the

N. B. The Reader is deſired to correct the Expreſſion of the Radical Symbols in Inſtitutions 3308 and 3311, thus; $\sqrt{\frac{1}{4}p^2 + \frac{\frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}}$.

the Motion 1 Minute *per* Day; for by the Theorem (3313) if we put $p = 1438$, we ſhall get $f = 1331,5$; if then we ſay, as $1440 : 1331,5 :: 392 : 362,5$, we have the Diſtance of the Weight D from the Center C, but 29,5, or not quite 3 Inches.

3318. Therefore in order to have a Scale of a ſufficient Variation for the Poſition of the Weight D, we muſt give to the Rod A C a greater Length in Compariſon of the Length (p) of a Pendulum vibrating Seconds, and conſider the Weight C or Bob of the Pendulum as *moveable upon* it in the ſame Manner as the Weight D is. Thus let $AB = a$, be the Length of the Rod or Wire; and $AC = e$, the new Diſtance of the Weight C upon it; then will the Theorem (3306) become $\frac{\frac{1}{3}a^2b + e^2c}{\frac{1}{2}ab + ec} = p$; and therefore $\frac{1}{3}a^2b + e^2c = \frac{1}{2}abp + ecp$, whence $e^2c + ecp = \frac{1}{2}pab - \frac{1}{3}a^2b$; and $e^2 - ep = \frac{\frac{1}{2}pab - \frac{1}{3}a^2b}{c} = \frac{\frac{1}{2}pa - \frac{1}{3}a^2}{c}$, by putting $b = 1$, as before. Then compleating the Square and extracting the Root, we get $e = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{\frac{1}{2}pa - \frac{1}{3}a^2}{c}}$. And the Theorem in (3312) for A D will now become $f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{1}{2}pa + pec - \frac{1}{3}a^2 - e^2c}$. Theſe Theorems for A D and A C afford a new Conſtruction of the Pendulum whoſe Nature and extenſive Uſe will be fully declared in the enſuing Chapter. (See the Plate of the *Univerſal Pendulum*, Fig. 1, 2.)

CHAP.

CHAP. XI.

The Theory *and* Construction *of an* Universal Pendulum *for measuring the Time of* Solar, Lunar, *and* Planetary Days *very correctly*.

3319. IN order to shew the Universality of the new Structure of the Pendulum premised in the foregoing Chapter, it will be necessary to consider that the Design of a Clock is to measure constantly, that equal Portion of Time which is called a *Mean Day*, and is divided into 24 equal Parts or Hours. As Time in itself flows equally, the Clock, if it could be kept to an equable Motion, would be an adequate Measure of it; and this would be the greatest Excellency and Use of such a Machine.

3320. But this, however, has, in my Opinion, been very indifferently provided for in the governing Principle of a Clock's Motion, *viz.* a *Pendulum of isochronal Vibrations*; and it must be observed, that after all the Precautions and Inventions for rendering this *Isochronism* of Vibration permanent, yet such a Discovery still remains too obviously the great *Desideratum* of Clock-work, and therefore Artists are obliged to have recourse to the best Methods of remedying such Irregularities, and of checking them even in their *Nascent* State.

3321. How gross and indirect the Methods of rectifying the Pendulum in general are, if compared with that of the secondary or moveable Weight D, invented by *Hugenius*, will be very evident on mature Consideration. And as it is here improved, I flatter myself it will be allowed the most easy and extensive that can be desired, when it is considered, that the *Scale of Variation* for the Corrector D is by this Means made so large as to render any Clock capable of measuring not only the *Mean Time* or *Day*, but also that measured by the Motion of any of the celestial

ſtial Bodies, whether *Moon*, *Planet*, or *Star*; as we now proceed to ſhew.

3322. The Time that intervenes the departing of any celeſtial Body from the Meridian and its Appulſe to it again, is called the *Day* peculiar to that Luminary; but ſince none of the Planets deſcribe *circular Orbits*, but *Ellipſes*, their Motions will be variable, and the Times of their *meridional Revolutions*, or Days, will be unequal; therefore the *Mean Revolution*, in all, muſt be taken for the *Mean Planetary Day*; and theſe collected from the aſtronomical Tables are for the ſeveral heavenly Bodies, as in the Table here ſubjoined.

		H.	′	″	‴	′	″	‴	′
3323.	A fixed STAR	23	56	3	28				1436
	SATURN	23	56	11	00	0	8	0	$1436\frac{1}{5}$
	JUPITER	23	56	23	00	0	20	0	$1436\frac{1}{3}$
	MARS	23	58	08	58	2	5	30	$1438\frac{1}{7}$
	SUN	24	00	00	00	3	56	32	1440
	VENUS	24	02	28	00	6	24	32	$1442\frac{1}{2}$
	MERCURY	24	12	25	28	16	22	0	$1452\frac{2}{5}$
	MOON Least	24	43	08	28	47	5	0	$1483\frac{1}{7}$
	MOON Mean	24	48	43	28	52	40	0	$1488\frac{7}{10}$
	MOON Greateſt	24	57	07	28	61	4	0	$1497\frac{1}{8}$

3324. The firſt Column in this Table ſhews the Hours, *Minutes*, *Seconds*, and *Thirds* in each planetary Day, the *Mean Solar Day* of 24 Hours being the Standard. In the ſecond Column are contained the *Minutes*, *Seconds*, and *Thirds*, by which each following Day exceeds the firſt or *Sidereal Day*; thus the *Mercurial Day* is longer by 16 Minutes, and 22 Seconds; and the greateſt *Lunar Day* by 61 Minutes, and 4″. In the third Column, the Quantity of each Day is expreſs in the Minutes of a *Mean Solar Day*.

3325. If therefore the Length of the iſochronal Pendulum (p) in the foregoing Equations be found for the Numbers in the 3d Column reſpectively, then will the Diſtances A D, and A C, or the Poſitions of the Weights D and C be found by the Theorems in (2318), ſo that the Clock ſhall keep Time with the Pla-

Planet, and Hour-hand point to the Hours when the Planet ſhall be on the correſpondent Hour-circles. So that by this Means any Clock may readily be adjuſted to ſhew the Time of any *Planetary Day*, or the Poſition of the Planet in any Part of its Revolution.

3326. But for this Purpoſe the Length of the Rod A B muſt be ſufficient to allow of a Scale of Poſition to the Corrector D, while the iſochronal Pendulum (p) encreaſes from its leaſt Length for the *Sidereal Day* of 1436′ to its greateſt Length for the *longeſt Lunar Day* of 1505′, in which Caſe the Weight or Bob C will poſſeſs the Extremity of the Rod, or $AC = e$ will become $AB = a$. And becauſe as the *Days* encreaſe, the *Hours*, *Minutes*, and *Seconds* increaſe in the ſame Proportion (as there is a conſtant Diviſion of each Day into the ſame Number, *viz.* 24 Parts), therefore, alſo, the Time of the Pendulum's Vibration will encreaſe with the Length of the Day. And conſequently the Numbers in the 3d Column of the Table will be as the Times of Vibration in the Pendulums appropriated to the Clocks for ſhewing the Time of the Days reſpectively.

3327. Therefore it will be neceſſary in the firſt Place to determine the Length of P the iſochronal Pendulum whoſe Time of Vibration is as 1500, by ſaying, as $1440^2 : 1500^2 :: 1440 : 1562\frac{1}{2} = P$. And becauſe the larger the Weight C is, the leſs will be the *Scale of Acceleration*, therefore we muſt take the ſaid Weight, ſuch as will allow this Scale to be ſufficiently large to comprehend all the Variation of the Moon's Motion; and by Trial, it will be found that 10*lb.* will be the greateſt Weight that can be allowed to the Bob; therefore $c = 10$; b and d being $= 1$, as before.

3328. Hence we determine A B, the Length of the Rod, by the Theorem $\frac{\frac{1}{2}P + cP}{\frac{1}{3} + c} = a = 1587{,}8$ as in (3306). Alſo by the Theorem in (2311) we find the Length of an iſochronal Pendulum, when a *Minimum*, to be $p = 1526{,}2$; therefore $1562{,}5 - 1526{,}2 = 36{,}3$; and half this Number, *viz.* 18 ſubducted from 1500 leaves 1482 for the Expreſſion of the Time of a Vibration of the Pendulum (p) compared to the Time 1500 of the Pendulum P.

3329.

3329. That the Reason of this Assertion may appear, we shall demonstrate this *Lemma*, viz. *when Numbers are very large, and their Differences very small, then any three or four of them that are in* geometrical Proportion *are also* in arithmetical Proportion. Thus let a, and $a-d$, be two large Numbers, for Instance 1440, and 1439, then $d = \frac{1}{1440}$ Part of a; and let it be $a : a-d :$ $a-d : z$; then $z = \frac{a^2 - 2ad + dd}{a} = a - 2d$, because (dd) vanishes in Comparison of the rest; therefore a, $a-d$, $a-2d$, are in *geometrical Proportion*; they are also evidently in *arithmetical Proportion*; and hence 1440, 1439, 1438 are Numbers that have the same Property; for since $1440^2 :$ $1439^2 :: 1440 : 1438$, therefore $\frac{1439^2}{1440} = 1438$, and consequently $1440 : 1439 :: 1439 : 1438$, and so they are geometrical and arithmetical Proportionals at the same Time.

3330. Since the Number 1483 answers to the shortest Lunar Day in the Table (3323) and the Difference between that and the longest Day is but about 14 Minutes; it is evident, this Scale of 18′ will serve for all Lunar Days. And the Values of P and f may be found for every Number betwixt 1483 and 1497 by the Theorems in (3307, 3308). And the Scale on the Rod of the Pendulum may be graduated for Use and the adjusting Weight D put to its proper Place for the due Rectification of the Clock to the Length of the respective *Lunar* Day proposed, which is always known from an *Ephemeris*.

3331. From the Table (3323) it appears, that the shortest *Lunar Day* 1483 exceeds the longest *Planetary Day* 1452; and consequently the *Lunar Scale* on the Pendulum will be of no Use for the Planets. A planetary Scale therefore must be constructed that shall have an Extent sufficient for the Movement of the Corrector D from the *Sidereal Day* of 1436′, to that of the Planet *Mercury* of 1452; whose Difference is 16 Minutes. And since the *Mercurial Day* of 1452 exceeds the *Solar Day* 1440 by 12′, if we allow 4′ more for the Diameter of the Weight C, we shall have 1456′ for the longest Day in the *Planetary Scale*, for which (having the Length of the Rod AB $= a = 1587,8$) we can find P and e, by the Theorems already premised.

3332.

3332. Thus *per* Theorem (3329) we have 1440 : 1456 :: 1456 : 1472 = P, the Length of the ſimple Pendulum proper for that Length of Day.

3333. Again, having given A B = 1587,8 and P = 1472, we have $e = \frac{1}{2}P \pm \sqrt{\frac{1}{4}P^2 + \frac{\frac{1}{2}Pa - \frac{1}{3}a^2}{c}} = 1494 = AE$, by the Theorem (3318). Therefore, A B — A E = E B = 94, which is a little more than an Inch, through which the Weight C is to be raiſed on the Rod of the Pendulum A B, from C to E.

3334. We are next to enquire the Length of an iſochronal Pendulum p correſponding to the greateſt Acceleration, when $f = \frac{1}{2}p$. In this Caſe we ſhall have $\frac{1}{2}p^2 + \frac{1}{2}pa + ecp = \frac{1}{3}a^2 + e^2c$, by the Theorem ſo often quoted in (3318). If we put $\frac{1}{3}a^2 + e^2c = s$, and $\frac{1}{2}a + ec = t$, we ſhall have $p = 2\sqrt{s + tt} - 2t = 1433\frac{1}{2}$, and of Courſe $AN = AD = f = \frac{1}{2}p = 716{,}7$.

3335. But the Time of Vibration of this Pendulum being a Mean proportional between 1440 and 1433.5 by the *Lemma* in (3329), that is, 1440 : x :: x : 1433,5, whence $x = \sqrt{1440 \times 1433{,}5} = 1436{,}75$, which ſubducted from 1456 the ſloweſt Vibration (3331) will leave 19′,3 which will be juſt large enough to take in the *Planetary Days* as far as they are ſenſibly different from each other: For the *Sidereal*, *Saturnian*, and *Jovian Day* differ ſo little, as not to be diſcernable in the Scale, as is evident from the Numbers in the third Column of the Table (3323).

3336. In this Scale the Length of the iſochronal Pendulum for the *Mean Day* of the SUN being 1440 = p, we ſhall find $f = AD = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + tp - s} = 929$, wherefore the Place of the Corrector D is given for regulating the Clock for common Uſe, or to ſhew *Mean Time*.

3337. In like Manner, by having the Numbers

1438 for *Mars*, you find $p = 1436$.
$1442\frac{1}{2}$ for *Venus* ——— $p = 1445$.
1452 for *Mercury* ——— $p = 1465$.

And

And from thence, by the above Theorem, the Position of D, or AD = *f*, will be found for each *Planet* respectively. And thus the *Lunar and Planetary Scales* of Acceleration are compleated; and the Clock fitted to shew Time *universally*.

N. B. For the better Illustration of what we have delivered concerning this new constructed Pendulum, we have (in Fig. 2. of the following Plate) added so much of the lower Part of the *Half-second* Pendulum, as contains both the *Lunar and Planetary Scales*; but these Scales will be four Times as long in a Pendulum that beats *Seconds*.

CHAP. XII.

Concerning the BOB *or* WEIGHT *of the* PENDULUM; *and the* THEORY *of such a* FORM *or* SHAPE *thereof, as shall meet with very small* RESISTANCE *from the* AIR.

3338. IN a Treatise on the *Rationale* of CLOCK-WORK, I judge it will be expected that something should be said concerning the FORM of the BOB, or WEIGHT of the *Pendulum*, with regard to the Resistance of the Air in which it moves, and the consequent Irregularity in its Motion, which will be thereby unavoidably produced; and the rather, because so great a Judge in these Matters as HUGENIUS, has asserted it to be a very interesting Point, *nam plurimum refert*, are his Words.

3339. But as we have shewn, the only Design of the Contrivance and Mechanism of a Clock is to annihilate the Resistance which the Pendulum meets with from the Air, and every other Cause, it is plain, if it were possible to construct a Clock with so much Accuracy as to do that, the Motion of the Pendulum would be as equable as if it moved without any Resistance at all; for the THEORY of *Clock-work* would be a very imperfect Thing, if it could not provide against the Effects of Re-

Resistance of every Kind, and cause the Clock to go with an *equable Motion*, whether the Pendulum moved *in Vacuo*, in *Air*, or in *Water*, and even in *Quick[illegible]* itself.

3340. But it may neverth[illegible] [illegible]e useful in some Cases of a Pendulum, to know what *Form [illegible] the Ball* will be most convenient and least resisted in its Motion through the Air, which though insensible in a few Vibrations, or in a short Time, may yet, by a constant Accumulation, produce Effects too sensible and molesting in long Run, maugre all the Ingenuity and Diligence of the Artist. This Subject therefore we shall now proceed to consider more particularly.

3341. It is well known that the Resistance of a Body in Motion regards the *Quantity and Figure of its Surface*, and not the Quantity of Matter moved; for a Cubic Inch *of Matter* may be disposed into the Form of a Sphere, whose Diameter will then be 1,247; and its Superficies = 4,836 *Cubic Inches* (836) whereas the same Matter in the Form of a Cube had a Quantity of Surface = 6 Cubic Inches. And the Sphere has the least Quantity of Surface that it can possibly be contained under; and therefore *has less Resistance in Proportion to its Weight*, than any other Body, and consequently is the *best Form*, for the Ball of a Pendulum in general. And because the Surfaces of Spheres are proportioned to the Squares of their Diameters (842), larger Spheres will have much less Surface than small Ones, in Proportion to their Weight, and will therefore meet with less Resistance.

3342. The Solid of *least* Resistance (whose Theory you will find in (2042, &c.) would undoubtedly merit the first Consideration, were it not, that its Construction will be much too difficult for Practice; and this Solid is in itself but little preferable to the *Frustum* of a *Cone* of *least Resistance*, which is not only made with the utmost Ease, but is at the same Time best fitted for the Purposes of the planetary Pendulum and Clock before described. The Theory of which we shall, therefore, here deliver.

3343. Let A B C (Fig. 3.) be the isosceles Triangle generating by its Revolution a Cone, whose *Frustum* C G F B has the least Resistance for a given Base C B, and Altitude D E. Let the *Frustum* be supposed to move in the *Medium* in a Direction parallel to its Axis; let S V be drawn parallel to A E, to represent the Direction in which the Particles of the Fluid strike the

Cone. Then V I being perpendicular to the Baſe C B will repreſent the Particles ſtriking the Baſe with their whole Force, ſuppoſing the Fruſtum moved from D toward E; but in moving from E towards D, the Particles S F will come obliquely on the Side A B, and therefore cannot ſtrike any Particle F with their full Force.

3344. Let any Line L F repreſent the whole Force of the Stroke againſt the Baſe B C, and through the Point F draw M H perpendicular to the Side A B; let M F = F H; and compleat the Parallellogram A L F M; then the whole Force L F will be reſolved into two Forces L M and M F, of which one, *viz.* L M being parallel to the Side A B of the Cone, cannot affect its Motion; the other Part M F is, therefore, all the Force of the Fluid on that Point, that can cauſe any Reſiſtance.

3345. We are next to enquire what the Effect of this Force M F or F H is, in the Direction D E; to this End let F H be reſolved into two Forces F D and D H, of which the former F D is equal to, and its Effect wholly deſtroyed by the antagoniſt Force G D, ariſing from the Action of the Fluid T G on the other Side; the Force D H, being all exerted in the Direction D E, is therefore all the Force of the oblique Stroke that directly oppoſes the Motion of the Body, and conſequently that can cauſe any Reſiſtance to the Point F in the Side A B.

3346. Therefore the Effect or Reſiſtance of a Particle of the Medium at the Point F in the Surface, is to that at the Point I in the Baſe, by a direct Stroke, as D H to L F or A H, and therefore as $AH \times DH$ to AH^2. But by ſimilar Triangles H F D, H A F, it is $HD : HF :: HF : HA$, whence $AH \times DH = FH^2$, therefore AH^2 is to FH^2, as the Reſiſtance at I to that at F: But $AH : FH :: AB : BF$; therefore the Reſiſtance at I is to that of F, as $AB^2 : BE^2$. And ſince what has been ſaid of the Points F and I hold equally true of all other Points in the Surface and Baſe of the Cone, *therefore the Reſiſtance to the Baſe will be to that upon the Surface of the Cone as* AB^2 *to* BE^2.

3347. Hence then the Reſiſtance to the Surface F A G of the Cone cut off, is to that againſt its Baſe F G, as FD^2 to AF^2; therefore $BE^2 - FD^2$ is as the Reſiſtance to the Surface of the *Fruſtum*; and by adding AF^2, (which is as the Reſiſtance upon

the Base or End F G) we have $BE^2 + AF^2 - FD^2$ for the Expression of the Resistance to the whole *Frustum*, when moving in the Direction E A. But $AF^2 - FD^2 = AD^2 = AE^2 - 2AE \times ED + ED^2$ (637). Therefore the Resistance of the Frustum will be $BE^2 + AE^2 - 2AE \times ED + ED^2 = AB^2 - 2AE \times ED + ED^2$.

3348. But since AB^2 expresses the Resistance to the Base BC (3346) and is a *constant Quantity*, it may be represented by Unity, or $AB^2 = 1$; and as any Quantity may be considered as divided by Unity, the Resistance of the *Frustum* may be also thus expressed $\frac{AB^2 - 2AE \times ED + ED^2}{AB^2}$, or thus, $1 + \frac{DE^2 - 2AE \times DE}{AB^2}$.

3349. Therefore putting $DE = a$, $BE = b$, and $AE = x$, we have the last Expression for the Resistance in Symbols thus $1 + \frac{a^2 - 2ax}{b^2 + x^2}$, whose Fluxion (when a *Maximum* or *Minimum*) is $\frac{2ax^2\dot{x} - 2a^2x\dot{x} - 2ab^2\dot{x}}{\overline{xx + bb}^2} = 0$; whence we have $x^2 - ax - bb = 0$; and by compleating the Square and extracting the Root, we have $x = \frac{1}{2}a \pm \sqrt{\frac{a^2}{4} + b^2}$. But x being the Height of the whole Cone must be greater than a, which is but a Part of that Height, therefore $x = \frac{1}{2}a + \sqrt{\frac{a^2}{4} + bb}$.

3350. Hence we have the following easy Construction; bisect DE in K, draw KC; and produce ED to A, so that KA may be equal to KC. Then will ABC be a Cone, whose *Frustum* BFGC shall meet with less Resistance, in an uniform Fluid, than any other of the same Base and Altitude. For because $AK = KC = x - \frac{1}{2}a$, and $KE = \frac{1}{2}a$, and $EC = b$, we have $KC^2 = \frac{1}{4}a^2 + bb = x^2 - ax + \frac{1}{4}a^2$, which gives $x^2 - ax - bb = 0$, as required in the Property of such a *Frustum* (3349).

 3351.

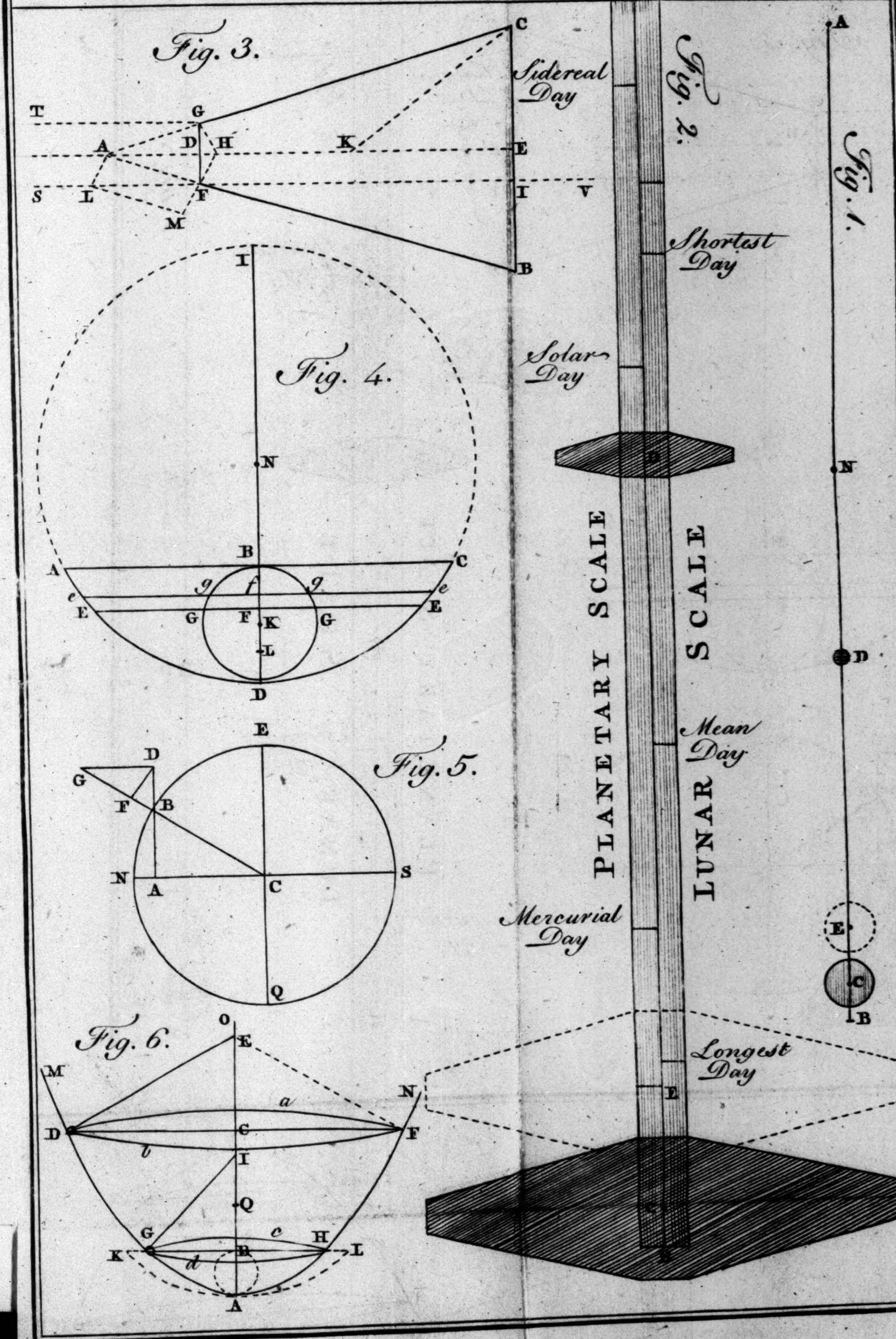
3351. In order to be certain that this Resistance is a *Minimum*, and not a *Maximum*, we are to consider this Principle, *that while a Quantity is increasing, its Fluxion will be positive; but negative while it is decreasing.* Now the Quantity $xx - ax - bb = 0$, is analogous to the Fluxion from whence it proceeds, and therefore when $x = a$, this Quantity becomes $-bb$, which is plainly negative.

3352. Since then the Resistance decreases from the Value of $x = a$, to $x = \frac{1}{2}a + \sqrt{\frac{a^2}{4} + bb}$, where it is a *Minimum*, it is plain that the *Frustum* BCGF is less resisted than a *Cone* of equal Base and Height, or than any other *Frustum* of a larger Cone, whose Base and Height are the same with these, *viz.* CB and DE.

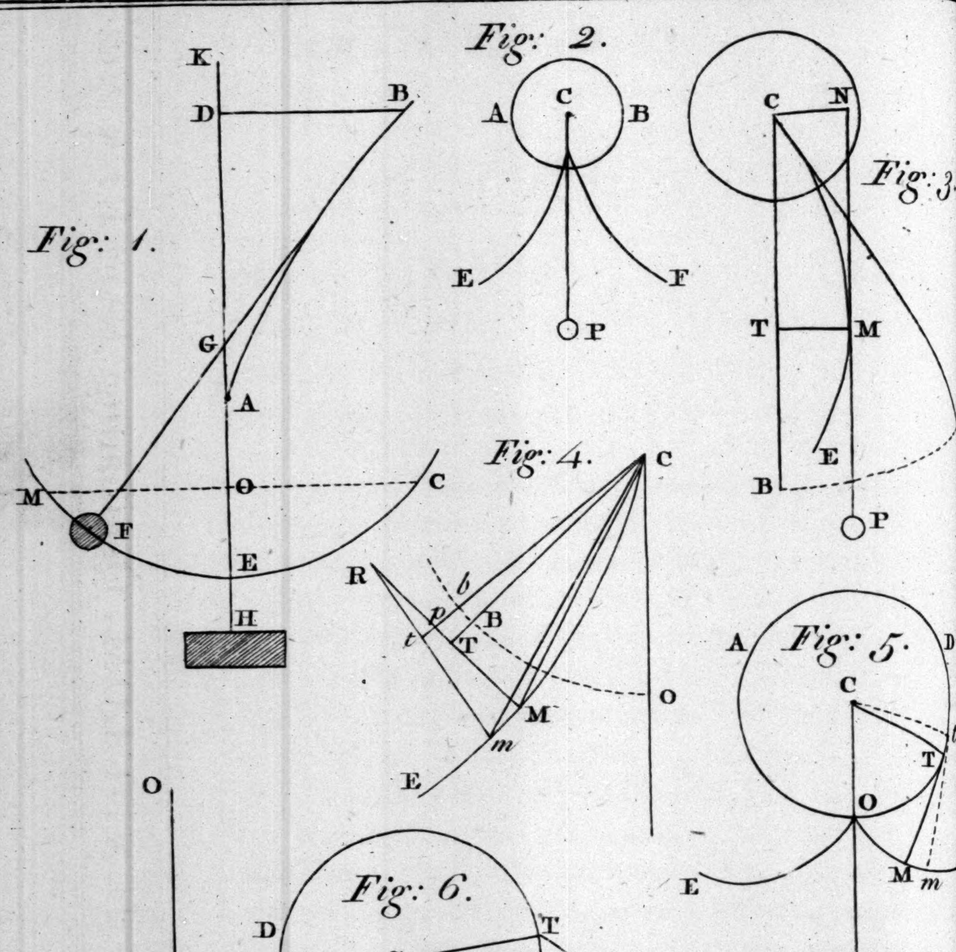
3353. The above Demonstrations will hold for any Proportion of CE to DE; and the less CE is in regard to DE, the less will AD be in respect of AE; so that at Length the Frustum of a Cone, of least Resistance, will come near to the Form of the *Solid of least Resistance*, and serve almost equally as well for the Purpose of the *Bob* of a Pendulum, and is excellently well fitted for that of the *planetary Pendulum* described in the preceeding Chapter, which consists of two such Frustums, whose Center is that of their common Base. And further, as such a *double Frustum* of a Cone is less resisted than a Globe of the same Weight, so the middle Segment or Proportion of such a *Frustum* cut longitudinally, will be better and more artificial, than a Bob of the common *lenticular* Form. ——— It is somewhat remarkable, that the *French* and *Spanish* ACADEMICIANS in the Department to PERU, for measuring the Length of a Degree of the Meridian at the Equator, had in their Clock a Pendulum of *two truncated Cones*, but placed just in a contrary Direction to that which the Theory above requires, and is here shewn in Fig. 2.

CHAP.

The Universal PENDULUM.



The THEORY of the Circulating PENDULUM, Oscillating ROTULA &c.



CHAP. XIII.

The THEORY *for finding the* TIME *of a* least OSCILLATION *of a given* PENDULUM *swinging in the* ARCH *of a* CIRCLE; *and also the Time of any other: with proper* CANONS *for* CALCULATION.

3354. WE have formerly shewn (1119) that the *Time* of the *least Vibration* of a given Pendulum is equal to the Time such a Pendulum would take to vibrate in the Arch of a *Cycloid*. Also, that when the Arch of a Circle, in which such a Pendulum vibrates, is so small, that it differs not sensibly from the Cycloid, the Time of a Vibration will not be affected thereby; and this Consideration makes *cycloidal Cheeks* unnecessary in long Pendulums, or such as swing *Seconds*, where the Arch of Vibration is not more than 3 or 4 Degrees of a Circle.

3355. But in *Table-clocks*, whose Pendulum is not more than about 6 or 8 Inches long, the Case is different; for here the Space can be but small, in which there can be any sensible Coincidence of the *Circle and Cycloid*, and yet the Arch of Vibration in these short Pendulums is generally much longer in Proportion to their Lengths than in the long Ones; whereas they ought to be proportionably shorter. These Clocks must therefore be subject to an Error that cannot be avoided but by *cycloidal Plates*, such as *Hugenius* invented and prescribes for this Purpose; the Nature of which see already largely explained (1120), (or by any of the *new Species of Pendulums* hereafter to be described) for the common Method of moving the Bob up and down upon the Rod, by a Screw at the Bottom, is very fallacious and inartificial, and can never procure an equable Motion to the Clock, unless the Arch of Vibration be very small indeed.

3356. When the circular Arch described by the Pendulum begins to deviate from the *Cycloid*, the Times of Vibration will begin to encrease or exceed the Time of a *least Vibration*, and therefore it will be necessary to shew how this Error arises, and in what Proportion it encreases, that the ingenious *Tyro*, in Clock-work,

work, may not be left in the Dark in that Part of the Theory which is most essential to his Art, or conducive to the Truth of his Work. And this we shall be able to do in the plainest Manner, by following the Footsteps of the late learned *Professor* SAUNDERSON.

3357. The general Problem is, *to find (independent of the Cycloid) the Time precisely of a least Oscillation of a Pendulum of a given Length swinging in an Arch of a Circle; and also to find, without any sensible Error, the Time of any other.* Let the Curve A D C be the Arch of a Circle, whose Diameter is D I, and in which the Pendulum N D is supposed to vibrate. Then in descending from C to any Point E, it will acquire a Velocity which will be as $\sqrt{EM} = \sqrt{BF}$ (having drawn the Chords A C, E E intersecting the Diameter I D in B and F). Now $\frac{1}{2}\sqrt{FD}$ is as the Velocity acquired in descending through $\frac{1}{4}$ I D, and by that Velocity it would describe uniformly a Space equal to $\frac{1}{2}$ I D in the Time of the said Descent through $\frac{1}{4}$ I D (993). Then the Space $\frac{1}{2}$ I D divided by the Velocity $\frac{1}{2}\sqrt{ID}$ will give the Time $\sqrt{ID}$ of the Descent through $\frac{1}{4}$ I D, *i. e.* through *half the Length of the Pendulum* (971.)

3358. Draw *e e* very near to E E; then may E *e* be considered as the Fluxion of the Arch D E; and $\frac{Ee}{\sqrt{BF}}$ will be the Time wherein the small Arch E *e* is described by the Pendulum, or the *Fluxion of the Time of a Vibration.* But $Ee = \frac{\frac{1}{2}ID \times Ff}{\sqrt{IF \times FD}}$ $= \frac{\sqrt{ID}}{\sqrt{IF}} \times \sqrt{ID} \times \frac{1}{2}\frac{Ff}{\sqrt{FD}}$; therefore we have the Fluxion of the Time $\frac{Ee}{\sqrt{BF}} = \frac{\sqrt{ID}}{\sqrt{IF}} \times \sqrt{ID} \times \frac{1}{2}$ $\frac{Ff}{\sqrt{BF \times FD}}$.

3359. Bisect B D in K, and K D in L; and when the Arch A D C is small, the Quantity I F cannot differ sensibly from I K,

IK, nor $\frac{\sqrt{ID}}{\sqrt{IF}}$ from $\frac{IL}{IK} = \frac{IL}{IF}$ (becauſe ID : IL :: IL : IK or IF, therefore ID : IF :: IL² : IK² (by 672). Therefore $\frac{Ee}{\sqrt{BF}}$ is very nearly equal to $\frac{IL}{IK} \times \sqrt{ID} \times \frac{1}{2}\frac{Ff}{\sqrt{BF \times FD}}$.

3360. Upon the Diameter BD deſcribe a Circle cutting the Chords EE and *ee* in G and *g* reſpectively, then will the Fluxion of the Arch DG be $Gg = \frac{\frac{1}{2}BD \times Ff}{\sqrt{FB \times FD}}$ (875) conſequently $\frac{Gg}{BD} = \frac{\frac{1}{2}Ff}{\sqrt{FB \times FD}}$; and therefore the Fluxion of the Time of Vibration through DE will be $\frac{Ee}{\sqrt{BF}} = \frac{IL}{IK} \times \sqrt{ID} \times \frac{Gg}{BD}$.

3361. But the Fluent of this laſt Fluxion is $\frac{IL}{IK} \times \sqrt{ID} \times \frac{DGB}{BD}$, which is therefore the Time of *half a Vibration* or Motion of the Pendulum from D to C. The Time of a whole Vibration therefore through ADC will be $\frac{IL}{IK} \times \sqrt{ID} \times \frac{BGDGB}{BD}$.

3362. Let the Arch ADC be indefinitely ſmall, then the Quantity $\frac{IL}{IK} = 1$, and the Time (T) of a leaſt Vibration will become $T = \sqrt{ID} \times \frac{BGDGB}{BD}$. And therefore BD : BGDGB :: $\sqrt{ID}$: T; that is, *as the Diameter is to the Circumference of a Circle, ſo is the Time* ($\sqrt{ID}$) *of the Deſcent through half the Length of the Pendulum to the Time of a leaſt Oſcillation of that Pendulum* which is the very ſame Analogy as we found from the

the *Cycloid* (1124). Wherefore the Time of Oſcillation in a *Cycloid*, and in an *indefinitely ſmall Arch of a Circle* is the ſame; *viz.* T = 1 *Second*, when the Length of the Pendulum is 39,2 Inches.

3363. Therefore the Time of an Oſcillation or Vibration in a circular Arch in general, is $T \times \frac{IL}{IK}$, or becauſe $IL = IK + KL$, therefore the general Expreſſion of the Time of Vibration through any Arch A D C will be $T + T \times \frac{KL}{IK}$.

3364. But the Exceſs $T \times \frac{KL}{IK}$ of the Time of Vibration in a circular Arch above the Time in the Arch of a *Cycloid*, or the *leaſt of all*, is next to be transformed and fitted for Computation. In order to this put the *verſed Sine* of the Arch of Vibration $BD = x$; then will $KL = \frac{1}{4}x$ (3359) $2L - \frac{1}{2}x = IK$ ($L = ND$, being the Length of the Pendulum). And $T \times \frac{KL}{IK} = \frac{\frac{1}{4}Tx}{2L - \frac{1}{2}x} = \frac{Tx}{8L - 2x}$, the Exceſs or Error above T, the Time of Oſcillation in the *Cycloid.*

3365. Let the Time be expreſſed in *Seconds*; and let $S = 3600''$ the Seconds in an *Hour*, or $86400''$ the Seconds in a Day or 24 Hours. Then $\frac{TSx}{8L - 2x} = r$, the Seconds loſt in an Hour or a Day, by the Pendulum vibrating in a *circular Arch* of a given verſed Sine x. For it is plain, that as $\frac{Tx}{8L - 2x}$ is the Increaſe of T the Time of Vibration in a Cycloid, ſo $\frac{TSx}{8L - 2x}$ will repreſent the Dimunition of the Number of Seconds S in a Day or an Hour; or $S - \frac{TSx}{8L - 2x}$ = Number of Vibrations in the ſame Time in the *Arch of a Circle.*

3366. If the Quantity (r) be given, we have the verſed Sine of the Arch of Vibration $x = \frac{8Lr}{TS - 2r}$. But in all theſe Theorems, the Value of x is found in Parts of L; and ſo muſt be reduced to the *tabular verſed Sine* by this Analogy as $L : x ::$

R:

$R : \frac{Rx}{L}$; therefore putting $B = 1$, we have $\frac{Rx}{L} = \frac{\frac{1}{2}TS - r}{4r}$ = tabular verſed Sine of the Angle required.

3367. That the young *Automatiſt* may ſee the Uſe of this Doctrine, I ſhall illuſtrate this Theorem by an Example. Therefore let it be required to find how many Vibrations are made in a *circular Arch* of 120 Degrees, by a Pendulum whoſe Length $L = 39,2$, and which vibrates 86400 Times in one Day, in the Arch of a Cycloid. Here $L = 39,2$, $T = 1''$, $S = 86400''$, and $x = \frac{1}{2}L = 19,6$, being the verſed Sine of 60 Degrees, then $\frac{TSx}{8L - 2x} = 6278''$, which deducted from 86400, leaves 80122, the Number of Swings required. Whence ſuch a Clock will looſe daily $6278'' = 1$ Hour : $44' : 24''$.

3368. Again let it be required to find in what Arch of a Circle a *ſecond Pendulum* muſt vibrate to looſe one Minue, or $60''$ *per* Day. Then $r = 60''$, $T = 1''$, and $\frac{1}{2}TS = 43200$. And therefore $\frac{4r}{\frac{1}{2}TS - r} = 0,0028367$, the tabular *verſed Sine* of $4° : 19'$; the Arch of Vibration therefore is $8° : 38'$ required. In like Manner it is found, that the *verſed Sine* of half the Arch in which ſuch a Pendulum will looſe but *one Second per* Day, or in 24 Hours, is 0,0000896, correſponding to $46'$ of a Degree, the Arch therefore is $1° : 32'$; which is equal to *one Inch* in Length.

3369. Hence it appears, that in *Second-pendulum* Clocks, if the Arch of Vibration does not exceed two or three Inches, the Error will be very ſmall, and may be eaſily corrected, either in the ſimple or compound Form, by the common Methods. But it is otherwiſe with *Table-clocks*, whoſe Pendulums are ſhort, and the Arch of Vibration long. Thus for Example, let the Pendulum of ſuch a Clock be 9,8 Inches, to ſwing *Half-ſeconds*. And let $z = \frac{4r}{\frac{1}{2}TS - r}$, then $\frac{\frac{1}{2}TSz}{4 + z} = r$, the Error of the Clock in $\frac{1}{2}$ Seconds *per* Day. Therefore $T = 1$, $S = 172800$, and $z = 0,060307$, the tabular verſed Sine of 20 Degrees; and $\frac{\frac{1}{2}TSz}{4 + z} = 1283$ Half-ſeconds, or $10' : 40''$ is the

Error of ſuch a Clock *per Day*; which is ſtill much greater when the Pendulum is but 6 or 7 Inches long, as in moſt of thoſe Clocks, and therefore no great Exactneſs in them can be expected.

N. B. The Arch of Vibration in Table-clocks is conſiderably more than 20 Degrees, and of Courſe the Error greater on that Account.

CHAP. XIV.

The phyſical THEORY *of the* FIGURE *of the* EARTH; *the ſame demonſtrated to be a* SPHEROID; *and from thence a* SOURCE *of* ERROR *in the* MOTION *of* PENDULUMS.

3370. BEſides the Oſcillation of Pendulums in *circular Arches* inſtead of *cycloidal Ones*, there are other Sources of Error, which are unavoidable, as they are founded in the very Conſtitution of the Earth itſelf, and its Appendages. That which ariſes from the Earth itſelf is *the different Force of Gravity on different Parts of the Earth's Surface*. For ſince the Velocity in a given Time is always as the accelerating Force (998); therefore, in ſuch Latitudes, where the Power of Gravity is ſtrongeſt, there the Velocity of the Pendulum will be quickeſt, and *vice verſa*; conſequently the ſame Length of a Pendulum will not perform its Vibrations in the ſame Time in different Regions of the Earth.

3371. This variable Power of Gravity ariſes from two Cauſes: (1.) The ſpheroidical Form of the Earth, by which Means a Body on different Parts of the Surface is not equidiſtant from the Center. (2.) The *centrifugal Force* ariſing from the Rotation of the Earth upon it Axis, by which the Force of Gravity is very unequally diminiſhed from the Equator to the Poles. A Pendulum, therefore, upon different Parts of the Earth will be vari-

ouſly

ously influenced by this Power, and consequently can never vibrate equally but in the same Place.

3372. We shall lay open these Sources of Error in the Motion of Pendulums by tracing them from their first Principles, and thereby convince the young *Automatist* that they necessarily arise from the natural Constitution and Disposition of the Earth, taking it for granted, that the Earth in its first Formation, and at the Commencement of its Motion, was in a fluid State, or at least so far, that its Parts could yield to the Force impressed upon them by that Motion, according to the *Laws of Nature* (964, *&c.*)

3373. Therefore let N S (Fig. 5.) be the Earth's Axis, and E Q the Diameter of the Equator, in the Earth considered as an *Ellipsoid*. And let B C be a Column of fluid or yielding Particles gravitating to the Center C, which (because of an *Equilibrium* between all the Parts) must be of equal Weight with any other Column of Particles C N or C E. Put $CE = a$, $CN = b$, $CB = x$, and $AB = s =$ Sine of the Angle B C N; and lastly, let the Power of Gravity (g) be every where as the (n) Power of the Distance from the Center C.

3374. Therefore the Gravity at E will be to that at B, as CE^n to CB^n; and therefore $\dfrac{g \times CB^n}{CE^n} = \dfrac{gx^n}{a^n} =$ the Power of Gravity at B.

3375. Again, the centrifugal Force (f) at E is to that at B, as CE to AB (1177). But $AB : CB :: s : 1 =$ Radius, whence $AB = s \times CB = sx$; therefore $CE : AB :: a : sx :: f : \dfrac{fsx}{a} =$ the *centrifugal Force* at B.

3376. But since this Force acts in the Direction A D, let its absolute Quantity at B be denoted by $BD = \dfrac{fsx}{a}$; this is resolvable into two other Forces F D and F B, of which the latter is all that Part which *opposes Gravity* in the Direction B C; whence, since $BD : BF :: 1 : s :: \dfrac{fsx}{a} : \dfrac{fs^2x}{a} =$ the centrifugal Force at B, by which Gravity is there diminished.

3377. Hence the Power of Gravity on a Particle of Matter $(\dot{x})$ at B will be $\frac{g\,x^n}{a^n} - \frac{f\,s^2\,x}{a}$, impelling it towards the Center C; therefore its *Momentum* or Weight will be $\frac{g\,x^n\,\dot{x}}{a^n} - \frac{f\,s^2\,x\,\dot{x}}{a}$ (1072); but this is the Fluxion of the Weight of the whole Column of Particles B C, and therefore the Fluent $\frac{g\,x^{n+1}}{\overline{n+1} \times a^n} - \frac{f\,s^2\,x^2}{2\,a}$ will be the Weight of the said Column of Particles B C.

3378. But this is likewise the Weight of every other Column of Particles (3373), and therefore of the Column C E; and because in this Case the Angle N C E is a right One, we have $x = a$, and $s = 1$; wherefore the Weight of the Column C E will be $\frac{g\,a^{n+1}}{\overline{n+1} \times a^n} - \frac{f\,a}{2} = \frac{2\,g - nf - f}{2 \times \overline{n+1}} \times a = \frac{g\,x^{n+1}}{\overline{n+1} \times a^n} - \frac{f\,s^2\,x^2}{2\,a}$; this Equation by proper Reduction, and putting $s = 0$, will give $\overline{2\,g - nf - f} \times a^{n+1} = 2\,g\,x^{n+1}$; and thence we get $a : x :: 2\,g^{\frac{1}{n+1}} : \overline{2\,g - nf - f}^{\frac{1}{n+1}} ::$ CE : CN :: *so is the* DIAMETER *of the* EQUATOR *to the Earth's* AXIS.

3379. If Gravity be supposed *uniform*, then $n = 0$, and then $2\,g : 2\,g - f ::$ CE : CN. But $g : f :: 294 : 1$, (1198); therefore CE : CN :: 588 : 587, in such a Case.

3380. If Gravity were *proportional to the Distance from the Center* C, then $n = 1$, and we should have $g^{\frac{1}{2}} : g - f^{\frac{1}{2}} ::$ CE : CN.

3381. But in the present Constitution of Nature, it is well known that $n = -2$ (1230); therefore $2\,g^{\frac{1}{-1}} : \overline{2\,g + f}^{\frac{1}{-1}} :: 2\,g + f : 2\,g :: 589 : 588 ::$ CE : CN.

3382. What has been hitherto said, shews the Figure of the Earth is not *spherical*, but must be that of a SPHEROID, flatted at the Poles. And when it is said, Gravity is inversely as the Square of the Distance from the Center (or as x^{-2}) it is to be understood that (x) is the Distance from the *Center of Curvature*, which is the true *Center of Attraction* to a Particle on the Surface of

of the Spheriod, and not the *Center of the Spheriod* itself, as is the Case of a Globe or Sphere.

3383. Now the *Radius of Curvature* at the Equator E is $\frac{2\,CN}{CE^2}$, and at the Pole N is $\frac{2\,CE}{CN^2}$ (933, 935.) but the inverse Ratio of the Squares of these Quantities is CN^6 to CE^6; and therefore Gravity at E and N is not inversely as the *Square*, as the sixth Power of C E and C N.

3384. Let D = Length of a Degree at the *Pole*, and d = a Degree at the *Equator*; now it is plain those Degrees will be as the Radii (R, r,) of Curvature in those Places, *viz.* $D : d :: R : r :: \frac{2\,CE}{CN^2} : \frac{2\,CN}{CE^2} :: CE^3\ CN^3$. Consequently $\sqrt[3]{D} : \sqrt[3]{d} :: CE : CN$.

3385. Therefore if a Degree could be accurately measured at the Equator E, and at the Pole N, the Ratio of the *Diameter* of the Earth at the Equator to that of its *Axis* would be known. Indeed, if two Degrees be measured in any two distant Parts, that Ratio is equally known from them. But a Degree has been measured at the Equator, and found to contain $56767\frac{3}{4}$ Toises or *French* Fathoms.* Also, a Degree has been measured under the arctic Circle, and was found to be 57438; lastly, a Degree in Latitude 45 was measured, and found 57050 Toises. From any two of these, the Ratio of C E to C N, may be found; and from a Mean of them all, it appears that $CE : CN :: 266 : 264\frac{7}{10}$.

3386. From a View of this Theory, it is plain, that there is a Diminution of Gravity arising from two Causes, *viz.* the *centrifugal Force*, and the *spheroidical Figure* of the Earth. Let C B be produced to G, so that B G may represent the centrifugal Force at the Equator, and B D that in the Latitude B. And draw G D parallel to C N, then it is $GB : BD :: CE : AB$. And $BD : BF :: (CB : AB ::)\ r : s$. Whence $BD = \frac{GB \times AB}{CE} = \frac{BF \times r}{s}$; whence $GB : BF :: r \times CE : AB \times s ::$ so is the *whole Diminution of Gravity under the Equator at* E *to the Diminution at the Latitude* B.

3387.

* See my new PRINCIPLES of GEOGRAPHY and NAVIGATION.

3387. As the *Diminution of Gravity* decreaſes from the Equator to the Pole, ſo there muſt be an *Increment of Gravity* conſtantly attending the ſame, which will be repreſented by $BG - BF = GF$, becauſe GB is the whole Diminution of Gravity at the Equator E. But becauſe $BG : GD :: GD : GF$ (660) we have $BG : GF\ BG^2 : GD^2$ (672) $:: BC^2 : AC^2$; that is, *the Decrement of Gravity at the Equator is to the Increment thereof at the Latitude* B, *as the Square of Radius to the Square of the Sine of the Latitude.*

3388. And therefore, becauſe BG and BC, are conſtant Quantities, the Increment of Gravity will ever be proportioned to the *Square of the Sine of the Latitude in the Sphere*; and in the *Spheroid* of the Earth, it will be very nearly the ſame, as AC will differ inſenſibly from the Sine of the Latitude. Therefore ſince by Menſuration, it appears that CE is to CN, as 266 to 264,7 (3385) which is very near the Ratio of 230 to 229, as determined by Sir ISAAC NEWTON; it will follow, that the Ratio of the centrifugal Force at the Equator is to Gravity as 1 to 230, or that the Decrement of Gravity at the Equator is $\frac{1}{230}$ Part of the Whole.

3389. Therefore if we put this Diminution of Gravity = 10000; then will the Gravities at the *Equator* E, *London* B, and the *Pole* N, be as 2290000, 2296124, and 2300000. And conſequently the Lengths of Pendulums, vibrating in *equal Times* at thoſe Places, will be reſpectively proportional to thoſe Numbers. For the Times of Vibration in Pendulums are in a given Ratio to the Times of Deſcent through half their Length (1124) and, of Courſe, to the Times of Deſcent through their whole Length. But if a Body deſcends through different Spaces in the *ſame Time*, the Forces impelling it will be as thoſe Spaces (999); conſequently, *the Lengths of Pendulums will increaſe with the accelerating Forces of Gravity* from the Equator to the Pole, for the Times of Oſcillation to be *equal.*

3390. For Example; it will be 2296124 : 2290000 :: 39,2 :: 39,1 = the Length of a Pendulum vibrating *Seconds* at the Equator; again, 2290000 : 2300000 :: 39,1 : 39,27 = the Length of a ſecond Pendulum at the Poles. And this Proportion in the increaſing Lengths of Pendulums has been verified by Experiments made in all Parts *from the Equator to the polar Circle.*

3391.

3391. Mr. MAUPERTUIS (in his TREATISE of the FIGURE of the EARTH) has given us a Table of the Length of an isochronal Pendulum for every 5th Degree of Latitude, as deduced from their Measures of a Degree in different Latitudes, and from a great Number of Experiments on Pendulums in all Parts from the *Equator* to the *polar Circle*; these I have reduced to *English Measure* for the Benefit of such who do not understand those of the *French*. If the Pendulum, which, at the Equator, is exactly adjusted by the Motion of a fixed Star, be carried to any other Latitude, it will be accelerated, or gain upon the mean Time (3389). And the Quantity of this Acceleration upon one Revolution of the Stars, is here also shewn to every 5th Degree of Latitude.

3392. TABLE

Of the ACCELERATIONS *of a* CLOCK, *and of the Lengthenings of a* PENDULUM *from the Equator to the Pole.*

LATITUDE of the Place.	ACCELERATION in one Revolution of the fixed Stars.	Length of the PENDULUM.
0°	0″	39,0754
5	1,6	39,0768
10	6,4	39,0811
15	14,3	39,0873
20	24,9	39,0980
25	38,1	39,1097
30	53,3	39,1236
35	70,2	39,1389
40	88,1	39,1551
45	106,6	39,1718
50	125,1	39,1903
51 : 30		39,2
55	143,1	39,2049
60	159,9	39,2202
65	175,1	39,2338
70	188,3	39,2459
75	198,9	39,2554
80	206,8	39,2626
85	211,6	39,2669
90	213,2	39,2684

CHAP. XV.

The EFFECTS *of* HEAT *and* COLD *in varying the* DIMENSIONS *of* BODIES; *and thereby proving another natural* SOURCE *of Error in* PENDULUMS.

3393. THE other *natural Source* or Cauſe of Error in the Time of a Pendulum's Vibration, is the *different Temperature of the Air or Climate*; for it is found by Experience that all Bodies are ſubject to be *expanded* by *Heat*, and contracted by *Cold*, in all their Dimenſions. And therefore if the Pendulum by theſe natural Cauſes be conſtantly varying its Length, the Time of Vibration will be always inconſtant; and ſo, on this Account, can never (without a proper Rectification) be an equable Meaſure of Time.

3394. For ſince the Lengths of Pendulums are inverſely as the Squares of the Numbers of Vibrations in a given Time, (a *Day*, for Inſtance), if we expreſs the Length of a *Second-pendulum* in 100ths of an Inch, it will be 3920; and then ſuppoſe it contracted by Cold $\frac{1}{100}$ of an Inch, its Length will be 3919; and ſince in one Day there are 86400″; we ſhall have, as $\sqrt{3919} : \sqrt{3920} :: 86400'' : 86411''$. Whence it appears, that a Clock will gain or loſe every Day 11 Seconds for every 100th of an Inch it is contracted or lenthened by Cold and Heat. (See Table 3392).

3395. This is by far the moſt general Cauſe of Error in Time-pieces; for the different Power of Gravity affects the Pendulum in this Reſpect only in *different Latitudes or Climates*, but this Effect of *Heat and Cold* equally affects it in every Region, and in any particular Place. And as ſuch minute Variations in the Length of Pendulums are productive of ſuch ſenſible Errors in the Account of Time, they ought to be at all Times accurately known that the Clock might be properly rectified to every variable Degree of Heat and Cold.

3396.

3396. Hence it appears, that a proper Regulation of *Time-Pieces* with Pendulums is closely connected with the Nature of THERMOMETERS, whose sole Use is to indicate, at all Times, the comparative Degrees of Heat and Cold by a proper Scale of Variation. And if by Experiments we can find how much a given Rod of *Iron* or *Brass* extends or contracts with given Differences of Heat and Cold, denoted by the *Thermometer*, then such an Instrument being fixed by the Clock, will always shew what Degree of Correction is necessary at all Times of the Year for keeping the Clock to true Time.

3397. Now by many Experiments made on purpose to determine this Affair,* it appears, that a flat Rod of Brass $\frac{1}{2}$ an Inch wide, and $\frac{1}{12}$ of an Inch thick, and in Length $38\frac{1}{2}$ Inches, will expand or extend in Length just $\frac{1}{50}$ of an Inch with such a Difference of Heat as will cause the *Mercury* in *Farenheit*'s Thermometer to move through $22\frac{1}{2}$ Divisions; therefore $\frac{1}{100}$ of an Inch will correspond to $11\frac{1}{4}$ Divisions; but the Extension of $\frac{1}{100}$ of an Inch will occasion the Clock to lose 11″ *per* Day. (3394.) Consequently each Division of *Farenheit*'s Termometer corresponds to an Extension of $\frac{1}{1000}$ Part of an Inch, in the Rod of the Pendulum, and to *one Second of Time* which the Clock, by that Means, loses *per Day*.

3398. Therefore, if the Clock be adjusted to equal Time, when the *Mercury* stands at 55, (the *Temperate* Degree) in the Thermometer; it will be seen in the said Instrument for any other Time, how much the Clock has lost or gained upon the equal Time, by the Height of the *Mercury* at that Time. For Instance; suppose the *Mercury* stands in a warm Summer's Day at 70, that is 15 Divisions above Temperate; therefore the Pendulum is lengthened $\frac{15}{1000}$ of an Inch, and consequently the Clock will lose 15″ *per Day*, if not properly rectified. On the other Hand, if the *Mercury* stands at 40, (as far below *Temperate*) then it will shew the Pendulum is contracted as much, and the Clock now gains 15″ *per Day*.

3399. Hence it appears that as the greatest Summer Heat in *England* never raises the *Mercury* higher than about 80 Degrees,

* See a Treatise entituled, OBSERVATIONS *Astronomiques & Physiques* par G. JUAN & ANT. de ULLOA. Page 86.

grees, our *Second Pendulum Clocks*, of *brass Rods*, can loose at most but 25″ *per* Day by Extension from Heat. But they may gain 30″ or 40″ in a Day by extreme Cold, contracting the same.

3400. If the Rod of a Second-pendulum be made of *Iron*, it will vary less in Length by *Heat*, and *Cold*, for by the same Experiments it was found, that (*cæteris paribus*) the Extension of *Brass* was to that of *Iron* as 20 to $13\frac{1}{4}$, which is extremely near the Proportion of 3 to 2. And the Extension of *Iron* to that of *Steel* was found to be as $13\frac{1}{4}$ to $12\frac{1}{3}$. It was also found that the Expansion of *Brass* to that of *Copper* was as 20 to $19\frac{1}{4}$.

3401. From what has been said it appears, there is no such Thing in Nature as an *equable Time-piece*; that a Pendulum-Clock is the only Machine which, by Art, can be made to approach near it; and lastly, that such a Clock will stand in constant Need of a proper Adjustment or Rectification by the fixed Stars, or other astronomical Means. To contrive to correct these Errors, as fast as they arise, by some artificial Construction of the Pendulum has been the Endeavour of many Artists, and with some Degree of Success; but none have come to my Knowlege that appears to answer this Purpose so well as the following Method, which I am informed is put in Practice by an ingenious Artist in the North of *England*; and is as follows.

3402. A Bar of the same Metal with the Rod of the Pendulum, and of the same Thickness and Length, is placed against the Back-part of the Clock-case; from the Top of this a Part projects to which the upper Part of the Pendulum is connected by two *fine pliable Chains*, or *silken Strings*, (as in the Cut at Page 374.) which just below pass between two Plates of Brass, whose lower Edges will always terminate the Length of the Pendulum at the upper End. These Plates are supported on a Foot fixed to the Back of the Case. This Bar rests upon an immoveable Base on the lower Part of the Case, and is braced into a proper Groove, which admits of no Motion any Ways but that of Entension and Contraction in Length by Heat and Cold.

3403. Now 'tis evident, that the Extension or Contraction of this Bar and the Rod of the Pendulum, will be equal, and in contrary Directions; and therefore, suppose by Heat the Pendulum is increased $\frac{1}{100}$ of an Inch in Length, below the Edge of the brass Plates or Cheeks, then because the Bar is lengthened

just

juſt ſo much upwards it will raiſe or draw up the Pendulum juſt $\frac{1}{100}$ of an Inch, and thereby make its Length below the Plates ſtill the ſame as before. The caſe is the ſame in regard to Contraction by Cold; for as the Pendulum is thereby ſhortened gradually, it is as gradually lengthened by being let down between the Plates, by the equal Contraction of the Bar behind. Whence it ſhould ſeem that this is a conſtant and adequate Rectification of the Pendulum by which it will always keep true Time.*

3404. If the Clock be of the common Conſtruction, *viz.* with a Pendulum conſiſting only of a Rod and Bob; then we muſt be content with a Method of correcting it as ſoon as its Quantity is diſcovered. Thus, if the Bob be made to reſt or depend on a *Nutt* and *Screw* upon the loweſt Part of the Rod; then if there be 25 Threads to an Inch, and the Nutt be of a circular Form, and its Perimeter divided into 45 equal Parts, it is evident from (3397,) that each of thoſe Parts will correſpond to a ſecond of Time in the Clock, and to a ſingle Diviſion of the Scale of *Farenheit's Thermometer*; and therefore by moving the Nutt one Way or the other, and ſo many Diviſions, as the Thermometer directs, we ſhall be able to correct the Error ſhewn thereby to a great Degree of Exactneſs.

3405. If the Pendulum be of the compound Sort, *viz.* with a Ball and a Corrector, and conſtructed as directed (in 2334, 2335, 2336,) then it will be eaſy for the Artiſt to divide the Rod of the Pendulum into *Minutes* and *Seconds* by the Theorems there given, and conſequently, if by the Revolution of a Star, he finds how much his Clock looſes or gains upon the Mean Time in a Day or ſeveral Days; or if by equal Altitudes of the Sun, he knows at any Time how much the Clock is too faſt or too ſlow, he can very readily adjuſt it to the Mean or true Time. It is not worth while here to take Notice of the Uſe of the Barometer in eſtimating the Errors of the Pendulum ariſing from the *different Weight or Denſity of the Air*, becauſe the Difference of Reſiſtance to the *Bob* is on that Account ſo exceeding ſmall as ſcarcely to come under the niceſt Obſervation, and which we have before provided againſt, (3342, *&c.*)

* The Reader may meet with a larger Account of this Invention in a *French Treatiſe* on Clock-work by Mr. Thiout, with a Print thereof; but as I have had no Opportunity of peruſing that Book, I do not know who was the Author of this Contrivance.

CHAP. XVI.

The THEORY *of the* CIRCULAR PENDULUM *demonstrated, with its peculiar* ADVANTAGES *in* CLOCK-WORK.

3406. WE have now delivered all we think neceſſary concerning the Structure of Clocks and Watches of the ordinary Sort, whoſe Motions are governed and regulated by Springs and Pendulums in the uſual Way. But here we are debarred from the Uſe of a *Second-pendulum Clock* of a portable Form; though from what we have ſaid, it appears, no other Sort of *Automata* can be regarded as *equable Time-keepers*, in any tolerable Degree. But Nature has ſupplied a Method, and Art has diſcovered it, by which a Pendulum may be made very eaſily to *circulate* Seconds in the ſame Length that a common Pendulum *vibrates* Half-ſeconds; which is as follows.

3407. We have formerly ſhewn, that a Weight at the End of a String might be made to move in ſuch a Manner, that the String ſhould deſcribe the *Surface of a Cone*, while the Weight moved through the Circumference of its *Baſe*, the *Vertex* of the Cone being the Point to which the String is fixed [See Fig. to 1193.] Or thus, let D [Fig. 6.] be a Weight at the End of a String D E fixed to the Point E; then if the ſaid Weight be moved in the Circle D *b* F *a* D, the String will deſcribe the Surface of the Cone D E F; and this Motion being continued would conſtitute a *Circular*, or rather *Conical Pendulum*.

3408. Now a Pendulum of this Form that ſhall perform its *Circulations* in a *Second of Time* is required to be but $\frac{1}{4}$ of the Length of a common Pendulum to *vibrate* in a Second, as we ſhall bye-and-bye demonſtrate. The Invention of this Form of a Pendulum the celebrated HUGENIUS claims alſo as his own, and tells us he found it nearly at the ſame Time he invented the common long Pendulum. His Words are, "*Unde aliud quoque horologii commentum deduximus, eodem fere tempore quo prius illud.* ———

3409. I

3409. I have been the more particular in mentioning this Gentleman's Claim to this Invention, because I find it is also claimed by Dr. Hook, who we have before observed was his *Competitor* for the Invention of *Watches*. (3324.) Dr. *Hooke* has not, I think, a very clear Title to the Invention, nor has said any Thing relative to the Theory thereof. Mr. *Hugens* has only given the Theorems on which it depends, but without any Demonstration. So that the *Rationale* of a Clock with a *Circular Pendulum* is yet a Novelty in our Tongue; nor has any foreign Author wrote expresly on this Subject, that I know of. I shall therefore proceed to explain the *Rationale* of such a Clock in as plain and concise a Manner as I can.

3410. Let N A M [Fig. 6.] be a *Parabola* inverted, its Vertex A, and Axis A O. Then suppose a Vessel or Bowl was excavated to the Figure of such a *Parabola*, so that its concave Surface might be that of a *Paraboloid*; if such a Bowl were properly agitated, a heavy globular Body within it might be made to circulate round it in any Part of the Surface as at D or G, and there to describe the Circles D *a* F *b*, or G *c* H *d*, whose Diameters are D F and G H.

3411. The Ball D is supported or sustained in the Circle by an Equilibrium of three different Forces, *viz.* (1.) The Force of *Gravity*, or that of its own *Weight*. (2.) A *Centrifugal Force* which is always a necessary Consequence of circular Motion: And, (3.) The *Re-action* of the Side of the Vessel in any Point D, which is always equal to the *Force* of its *Pressure* against it, in a perpendicular Direction, arising from the other two Forces.

3412. The Circle which the Ball describes being supposed parallel to the Horizon, O C will be perpendicular to its Plane in the Center C; and let D E be drawn perpendicular to the Side of the Vessel in D. Then because *Gravity* acts perpendicularly to the Hooizon, let it be represented by the right Line E C. Again, since the *Centrifugal Force* is in a horizontal Direction always from the Center C, it will be properly represented by the right Line C D; then, lastly, because the Body D is sustained *in Equilibrio* at D, its *Pressure* or *Re-action* directly against the Side at D will be denoted by the right Line

D E.

DE. All this is very evident from the *Mechanical Doctrine* of *Compound Forces*. (1027, *&c.*)

3413. In like Manner, if the Body circulates in any other Part of the paraboloid Veſſel, as at G, the three Powers will there alſo be repreſented by the three Sides of the Triangle IGB; that is, *Gravity* by the Side IB; the *Centrifugal Force* by the Side BG; and the *Reſiſtance* of the Veſſel by the Line GI, perpendicular to the Tangent at G.

3414. As by theſe three Forces the Body is kept in a conſtant *Equilibrium* in every Point of the Circumference it deſcribes, it will be ſuſceptible of any Degree of projectile Force impreſſed upon it by an horizontal Agitation of the Veſſel, and thereby be put into *Circular Motion*, every Way ſimilar to that by a central Attraction (1170,) or to that of a Body annexed to the End of a String, (1192,) and will therefore be ſubject to all the ſame Laws of Motion.

3415. Conſequently, ſince the Body in deſcribing the Circles D *a* F *b*, and G *c* H *d*, has Centrifugal Forces proportional to the Radii CD, and BG, it will deſcribe thoſe Circles in *equal Times*, (1177.) Therefore, *all circular Revolutions in every Part of ſuch a Veſſel are iſochronal, or made in equal Times*.

3416. If a Plane be freely moveable about a Center, and a heavy Body lies upon it, connected by a String to the Center, then as the Weight of the Body is ſuſtained by the Plane it may be conſidered as without Weight, and in Equilibrio. Therefore if the Plane be moved horizontally, it will impreſs a projectile Force on the Body, and it will begin and continue to move round the Center, with a circular Motion. Now this projectile Force being in the Direction of a *Tangent*, the Body will endeavour to proceed in ſuch a Line, but it is conſtantly checked and drawn therefrom into the Periphery of a Circle by the String, which String will therefore be ſtretched with a certain Force by which the Body endeavours to receed directly from the Center; and which is therefore the *Centrifugal Force* generated by the Motion of the Plane.

3417. As the Velocity of the Plane or circular Motion is greater, the Centrifugal Force will encreaſe and that in Proportion to the *Square of the Velocity*, (1175.) So that the Centrifugal Force will begin from Nothing and encreaſe to any Degree; and

and consequently will, in one Degree of Velocity, be equal to the *Weight of the Body*, or the String will, in that Case, be stretched just as much as it would be by the Weight of the Body hanging freely to it at rest.

3418. If, therefore, in this particular Case, a Power of Attraction to the Center, equal to the Weight of the Body, was substituted instead of the String, it would make no Alteration in the Motion of the Body; and hence it appears, *that when a Body is moved in a circular Orbit by a projectile and gravitating Force, there will be a Centrifugal Force produced just equal to the Power of Gravity.*

3419. But, in such a Case, the Velocity of Motion is such as the Body would acquire by descending through half the Radius of the Circle (1187.) Therefore in the Time of that Descent, the Body (with the said Velocity uniformly continued) would describe twice that Space, or a Space equal to the Radius, (993.) *Therefore the Time* (t) *of Descent through* $\frac{1}{2}$ *of the Radius* ($\frac{1}{2}$ R) *is to the periodical Time* (T) *in the Circle, as Radius* (R) *is to the Periphery* (P) *of the Circle, or as the Diameter* (D) *is to twice the Periphery* (2 P). That is, $t : T :: D : 2P$.

3420. Now in the paraboloid Vessel, since CE, or BI, is a constant Quantity, being ever equal to half the *Latus Rectum* of the Parabola (747,) there will be one Part where the Centrifugal Force or Radius of the Circle BG will be equal to Gravity BI. And it is evident this must be when the Diameter of the Circle GH passes through the *Focus* B of the Parabola, because then $BG = BI = \frac{1}{2} GH$, the *Latus Rectum.*

3421. By the Nature of the *Parabola* we have AB equal $\frac{1}{2} BI = \frac{1}{2} BG$ (742.) Therefore, the Time of describing the Circle GcHd (or any other) will be to the Time of Descent through $\frac{1}{2} BI = AB$, as 2 P is to D. (3419.)

3422. Bisect BI in Q, then $AB = BQ$; and let AB be the Diameter of the generating Circle of the Cycloid KAI. Now we have shewn that the Time (T) in which a Pendulum QA vibrates through the Cycloid KL is to the Time (t) of Descent through half its Length AB as the Periphery of a Circle (P) to its Diameter (D,) (1124.) Whence we have $t = \frac{\mathcal{T} D}{P} = \frac{T D}{2 P}$, which gives $2 P \mathcal{T} = T P$, or $T : \mathcal{T} :: 2P : P ::$

2 : 1;

2 : 1; consequently, *the Time of revolving in any Circle in the* Paraboloid *is double the Time of Vibration in the* Cycloid, *in a Pendulum whose Length* AQ *is* $\frac{1}{2}$ *the Latus Rectum of the* Parabola.

3423. It remains now to shew how a Pendulum may be constructed so that it may always describe a *Conical Surface*, and its Ball perform its Gyrations in a *parabolical Superficies*. To this End let KH* be a Verge or Axis perpendicular to the Horizon with a Pinion at K moved by the last Wheel in the Train of the Clock; at H it has a hardened steel Point in a Pivot of Agate, to render the Motion as free as possible.

3424. Let it be proposed that the Pendulum shall perform each Revolution in a *Second* of Time; then it is plain, the Paraboloid Superficies it moves in, must be such whose *Latus Rectum* is double the Length of a Pendulum vibrating *Half-Seconds* in a Cycloid, (3422.) Let O be the Focus of the Parabola MEC, and MC the *Latus Rectum*; and make $AE = MO = \frac{1}{2}MC$ = the Length of a common *Half-second* Pendulum.

3425. At the Point A of the Verge, let a thin Plate AB be fixed at one End, and at the other End B let it be fastened to a Bar or Arm DB, standing out from the Verge at right Angles and to which it is fixed at D. This *Lamina* or Plate AB is the *Semi-cubical Parabola* or *Evolute* of the given *Parabola* MEC, such as described (921 to 926.)

3426. The Equation of this *Cubical Parabola* AB was $\frac{27}{16}pxx = y^3$. Let $\frac{27}{16}p = P$, then $Pxx = y^3$ and in the Focus, $P = 2y$; in that Case, $2xx = y^2 = \frac{1}{4}P^2$; therefore $x^2 = \frac{1}{8}P^2$, and $x = P\sqrt{\frac{1}{8}} = \frac{27}{16}p\sqrt{\frac{1}{8}}$ = the Distance of the Focus from the Vertex A. By assuming the Value of x (or putting $x = \frac{1}{10}$ of an Inch) you will find all the correspondent y's or Ordinates of the Curve AB, by which it may very easily be drawn.

3427. If the Pendulum is to make its Gyrations in $\frac{1}{2}$ Seconds, then the Parameter is MC = 4,9 Inches, or 49 Tenths. $\frac{27}{16}p = P = 82,7$ Tenths, and by assuming $x =$ to the Number in the first Column of the following Table, we shall have $y =$ to those in the second respectively.

Abscissæ

* Fig. 1. in the PLATE of the THEORY of *Circulating* PENDULUMS, &c.

Abscissæ		Ordinates.	*Abscissæ*		Ordinates.
0,05	——	0,274	1,4	——	2,531
0,1	——	0,435	1,7	——	2,880
0,2	——	0,692	2,0	——	3,213
0,3	——	0,906	2,3	——	3,526
0,4	——	1,098	2,6	——	3,823
0,5	——	1,274	2,9	——	4,113
0,6	——	1,438	3,2	——	4,392
0,8	——	1,743	3,6	——	4,748
1,0	——	2,023	4,0	——	5,096
1,2	——	2,284			

3428. The String of the Pendulum muſt be of ſuch a Length that when one End is fixed at B it may lie over the Plate A B, and then from A hang perpendicular with the Center of its Bob in the Point E (or Vertex of the Parabola M E C) when at reſt. Then the Verge K H being put into Motion, the Ball of the Pendulum will begin to gyrate, and thereby conceive a *Centrifugal Force* which will carry it out from the Axis to ſome Point F, where it will circulate *Seconds* or *Half-Seconds*, according as the Line A E is 9,8 Inches, or $2\frac{1}{4}$ only; and A B anſwerable to it.

3429. Hugenius tells us, many Clocks of this Conſtruction were made, and with Succeſs; but that they prevailed not ſo much as the common Sort, on Account of their not being ſo *eaſily and expeditiouſly made* ——— That they depend on a Principle of Equable Motion, which the long Pendulum in a Circle has not perfectly ——— That the Index ſhewing Seconds moves with a moſt regular and uniform Motion, and not *(Subſultim)* by *Jerks* and *Stops* as in common Clocks ——— That this Pendulum is *entirely ſilent*, or without that conſtant and melancholy *Click Clack*, which neceſſarily attends the long Pendulum. ——— Moreover, it may be obſerved that the Pendulum to *circulate Seconds* is but a *fourth Part* of the Length to *vibrate Seconds*, or but juſt the Length of a *common Half-Second Pendulum*. ——— And, laſtly, a Contrivance may be added to ſtop the Pendulum in any Part of the Circumference, and thereby render it capable of ſhewing *Thirds*.

3430. This Conſtruction of a *conical Pendulum* is undoubtedly the beſt Form for a *Temporary* Chronometer, for meaſuring

 the

the ſmalleſt Parts of Time occaſionally. But the Part A B ſhould be divided into two from the Point A, and proceed divaricating to the End B, where the Ends of two fine Chains (uſed in *Watches*) or ſilken Strings are to be fixed, which Strings ſhould alſo unite after they paſs the Point A in the Verge or a little before they come to the Bob F. By this Means its Motion will be rendered more ſteady and certain than by a *ſingle Chain or String*. The Arm D B ſhould alſo be nicely balanced by a Counterpoiſe on the other Side. And an *endleſs Screw* at K will do better than a *Pinion* with Teeth; but this is ſubmitted to the Experience and Ingenuity of the Artificer.

CHAP. XVII.

The THEORY *of Mr.* SULLY'S INVENTION *of a Horological* ROTULA, *inſtead of a* PENDULUM, *for regulating* CLOCK-WORK; *with an* IMPROVEMENT *thereof.*

3431. AS this INVENTION of Mr. SULLY is upon an entire new Principle, found out by Trial and Experiments directed by a natural Sagacity, without any Aſſiſtance from a *Phyſico-mathematical* THEORY, I have Reaſon to think it may prove a *Novelty* equally acceptable, as ſingular, to the *Engliſh Automatiſt*; not only ſo, but the Omiſſion of ſo curious an Improvement in *Clock-work*, muſt certainly be thought a Defect in a Treatiſe wrote profeſſedly on the THEORY thereof.

3432. Mr. HENRY SULLY was an *Engliſh Watch maker*, and lived many Years at PARIS; he wrote, in the *French* Language, a Treatiſe on Clock-work, which he entitled *Regle artificielle de Temps*, printed at *Paris*, *A. D.* 1717. I never heard that this *Artificial Rule of Time*, was ever tranſlated into *Engliſh*, nor have I ſeen the Original; it ſhould ſeem by the Title as if it was a Deſcription of his new Clock; but we are told by *Profeſſor* EULER (in the *Petropol. Comment.* for the Year 1727,) that Mr. *Sully* had publiſhed an Account of his new Clock but a little more

more than a Year before, *viz.* in the Year 1725. But leaving the Date of its first Publication, this learned Professor thought it an Invention worthy of an Explication, which he has accordingly given us under the Title of *A Dissertation on a* certain *new Kind of Tautochronal Curves*; and which I shall here translate and abridge for the Use of the *English* Artists.

3433. SULLY's *Invention* consisted in a Kind of *Rotula* or small Wheel * A B made to oscillate about its Center C (Fig. 2.) by Means of a String with a small Weight C P, playing between two curved Plates C E, C F, fixed to the Surface of the *Rotula*. But what Degree of Curvature was necessary for those Plates, that the Oscillations might be all *isochronal* among themselves, and to the Vibrations of a Pendulum of a given Length, was what *Sully* wanted to know, and could only find by frequent Trials, and that very imperfectly.

3434. But to shew what this particular Species of Curvature is from *Physico-Mathematical Principles*, the Process is as follows. Let C M (Fig. 3.) denote one of the Plates in a Situation not natural, or when the *Rotula* is turned on one Side; and let C B be a vertical Line, parallel to the Direction of the Thread M P touching the Curve in M; then from the Point of Contact M draw M T perpendicular to C B, which will also be perpendicular to the Curve in M. Draw the Right Line C O containing the Angle B C O by which the *Rotula* is carried out of its natural Position. On the Center C with any Radius C B describe the Arch B O, measuring the Angle B C O.

3435. Then it is evident the Weight or Power P at the End of the String will endeavour to restore the Rotula to its natural Situation by a Force which will be as $P \times TM$ (1049.) But as the Application of this Force is continually altering with respect to the *Fulcrum* C, we must find one equal to it to be applied to the Radius C O, and acting normally at the Point O. Produce P M to N in a Right Line drawn horizontally through the Center C. Then because $CN = TM$, the Effect of the Power P to turn the *Rotula* will be the same as if it was applied to the Radius C N in N, and acting normally thereto.

 3436. But

* See Plate entituled "The THEORY of the Circulating PENDULUM, Oscillating ROTULA, &c."

3436. But from the Nature of the *Lever* (1051) we have $CO : CN :: P : p =$ the Weight to be applied at O, whose Force shall be equal to that of P applied at N. But $CN = TM$, therefore $CO : TM :: P : p$, and so $p = \frac{P \times TM}{CO}$; therefore because P and CO are constant, we have p every where proportional to TM.

3437. We are next to consider that the Power (p) acting normally at O, will make CO a Pendulum; and farther, that if the said Power (p) (which is the accelerating Force,) be proportional to the Space passed over, the Oscillations of the Pendulum will be *isochronal* (999.) But the Space to be run over is the Arch OB, or the Angle OCB; therefore in order to produce this *Isochronism* in the Pendulum (and consequently in the *Rotula)* the Affair is reduced to this *Problem, viz. To find a Curve* CME *of such a Property, that having a Right Line* CO *given in Position, and in it, from a given Point* C, *another Right Line* CT *be drawn perpendicular to the Radius of Curvature in any Point* M, *the Part* TM *shall be always proportional to the Angle* TCO.

3438. Now when any Quantities are proportional, their Fluxions have the same Ratio (789.) Therefore let M and m (Fig. 4.) be two Points in the Curve indefinitely near to each other; and draw CM, Cm, and the two Normals MT and mt, occuring in R the Center of Curvature at the Part M. Also let fall the Perpendiculars CT, Ct, on the said Normals. Then is TCp the Fluxion of the Angle TCO; and Tp, the Fluxion of the Normal TM. Now it is plain, that in Regard to the fluxionary Angle TCp, the Side TC is Radius, and Tp, the Tangent. Therefore the Fluxion of the Arch OB, which is Bb will constantly be as Tp, the Fluxionary Tangent of the same.

3439. Let CB be equal to $CT =$ Radius, and as the Radius of a Circle is constant, the Fluxion thereof is nothing, that is $pt = 0$; therefore CT is perpendicular to TM in the Point R, or Center of the Circle of *Osculation* in the Point M.

3440. In this Case, the Angle OCT has its Fluction Bb not only *proportional* but *equal* to the Fluxionary Tangent Tp. For the Fluxion $(\dot{z})$ of any Arch z and the Fluxion of the Tangent $(\dot{t})$ thereof, are expressed by this

Equa-

Equation $\dot{z} = \frac{a^2 \dot{t}}{a^2 + t^2}$ (823.) and therefore when $t = 0$, or the Tangent is in its nascent or fluxionary State, that Equation becomes $\dot{z} = \frac{a^2 \dot{t}}{a^2} = \dot{t}$.

3441. But the Fluxions of *contemporaneous Fluents* being equal, shew those Fluents are also equal, or that TM the Radius of Curvature is ever equal to the Arch of the Circle TO described in the same Time.

3442. Hence the Nature of the Curve CM is manifest, it being no other than the EVOLUTE of a *Circular Arch* OT (Fig. 5.), whose Beginning is at the Point of the Circle O, and is described by the Evolution of that Circle whose common *Radii* are CT = CO. ——— In this Process, we have used a different Method of Demonstration from that of *Professor* EULER, which we think is more natural, clear, and concise.

3443. Hence then it appears that the *Laminæ* or Plates OF, OE, [Fig. 5.] are formed by the Evolution of the Circle ADO equal to the Periphery of the proposed *Rotula*, and to begin from the Point O in the Circumference, and not from the Center C, as in *Sully*'s Construction. In the same Point O likewise the String OP is to be fixed, which by its Weight P will move between the Plates OE, OF, (like a Pendulum between the *Cycloidal Cheeks*) and thereby continue the Motion of the *Rotula*, once begun; and *render the Oscillations isochronal, by a Force ever proportional to the angular Space described in each Oscillation.*

3444. The next thing to be determined, is, the *absolute Time of an Oscillation* in a given *Rotula*; this is done by finding the Length of a *Pendulum* vibrating in a *Cycloid*, *isochronal* to the *Rotula*. The Method for doing this our celebrated *Professor* has rather *pointed out*, than *demonstrated*; we shall therefore here supply the Principles which he assumes as known, and on which this Part of the Theory depends.

3445. To this End we must consider that as in respect to the Endeavour or innate Force of Bodies to descend, it is all one whether we consider the Quantity of Matter as diffused through any Space or all collected into a Point, this Effort to descend being constantly proportioned to the *Mass of Matter into*

the

the Force of Gravity (1000.) So with Regard to an *Oscillatory* or *Angular Motion* of Bodies about a Center or Axis, it matters not how the Mass of Matter is disposed, whether we consider it in the Form of a *Circular Area*, or in that of a *Right Line*; but this latter is most natural, as it is that by which the said *Angular Motion* is described.

3446. If therefore the Radius of the *Rotula* be considered as charged with its whole Weight (Q) we shall find its Force of Resistance to such Angular Motion will be expressed by $Q g n$, (2286,) where $g = \frac{1}{2}$ = Distance of the *Center of Gravity*, and $n = \frac{2}{3}$ = Distance of the *Center of Oscillation*, the Length of Radius being = 1. The whole innate Force, therefore, of the Radius or *Rotula* to resist to a *Circular Motion* is as $\frac{1}{3}$ Q.

3447. But if we consider the *Rotula* in Motion its *active Force* then (called the *Vis viva*) will be as the innate Force multiplied by the Velocity of the *Center of Force*. But this Center of Force, in the Sector of a Circle is distant from the Center of the Circle, $\frac{3}{4}$ of the Radius (1098,) it is therefore the same in the Circular Area, or *Rotula*; therefore if the Velocity of a Point in the Circumference be such as is acquired by the Descent of a heavy Body through the Space S, which will be equal to 2 S (993;) then the Velocity of the Center of Force will be $\frac{3}{2}$ S (for $1 : 2\,S :: \frac{3}{4} : \frac{3}{2}\,S$), consequently the whole Force of the moving *Rotula* will be $\frac{1}{3} Q \times \frac{3}{2} S = \frac{1}{2} QS$, and therefore the Fluxion of its Motion will be $\frac{1}{2} Q\dot{S}$.

3448. But this Fluxion of Motion in the *Rotula* is the Effect of the Action of the Weight P by Means of the String P O upon the Plate O E, Let v = Velocity, (or Space described) in an infinitely small Particle of Time by the Weight P; then its momentary Force is $P v$; and as this produces the momentary Motion $\frac{1}{2}$ Q $\dot{S}$ in the *Rotula*, it must be equal to it, that is, $P \times v = \frac{1}{2} Q\dot{S}$, and so $v = \frac{Q\dot{S}}{2P}$.

3449. This is the Case of the *Rotula* and its Weight P, at the Commencement of an Oscillation, when the Position of the Whole is as shewn in Fig. 6. Where B D T is the *Rotula*, T A one of the Plates, when the Radius of Curvature B A is in a Horizontal Position, A P is the perpendicular Direction of the Weight P; and let fall C B perpendicular to A B. Then let

let the *Rotula* by the Action of the Power P move in the first Moment with the angular Motion T C t, and then will $t\,a$ be the Part of the Plate on which the Power acts, and B a the horizontal Radius of Curvature; the Line of Direction is now $a\,p$, and the Power P has moved from P to p, through the Perpendicular Space P $q = v$.

3450. On the Center C with the Radius C a describe the small Arch $a\,d$, and draw C a, C d; then is A d = P q; and the right-angled Triangles A $a\,d$ and B a C are similar; therefore A d : A a :: B a : B C; hence $Ad = v = \frac{Ba \times Aa}{BC}$:

But A a = T t, whence $\frac{Ba \times Tt}{BC} = v = \frac{Q\dot{s}}{2P}$ (3448;) therefore $2P \times Ba \times Tt = BC \times Q \times \dot{s}$ = (because of B a = BT) $2P \times BT \times Tt$. Therefore $\frac{\dot{s}}{Tt} = \frac{2P \times BT}{Q \times BC}$.

3451. But T t is the Space described by the Point T in the first Particle of Time, and is therefore as the inceptive Velocity $\dot{v}$; consequently $\frac{\dot{s}}{Tt}$ is as $\frac{\dot{s}}{\dot{v}}$; but $\frac{\dot{s}}{\dot{v}} = \dot{T}$, the Fluxion of the *Time of an Oscillation*, since in all Cases of descending Bodies, we have $\frac{S}{V} = T$ (991.)

3452. Having thus determined the Fluxion of the Time of Oscillation in the *Rotula*, we shall readily discover the Fluent, or Time itself, by comparing such an Oscillation with a *Synchronal Vibration* in the *Cycloid*. Thus (Fig. 7.) let A N be the *Semi-cycloid* just equal to the Semi-arch of Oscillation B T, and therein take N n = T t and through N, n, draw P N, $p\,n$, perpendicular to the Axis of the Cycloid, and let A O be the Length of the Pendulum describing the same by Evolution from the *Cycloidal Cheeks*, as taught (1120.)

3453. Then from (n) draw $n\,t$ perpendicular to P N, and the Fluxion of the Time of a Vibration will be as $\frac{tn}{Nn}$ (3451,) which therefore must be equal to the Fluxion of the Time of the Oscillation, *viz.* $\frac{tn}{Nn} = \frac{\dot{s}}{Tt} = \frac{2P \times BT}{Q \times BC}$. But from the Nature

of

of the *Cycloid* (1117 to 1126,) we have $\frac{nt}{Nn} = \frac{AN}{AO}$; therefore because $AN = BT$, it is $\frac{1}{AO} = \frac{2P}{Q \times BC}$ (3450) which gives $Q \times BC = 2P \times AO$, and therefore this Analogy $Q : 2P :: AO : BC$; that is, *The Weight of the* Rotula *is to twice the Weight* P, *as the Length of a Pendulum vibrating in a given Time, is to the Diameter of the* Rotula, *which shall oscillate in the same time.*

3454. For Example, suppose it be required to find the Radius BC of the Rotula that shall oscillate precisely in a *Second of Time.* Then $AO = 392$ Tenths of an Inch; and suppose $Q : P :: 100 : 1$, then $BC = \frac{2P \times AO}{Q} = \frac{784}{100} = 7{,}84$ Tenths of an Inch. Or the Diameter $2\,BC = 1{,}568$ or a little more than $1\frac{1}{2}$ Inch.

3455. It was necessary to suppose the Weight P very small in Comparison of the Weight Q of the Wheel or *Rotula*, that its own *Vis Inertiæ* might be inconsiderable; for where that to be taken into the Account, it must be deducted from the Force which we have all along appropriated to move the *Rotula*; and this would embarrass the Theory, or leave the practical Execution of it imperfect.

3456. Another Caution necessary in a Regulator of this Nature, is, that the Direction of the String below the Cheeks be always *perpendicular*, or that the Weight P has not the least oscillatory Motion in itself, as it would make a Difference in its Force upon the *Rotula*, and induce an Irregularity in its Motion, and Time of its Oscillations. A principal Thing to prevent this is to have the String of a considerable Length below the Plates that the Radius BC of the Wheel may bear but a small Proportion to it.

3457. But as a long String will be inconsistent with the Design of a Clock in a *portable Form*, there is another Remedy for both the forementioned Evils, and which will at the same Time be a great Improvement upon SULLY's Invention. This consists in adding a very large Wheel or *Rotula* ADG [Fig. 8.] to *Sully*'s small *Rotula* BHI; and then as a much greater Force will be required to move the compound *Rotula*, so there will be

a greater

a greater Disparity between its Weight and that of P, and a much less Motion of the String B P because of the very slow Motion of the *Rotula* thus altered.

3458. That the Reason of this may better appear, let T C T (Fig. 5.) be the fluxionary Sector of the *Rotula*, and $Tt = \dot{z}$ the Fluxion of the Periphery, then the Area of the said Sector will be $\frac{1}{2}CT \times \dot{z}$, the Distance of the Center of Gravity $\frac{2}{3}CT$, and that of Oscillation $\frac{3}{4}CT$; then (putting $CT = r$) we have its Force to resist circular Motion (3287,) thus expressed, $Qgn = \frac{1}{2}r\dot{z} \times \frac{2}{3}r \times \frac{3}{4}r$ (1098) $= \frac{1}{4}r^3\dot{z}$. Hence the Force of the whole *Rotula* will be $\frac{1}{4}r^3z$. But it is $1 : C :: r : z ::$ Radius : Circumference; therefore $z = Cr$, and so the Force of any Wheel or *Rotula*, will be $= \frac{1}{4}r^4C$, or as the 4th Power of the Radius.

3459. Therefore if R = Radius of the large additional *Rotula*, the Force of the Whole will be increased in Proportion to R^4, that is, the Length of the *Synchronal* Pendulum will be as

$$AO = \frac{Q \times BC \times R^4}{2P} = AO \times R^4. \ (3453.)$$

3460. Since the Time of a Vibration of the Pendulum A O is 1, we shall have the Time T of a Vibration in the Pendulum $AO \times R^4$, thus $1^2 : T^2 :: AO : AO \times R^4 :: 1 : R^4$, therefore $T^2 = R^4$, and $T = R^2$. So that if $T = 2$, $R = \sqrt{2}$, if $T = 3$, $R = \sqrt{3}$; if $T = 4$, $R = 2$; and so on. Whence it appears how very slow the *Rotula* will move when thus enlarged, and how little the Spring B P will on that Account be liable to be put out of a vertical Position.

3461. Such is the THEORY and CONSTRUCTION of the *Automaton Sullianum* with its improved *Rotula*; and besides this Kind of Regulation, which is entirely new and peculiar to this Piece of Clock-work, there is one Property of it which no other Time-Piece has, and that is, the slow Motion of the *Rotula*, by which the Time of the Oscillations become very long, and consequently by that Means more equable; which induced Mr. EULER * to think a *Time-Piece* constructed in this Manner

* For referring to this Improvement of an additional large *Rotula*, he says,— *Et tandem alium evolvam Casum, qui non contemnendum in* RE NAUTICA *usum mihi præstare videtur.*

might be of Service at Sea in finding the *Difference of Time* or *Longitude*, beyond any other Sort of *Clock-work* yet attempted.

CHAP. XVIII.

The Construction *of an* Angular Pendulum, *of a general* Nature, *and adapted to a* Planetary Time-Piece, *which, in the same* Length, *shall vibrate in any given* Time.

3462. THERE has been, as yet, no other Sort of Pendulum in Use, but such whose *Time of Vibration depends upon its Length only*; for this is the Case of the *Circular*, as well as the *Cycloidal Pendulum*. However the *Physical Theory* leaves us not destitute of a Method of regulating Clock-work, by a *Pendulum*, which shall in *any given Length* vibrate *in any Time required*.

3463. To demonstrate this we need only consider the Nature of a common Pendulum of the most simple Form, *viz.* that of an *uniform Rod or Parallelogram* whose Length suppose $= a$, and Weight $= w$; then the Force of the Rod is $\frac{1}{3} a^2 w = w \times \frac{1}{2} a \times \frac{2}{3} a$. (3286.) Now if the Pendulum be transformed, or divided into two equal Parts, and both these Parts move or oscillate about the same Center as before, then if those Parts be separated from each other to an equal Distance from the Vertical Line, the Center of Gravity (which we will now called x) will approach nearer to the Point of Suspension, and the Center of Oscillation (n) will recede farther from it in the same Proportion, for it will always be $\frac{\frac{1}{3} a^2 w}{x w} = \frac{\frac{1}{3} a^2}{x} = n$; and so $x n = \frac{1}{3} a^2$, a given Quantity. Therefore (n) will be inversely as (x).

3464. Therefore also the Time of a Vibration will be greater, in the same Pendulum of two Parts, as those Parts contain a greater Angle with each other; or strictly speaking, *the Time*

of a Vibration will be in the inverſe duplicate Ratio of the Co-Sine of half that Angle.

3465. To illuſtrate this by an Example; let A C B (Fig. 9.) be a common *Joint-Rule*, whoſe two Parts A C D and B C D are cloſe together; through the Center C of the *Joint*, let a ſmall Hole be drilled, and the Rule ſuſpended thereby on a poliſhed Pin or Wire, to vibrate freely. Let A C $= a$, the Diſtance of the Center of Gravity C G $= \frac{1}{2} a$; and that of the Center of Oſcillation C N $= \frac{2}{3} a$. Alſo let the Weight of the whole Ruler be $= w$. Then (as was ſaid) the Length of an iſochronal ſimple Pendulum will be $\frac{\frac{1}{3} a^2}{\frac{1}{2} a} = n = \frac{2}{3} a$. (1095.)

3466. But now let the Ruler be opened, or the two Legs A C and B C removed from the Perpendicular C D to an equal Diſtance on each Side, *viz.* to H C and I C; and let C K $=$ C L ($= \frac{1}{2}$ A C) $=$ C G; and on the Center C deſcribe the Arch K G L; and draw K L cutting the vertical C D in O. Then is the Point O the new Center of Gravity in the Rule thus opened; and C O $= x$; and the new Diſtance of the Center of Oſcillation will be $\frac{\frac{1}{3} a^2 w}{x w} = \frac{\frac{1}{3} a^2}{x} = n$.

3467. Examples will make this Doctrine plain; ſuppoſe it required to open the Legs of the Ruler to ſuch an Angle that the Time of a Vibration ſhall be juſt equal to that of a ſimple Pendulum whoſe Length is equal to that of the Rule C D; in this Caſe we have $n = a$, therefore $x = \frac{1}{3} a =$ C O; which will give this Analogy, as C O : C K (: : $\frac{1}{3} a : \frac{1}{2} a$) : : 10 : 15 : : Radius : Secant of 48° 12′ $=$ H C D, half the Angle H C I, as required.

3468. For a ſecond Example, ſuppoſe C N $=$ 9,8 Inches or C D $=$ 14,7; then will the Ruler cloſed vibrate *Half-Seconds* preciſely; let it now be required to find what Angle it muſt be opened to, that each Vibration may be performed in *One Second?* In this Caſe it is evident $n =$ 39,2 Inches. Conſequently $\frac{1}{3} a^2 = 39{,}2\, x$; therefore $a^2 = 117{,}6\, x$, conſequently 117,6 : 14,7 : : 14,7 : 1,837 $= x =$ C O. Then C O : C K : : 1,837 : 7,35 : : *Radius* : *Secant* of 75° 32′ $=$ H C D, half the Angle required.

 3469. When

3469. When $x = o$, the Distance of the Point N becomes infinite, and the Time of Vibration also; that is, the two Legs CH and CI are then in a Right Line, and at *Rest*. But this Form of a Parallelogram angular Pendulum will meet with too much Resistance from the Air; and besides, is a *third Part* longer than necessary; therefore that such a Pendulum may have all Advantages it must be in the common Form of a *Rod and Ball* at the End.

3470. The *Rod* and *Bob* of this new Pendulum must therefore be double. The Bob may consist of two Hemispheres or two Plano-convex Lenses, put together by their plain Sides, so as to make a compleat Sphere, or else a double and equally *Convex Lens*. Let AB, AC, (Fig. 10.) be the two Parts of such an angular Pendulum suspended on the Point A, and opened to the Angle BAC, exactly bisected by the vertical Line AD. But when the two Parts are closed, they make the simple Pendulum AE. And here let the Weight of the Rod AE ($=$ Weight of AB + AC) $= b$, and the Weight of the Bob $= c$.

3471. Then when the Pendulum is opened to the Angle BAC, the Center of Gravity is raised from a Point near E to O, in the vertical AD, as appears by drawing the Line BC; then the Center of Oscillation (n) is removed to a greater Distance in Proportion; for putting AE ($=$ AB) $= a$, and AO $= x$; we have $\frac{\frac{1}{3} a^2 b \times a^2 c}{x \times \overline{b + c}} = n$ (1094,) whence $\frac{\frac{1}{3} b + c}{b + c} \times aa = xn$.

3472. Put BO $= y$, then $a^2 = (AB^2 =)\ x^2 + y^2$, and so $\frac{\frac{1}{3} b + c}{b + c} \times \overline{x^2 + y^2} = xn$; therefore it will be every where $x^2 + y^2$ as xn; consequently $y^2 = xn - x^2$ which is an Equation to *a Circle* (819,) and shews the *Locus* of the Bob B is the Periphery of a Circle whose Diameter is $n =$ AD the Length of a Pendulum isochronal to the angular Pendulum BAC.

3473. Hence also it appears that the whole Circle ABDC, (supposed heavy) or any Part BAC (the Points B, C, being equidistant from A), will vibrate in the same Time with a Pendulum equal to the Diameter AD. All which *Hugenius* has demon-

demonstrated from far different Principles, which he himself understood very well.

3474. Now it is manifest the Time of Vibration in this angular Pendulum may be altered at Pleasure, and be made greater or less as occasion requires; and consequently, that it is capable of being adapted to measure Time universally in a Clock from the *Sidereal* (or shortest) Day, to the longest *Lunar One*, and thereby to answer the same Purposes as the universal Pendulum before described in Chap. X. and XI.

3475. In order to this, we are to consider the Value of x, when the Pendulum is closed in A E, is $\frac{2ac + ab}{2 \times \overline{c + b}} = g$, (3027) but it is always equal to $\frac{\frac{1}{3}b + c}{n \times \overline{b + c}} \times aa$ (3471,) therefore $\frac{\frac{1}{3}b + c}{n \times \overline{b + c}} \times a^2 = \frac{2ac + ab}{2 \times \overline{c + b}}$, which will give this Analogy $a : n :: c + \frac{1}{2}b : c + \frac{1}{3}b$. Also by the first Equation it is $a : g :: c + b : c + \frac{1}{2}b$; the same Analogies as before in (3029.)

3476. Let t = Time of the Pendulum's Vibration when accommodated to the *Sidereal Day* of 1436′; and T = Time of its Vibration when adapted to the *Solar Day* of 1440′ (3313.) Then it is $t^2 : T^2 :: t : \frac{T^2}{t} = N$, = the Length of the isochronal Pendulum for the *Solar Day*.

3477. Let $p = \frac{c + \frac{1}{3}b}{c + b}$; then $pa^2 = Nx$; therfore $\frac{pa^2}{x} = N = \frac{T^2}{t}$; therefore $tpa^2 = T^2 x$; but we have $g : x ::$ Radius : Co-Sine of the Angle BAO $:: r : c$; hence $\frac{gC}{r} = x = \frac{tpa^2}{T^2}$; therefore $gCT^2 = rtpa^2$; consequently $\frac{tpa^2}{gT^2} = C$, the Radius (r) being Unity. Hence since t, p, a, g, are given Quantities, *we have* C as $\frac{1}{T^2}$, or the *Co-Sine of the Angle* BAO *as the Square of the Time of Vibration inversely.*

3478. If

3478. If we make $b : c :: 1 : 10$ then we shall have $c + \frac{1}{3}b : c + \frac{1}{2}b :: t : a :: 1436 : 1463{,}4 = a$ (3313) = Length of the Rod A E, when the Pendulum is closed; and the Length as well as Time of Vibration of an isochronal Pendulum are both represented by (t) for the Case of the *Sidereal Day*; and T is $= 1440$.

3479. Again, since in that Case, $c + b : c + \frac{1}{2}b :: a : g$ (2475) $:: 1463{,}4 : 1397 = g$. Whence we have $\frac{tpa^2}{gT^2} = C = 83° 58' = ABO$, whose Complement $BAO = 6° 02'$; therefore $BAC = 12° 04'$, the Angle to which the *Sidereal* Pendulum must be opened to vibrate *Solar Time*, for Use in common Clocks.

3480. Let the Time of Vibration of the Pendulum for the longest *Lunar Day* be $T = 1500$; then the Angle to which the Pendulum must be opened will be $C = \frac{T^2 C}{T^2} = 66° 25'$, whose Complement is 23° 35′; the double of which is 47° 10′ = B A C, the Angle to which the Pendulum must be opened for the *longest Lunar Day*.

3481. By the same Theorem, putting T = to any Number of Minutes in the Day proposed for a given Planet (3323,) you find the Angle to which the Pendulum must be opened to vibrate true Time for that *Planetary Day*. But to save the Reader Trouble, I have here subjoined a Table by which the *Planetary and Lunar Scales* may be added to a Pendulum of this new Form.

′		°	′	′		°	′
1436	——	0	0	1455	——	13	5
1436¼	——	1	31	1460	——	14	40
1436½	——	2	8	1465	——	16	5
1436¾	——	2	37	1470	——	17	24
1437	——	3	3	1475	——	18	36
1438	——	4	17	1480	——	19	42
1439	——	5	14	1485	——	20	45
1440	——	6	2	1490	——	21	45
1445	——	9	2	1495	——	22	41
1450	——	11	15	1500	——	23	35

3482. Ac-

3482. According to this Table the Arch of a Circle may be divided into *Degrees* on one Part, to 23° 35′ on each Side the Vertical Line; and then into 64′ of Time on another Part, contiguous to the Degrees; by which a proper Scale will be constructed for the Pendulum which may thereby be adjusted to the Time or Length of any given *Planetary Day*.

3483. It must be left to the Ingenuity of the Clock-maker to contrive its Application in the best Manner, and to determine the Length of the Pendulum, which is done by taking the Length in *Tenths* of an Inch proportional to the Numbers before made use of for the Lengths of simple and compound Pendulums. Thus the Length of $\frac{1}{2}$ Second Pendulums being 98, you say, as 1444 : 1463,4 :: 98 : 99,32 = a = A B, in Tenths of an Inch; the Length A B is therefore 9,$\frac{93}{100}$ Inches.

3484. Hence it appears how well adapted this *Angular Pendulum* is for measuring *Planetary Days*; and perhaps it may be found in some Respects more useful for that Purpose, than that we have formerly described (Chap. XI.) However, this has a Property which that had not, *viz.* of having the Time of a Vibration protracted or augmented to what Degree you please. For Example; suppose the above Pendulum A B were required to vibrate *Seconds* exactly. Then $T : \mathcal{T} :: 1 : 2$; and $T^2 : \mathcal{T}^2 :: 1 : 4$; therefore $\frac{T^2 C}{\mathcal{T}^2} = C$ (3480,) the Co-sine of the Angle BAO = 76° 54′ in this Case; in that of the uniform Rod or Ruler it was 75° 32′. (3468.)

CHAP. XIX.

The THEORY *of several new* Eliptical *and* Horizontal PENDULUMS *of different* FORMS, *which shall vibrate in any given* TIME.

3485. BESIDES the Methods laid down for constructing a *Universal* or *Planetary Pendulum*, there is yet another, which as it contains some other Properties very singular and

and curious, I imagine the ſpeculative as well as the practical *Horologiſt* will be pleaſed with the following Account of it.

3486. Suppoſe A B an inflexible Line (without Weight;)* and in any Point taken therein, as L, let it be propoſed to affix thereto at right Angles, a heavy Rod or Bar F L G, biſected in L, which being moved *(in latus)* ſideways, ſhall oſcillate in the ſame Time with the ſimple Pendulum A B, the Point A being the Point of Suſpenſion to both. Let m be the generating Point by which the Rod is deſcribed; and join A m. Put A L $= x$, L $m = y$, A $m = d$, and A B $= n$.

3487. Then $dd = x^2 + y^2$; and $d^2 \dot{y} = x^2 \dot{y} + y^2 \dot{y} =$ the Fluxion $\dot{y}$ of the Particles in the Rod F G multiplied by the Square of its Diſtance; the Fluent whereof is $x^2 y + \frac{1}{3} y^3$ which therefore is as the Sum of all the Particles, each multiplied by the Square of its Diſtance from the Point of Suſpenſion A; the Weight of the Rod is as its Length, or half its Length, *viz.* as (y;) therefore $\frac{x^2 y + \frac{1}{3} y^3}{xy} = n = \frac{x^2 + \frac{1}{3} y^2}{x}$ $=$ A B. (1094.)

3488. Hence we have $x^2 + \frac{1}{3} y^2 = nx$; and therefore $\frac{1}{3} yy = nx - x^2$. Now this Equation is evidently that of an Ellipsis (767,) as that of the Angular Pendulum was an Equation to a *Circle* (3472.) Whence it appears the two Extremes F and G of the Rod are in the Perimeter of an *Ellipſis*.

3489. To determine the Species of this Ellipſis, in the Equation $y^2 = 3nx - 3x^2$, let $x =$ A C $= \frac{1}{2} n$, the Semi-Conjugate Diameter, and then $y =$ C E $= \frac{1}{2} a$, the Semi-Tranſverſe, which Values ſubſtituted for x and y in the Equation give $3n^2 = a^2$, whence $a^2 : n^2 :: 3 : 1 :: a : p =$ *Parameter* of the Ellipſis (764.) Therefore $p = 1$; $a = 3 =$ D E; and $n = \sqrt{3} =$ A B; by which the Ellipſis is truly limited or ſpecified.

3490. It may be proper here to obſerve to the young Reader that there is a Diſtinction made (by Mathematicians) of Analytic Problems, *viz.* into *Plane* and *ſolid Problems.* They call that a Plane Problem when the Quantities in the Equation are but of *two Dimenſions* (as a *Plane* has only *Length* and *Breadth*;) thus the

* See Fig. 1. in the Plate entituled, New Pendulums of different Kinds.

the Equation $y^2 = xn - x^2$, is a *Plane one*, because $y^2 + x^2 = nx =$ Rectangle of two Dimensions, n and x. But the Equation we have just now found $yy = 3nx - 3xx$ (or $y^2 + 3x^2 = 3nx$) (2488,) has three Dimensions, (*viz.* 3, n, x;) and is therefore a *solid Problem*, as a *Solid* has *Length*, *Breadth*, and *Thickness*.

3491. The *Plane Problem*, we observed (3473) makes the whole Circle, or any two Points or Parts of it equidistant from A (Fig. 11.) isochronal Pendulums. But the present *solid Problem*, shews that any Line terminated by the Ellipses at Right Angles to the Conjugate A B, and consequently that the Plane of the Ellipsis itself, or any Part of it between one or two right Lines, (normal to the conjugate Axis A B) will all perform their Oscillations in equal Time.

3492. Now this Form of a Pendulum is very well suited for short Vibrations, and may therefore be applied to Clocks of a *portable* or *Table Form*, and in a Size as small as you please. For there are three different Sorts of Pendulums that you may make out of this *Elliptic Plane*.

3493. The *First* is a Pendulum A M N with *horizontal Length* only, oscillating on the Point A; and which therefore has this peculiar Excellence, that its Vibrations are not liable to be disturbed by either the *different Force of Gravity* in different Regions of the Earth, or by different *Degrees of Heat and Cold*, as those of all long Pendulums are. Consequently this Pendulum does not stand in Need of that constant Correction necessary in all that have vertical Length. It is also on this Account better adapted to Sea-use; and is much less liable to be agitated or put out of Order than long Pendulums by the Motion and Tossings of the Ship.

3494. The *Second Pendulum* which this elliptic Plane affords, has the Length of the Conjugate A B, and a Bob O P B, which is a Segment of the Plane on its lowest Part. This Bob has two Peculiarities, *viz.* (1) it has properly no Center of Magnitude or Oscillation which the Clock-maker has any Need to concern himself about, as in common Bobs (3292 to 3296.) Then (2) it has a Form extremely well suited to avoid the Resistance of the Air, as is evident by Inspection. On both these Accounts

 this

this Pendulum (which is both *vertical* and *horizontal*) may claim Preference of all simple Pendulums of the same Length A B.

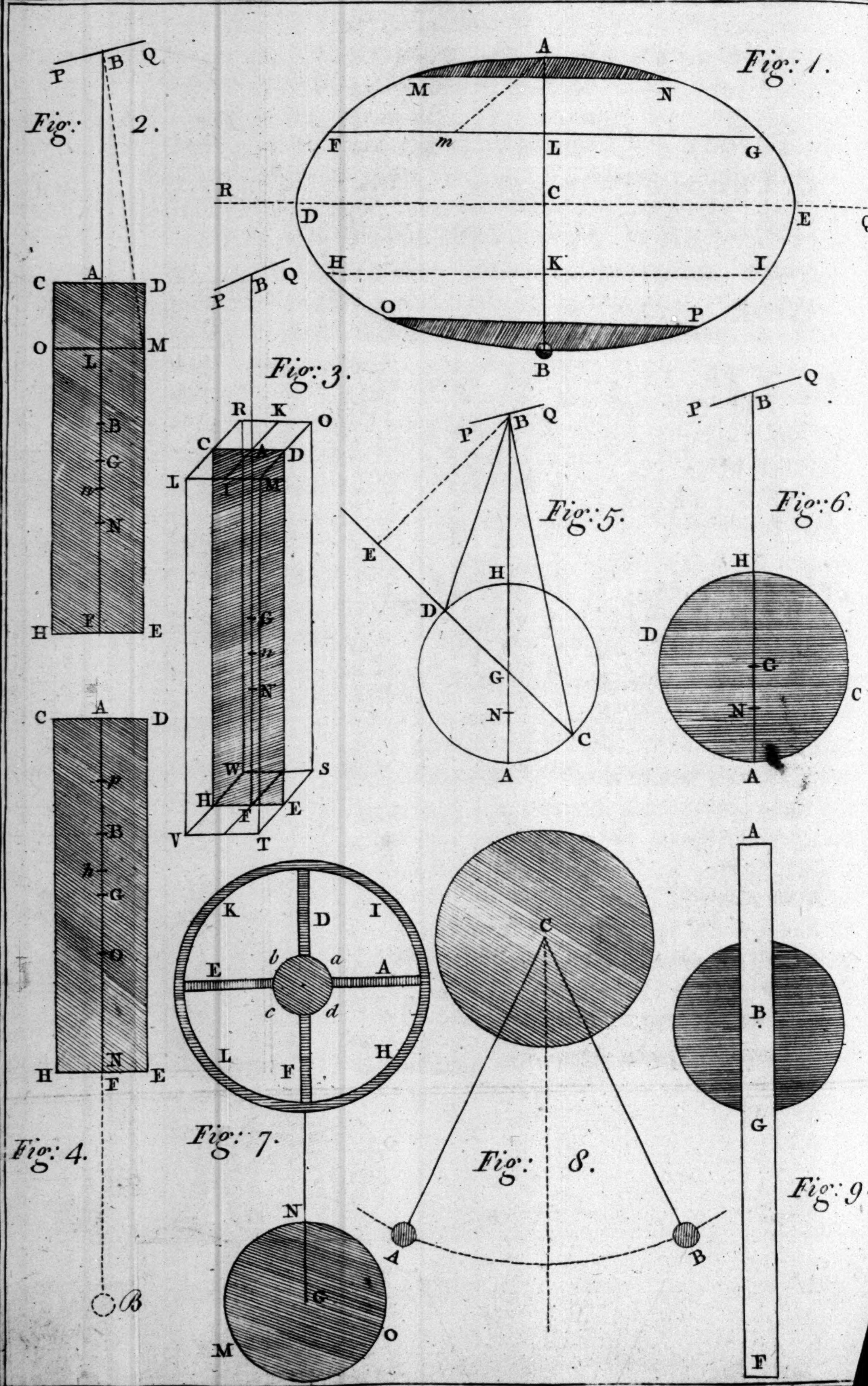
3495. The *Third Pendulum* derived from the Ellipsis is one of the *universal* Kind, or that may be adapted to a Clock for shewing *Planetary Time*. For this Purpose we are to take the longest Line or *Transverse Diameter* D E of the Ellipsis. For it is evident from the Equation (2487,) that while $AC = x$ continues the same, the Length of an isochronal Pendulum (n) will encrease with an Increase of $CE = y$; for Example, if $AB = 98$ Tenths of an Inch, (to swing $\frac{1}{2}$ Seconds,) then $CE = 84,7$; and $AC = 49$. for *Solar Time*; but if it be required to shew the Time for the longest *Lunar Day*, the Time of Vibration must be increased in the Ratio of 1440 to 1500 (3323,) and consequently the Lengths of the isochronal Pendulums in the duplicate Ratio thereof, or as 1440^2 to 1500^2, as shewn (1116,) this will give $AB = 106 = n$, for the said Lunar Day when we get $y = CQ = 91$; then $CQ - CE = EQ = 6,3 =$ the Length of the Part to be added at each End to accommodate the Pendulum for the Extent of the *Lunar Scale*. And thus may the Increase of Length be found for *Mercury*, *Venus*, *Mars*, *&c.*

3496. I have already observed, that a *Fourth Sort of Pendulum* resulting from this Elliptic Equation is the Plane A E B D of the Ellipsis itself; nicely suspended on the Point A; for then vibrating Sideways from C towards D and E, it will be perfectly isochronal to the simple Pendulum, whose Length is A B. And thus much, at present, may suffice for the *Theory of Elliptical and Horizontal Pendulums*, of which we shall make some further Use hereafter.

CHAP.

N. B. *The Reader is desired to correct an Error in Institution 3384, by writing* $\frac{CE^2}{CN}$ *for* $\frac{2CN}{CE^2}$, *and* $\frac{NC^2}{CE}$ *for* $\frac{2CE}{NC^2}$. *Also to erase Instit. 3383 entirely, as being an Over-sight.*

New PENDULUMS of different KINDS.



CHAP. XX.

The Theory *and* Construction *of different Sorts of* Uniform Pendulums, *which, in a* given Length, *shall vibrate in any* given Time; *with several curious Particulars relative to* Clock-Work, *regulated by such* Pendulums.

3497. MR. Graham (ever memorable for his singular Skill and Execution in *Clock and Watch Work*) having tried many Experiments with different Sorts of Metals combined in the Rod of a Pendulum to correct their Irregularities by their different Expansion or Contraction with Heat and Cold, found them all to little or no Purpose; he thought of another Method of effecting the same, which was by Means of a Pendulum consisting of one Part solid and the other Fluid, *viz.* a *Glass Tube filled with Quicksilver.*

3498. This *Mercurial Pendulum*, 'tis easy to understand, would have its Dimensions encreased or contracted each Way from the Center of *Oscillation* by Heat and Cold; for the Tube, as a *Solid*, would encrease a little downwards, and the *Mercury*, being a Fluid, would have its Column encreased upwards from the same Cause; and therefore the Center of Gravity in one being carried down, and in the other raised upwards, 'tis plain, if these Increments different Ways of the solid and fluid Parts could be nicely proportioned as they ought, the Center of Oscillation, would not be in the least affected thereby, but would constantly be at the same Distance from the Point of Suspension, and the Pendulum vibrate, of course, in equal Time.

3499. Accordingly he constructed a Pendulum of this Sort, and after many Trials, succeeded so well, that when he compared one with a Clock which he kept for a *Standard* or *Regulator* (whose Weight of Pendulum was 60 ℔) and which he observed

altered not more than 12 or 14″ in 24 Hours between Winter and Summer, he found by 3 Years and 4 Months constant Observation, that the Irregularity of the Clock with the *Quickfilver Pendulum*, did not, when greateft, exceed a 6th Part of that of the other Clock, and for the moft Part of the Time it did not exceed an 8th or 9th Part. In this Clock, the Tube was Brafs and varnifhed on the Infide.

3500. But as a Pendulum of this Form, to beat Seconds, muft be near 5 Feet long, it will be worth while here to fhew how any *uniform Solid Pendulum* may be conftructed, that, in a *given Length*, fhall vibrate in a *given Time*; efpecially as many curious Particulars will refult from the Demonftration of the Theory, and fuch as we think not a little interefting in a *Treatife* of *Clock-work*.

3501. Let CDEH (Fig. 2.) be a *Rectangle* fufpended on the Point B by a Thread AB, in its middle Point A; produce BA to F; and drawing OLM parallel to CD; put AB $= a$, AL $= x$, LM $= y$, and BM $= d$. Then, fuppofing the Rectangle DH to vibrate *(in latus,* or) Sideways about the Axis PQ, the Force of all the Particles in the Line LM will be $\overline{a+x}^2 \times y + \frac{1}{3}y^3$ (3487) $= a^2y + 2axy + x^2y + \frac{1}{3}y^3$, which multiplied by $\dot{x}$ gives $a^2y\dot{x} + 2ax\dot{x}y + \dot{x}x^2y + \frac{1}{3}y^3\dot{x}$, whofe Fluent $a^2yx + ax^2y + \frac{1}{3}x^3y + \frac{1}{3}y^3x$ is the Force of the Rectangle CM, or of the Rectangle CE, when x or AL becomes AF.

3502. Then let G be the Center of Gravity in the Rectangle, and becaufe the Fluxion of the Weight $\dot{x}y$ multiplied by the Diftance $\overline{a+x}$ is $a\dot{x}y + yx\dot{x}$, the fluxionary Moment of the Rectangle, therefore the Moment itfelf will be the Fluent $axy + \frac{1}{2}yx^2$. Confequently $\dfrac{a^2yx + ayx^2 + \frac{1}{3}x^3y + \frac{1}{3}y^3x}{ayx + \frac{1}{2}yx^2}$ $= \dfrac{a^2 + ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{a + \frac{1}{2}x} =$ Bn, the Diftance of the *Center of Ofcillation* from the Point of Sufpenfion B, when $x =$ AF.

3503. Let $a = 0$, or let the Rectangle be fufpended in its Vertex

Vertex A, then will the Diſtance of the Center of Oſcillation $AN = \frac{\frac{1}{3}x^2 + \frac{1}{3}y^2}{\frac{1}{2}x} = \frac{2}{3} \times \frac{x^2 + y^2}{x} = \frac{2}{3}x + \frac{2}{3}\frac{y^2}{x}$, and when the Breadth of the Rectangle is infinitely ſmall, or $y = 0$, it becomes a Right Line A F, and then $\frac{2}{3}\frac{x^2}{x} = \frac{2}{3}x = \frac{2}{3}AF$, as before ſhewn (1129.)

3504. If the Point of Suſpenſion B be within the Rectangle, then $AB = a$, will be negative, but a^2 will be poſitive; and the Diſtance of the Center of Oſcillation will be $\frac{a^2 - ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{\frac{1}{2}x - a}$.

3505. If $2y = x$, then the *Rectangle* becomes a *Square*; and the Equation for the *Center of Oſcillation* becomes $\frac{a^2 \pm ax + \frac{5}{12}x^2}{\frac{1}{2}x \pm a}$; and if $a = 0$, or the Square be ſuſpended in the vertical Point A, the Theorem is $\frac{\frac{5}{12}x^2}{\frac{1}{2}x} = \frac{5}{6}x =$ Diſtance of the Center of Oſcillation in the Line A F, from the Point A.

3506. If we ſuppoſe x leſs than $2y$, then is the Rectangle to be conſidered as a *Horizontal* one; but the ſame Equation (3504) gives the Diſtance of the Center of Oſcillation in the Line A F, continued out, if required.

3507. If $x = 0$, this Rectangle then becomes an *horizontal Line* $CD = 2y$; and the Equation (3502) becomes $\frac{a^2 + \frac{1}{3}y^2}{a}$, the ſame Expreſſion as was found for the Center of Oſcillation in ſuch a Line before. (3488.)

3508. Let us now ſuppoſe the Plane CDEH (Fig. 3.) to move parallel to itſelf in the Direction of the Line I K, and thereby generate the Solid or Parallelopiped C S in flowing from A to K, or the Solid C T in moving from A to I. Put A I, or A K, $= z$; then the Force of the Rectangle or Plane C E, which is $a^2yx + ax^2y + \frac{1}{3}x^3y + \frac{1}{3}y^3x$ (3501) being multiplied by $\dot{z}$, will be the Fluxion of the Force of the Solid, *viz.* $a^2yx\dot{z} + a^2x^2y\dot{z} + \frac{1}{3}x^3y\dot{z} + \frac{1}{3}y^3x\dot{z}$. And $axy\dot{z} + \frac{1}{2}x^2y\dot{z}$ is the Fluxion of the *Momentum* thereof; therefore the Fluent of the former divided by that the latter, is

$$\frac{a^2xyz + ax^2yz + \frac{1}{3}x^3yz + \frac{1}{3}y^3xz}{axyz + \frac{1}{2}x^2yz} = \frac{a^2 + ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{a + \frac{1}{2}x}$$

the Diſtance of the Center of Oſcillation from the Point B, and is the very ſame as was found for the Rectangle or Plane CE (3502.)

3509. From all which it is evident that the Center of Oſcillation is not affected by the *Thickneſs* of the Parallellopiped, but only by its *Length* AF $= x$, and *Breadth* CD $= 2y$; alſo, that the Increaſe of its Diſtance from the Point of Suſpenſion is always as $\frac{2}{3}\frac{y^2}{x}$, or $\frac{2}{3}$ of a *Third Proportional to its Length and half Breadth* (2503.) Therefore in an *horizontal* Plane, when $y = x$, we have its Diſtance $= \frac{2}{3}x + \frac{2}{3}\frac{y^2}{x} = \frac{4}{3}x$, or $\frac{1}{3}x$ below the Plane in the Line AF continued out.

3510. If $z = 0$, and $y = 0$, the Solid degenerates into a Line or Rod AF, ſuſpended by a String (without Weight) AB $= a$; and then the Diſtance of the Center of Oſcillation will be $\frac{a^2 + ax + \frac{1}{3}x^2}{a + \frac{1}{2}x}$ (3502.)

3511. When the Rod AF is ſuſpended at one End A, then the Diſtances of this Point A and the Center of Oſcillation N from the Center of Gravity G, we called g and d (3288.) And if the ſaid Rod be ſuſpended from the Point B by the inflexible Line AB, put the Diſtance BG $=$ G and the Diſtance Gn $=$ D; then will GD $= gd$, in all Caſes. For $\frac{a^2 + ax + \frac{1}{3}x^2}{a + \frac{1}{2}x}$ $=$ Bn $= N$. Therefore, $\overline{a^2 + ax + \frac{1}{3}x^2} = GN$; and $\frac{a^2 + ax + \frac{1}{3}x^2 - \overline{a + \frac{1}{2}x}^2}{a + \frac{1}{2}x} = \frac{GN - G^2}{G} = D$; whence $GN - G^2 = GD$, but $GN - G^2 = \frac{1}{12}x^2 = \frac{1}{2}x \times \frac{1}{6}x = gd$; therefore GD $= gd$; and conſequently G : g :: d : D.

3512. After the ſame Manner it is proved for the Parallellogram CE; and all other vibrating Bodies, *that the Diſtances of the Points of Suſpenſion and of the Centers of Oſcillation from the Center of Gravity are in an inverſe Ratio*, and are therefore, *mutually interchangeable, or convertible into each other.* Thus for Example, if

if the Rectangle C E was to be suspended upon the Point N (which is the Center of Oscillation when suspended at A) then the Plane will be inverted, and the Center of Oscillation will be in the End or lowest Point A of the middle vertical Line.

3513. Or thus; Let O be the Center of Oscillation in the Plane or uniform Pendulum CE; then $AO = \frac{2}{3}AF$, when the Point of Suspension is in the Vertex A. Then let the Pendulum be suspended on the Point B, so that $AB = \frac{1}{3}AF$ (Fig. 4.) then will the Center of Oscillation be removed from O to the End or Point F in N, and the Distance B N being equal to A O, the Time of Vibration on the Centers A and B will be the same.

3514. If the Point of Suspension be taken nearer to the Center of Gravity G, the Center of Oscillation will go out of the Plane to some distant Point in the Right Line A F produced. Thus if *(b)* be the Point of Suspension, *B* will be the Center of Oscillation. And if $Gb = Gn$ (in Fig. 2.) then will GB here be equal to G B there (according to 3511, 3512.) As it will ever be $AG \times GN = BG \times Gn = Gb \times GB$, and so on.

3515. Hence it is evident, since in the Fraction $\frac{AG \times GN}{Gb} = GB$, the Numerator is constant, we shall have GB as $\frac{1}{Gb}$; and therefore GB will be infinite, when $Gb = 0$, or when the Plane is suspended on the Center of Gravity G. Consequently, *in any given uniform Pendulum* C E *a Point of Suspension (b) may be found or assigned between* B *and* G, *such that the Time of Vibration shall be equal to that of a common Pendulum of any given Length not less than the Minimum of this Sort.*

3516. As the Time of a Vibration in a Pendulum of this Form, is, when a *Maximum*, infinite; it may be proper to determine its Quantity when a *Minimum* or the least it can be. To this End put $BG = x$, and $Gn = y$ (Fig. 2.) and $x + y = p$, an isochronal Pendulum; this, when a *Minimum* gives $\dot{x} + \dot{y} = 0$; and therefore $\dot{x}y = -y\dot{y}$; but since $xy =$ a constant Quantity, (3514,) we have $\dot{x}y + x\dot{y} = 0$; and hence $x\dot{y} - y\dot{y} = 0$, and so we find $x = y$ or $BG = Gn$, when the Vibration is of the least Time possible.

3517. But

3517. But it is always $xy = dg$ (3511,) and therefore in this Case $x = y = \sqrt{gd} = \sqrt{\frac{1}{12}S^2}$ (3287,) therefore $x + y = p = 2x = 2\sqrt{\frac{1}{12}}$, putting $= S = AF = 1$; that is, the whole Length AF is to the Length of the shortest Pendulum (p) as 1 to 0,577.

3518. That this Form of a Pendulum may be adapted to general Use, or to measure any Time from the *Sidereal* to the *Lunar Day*, in a given Length thereof S, Let $\frac{a^2 - aS + \frac{1}{3}S^2 + \frac{1}{3}W^2}{\frac{1}{2}S - a} = p$, the Length of an isochronous Pendulum proper for the Time of the given Day (3504.) (Here $S = x = AF$ the Length, and $W = y = \frac{1}{2}CD$, half the Breadth of the given Pendulum.) Then $a^2 - aS + \frac{1}{3}S^2 + \frac{1}{3}W^2 = \frac{1}{2}pS - ap$; and $a^2 - aS + ap = \frac{1}{2}pS - \frac{1}{3}S^2 - \frac{1}{3}W^2 = s$; and putting $p - S = t$, we have $a^2 - ta = s$; and $a^2 - ta + \frac{1}{4}t^2 = s + \frac{1}{4}tt$; and therefore $a = \sqrt{s + \frac{1}{4}t^2} - \frac{1}{2}t^2$; and thus the Distance of the Point of Suspension from A is found for any Value of (p) from that of a Pendulum of 1436′ for a *Sidereal*, to 1500′ for the longest *Lunar Day* (3323.)

3519. Hence then it appears, (1) That an uniform Pendulum entirely solid, may be made to vibrate in a given Time. (2) That it may be of any given Length and Width. (3) That it may be adapted to regulate Clocks for measuring different Lengths of Days. (4) And that the Errors they are subject to from *Heat and Cold*, or *Difference* of *Gravity* in different Regions of the Earth, are very small, if not altogether inconsiderable; as the Point of Suspension is here at so small a Distance from the Center of Gravity; and the Length of the Pendulum itself so much less than an isochronal One of the common Sort of 58,8 Inches (3298.)

CHAP.

CHAP. XXI.

The THEORY *of* Rotulary PENDULUMS, *constructed in a new* METHOD, *which, in very* short LENGTHS, *shall vibrate in any* given TIME.

3520. WE cannot dismiss the Subject of Pendulums, till it is rendered more compleat by a few Theorems and Observations which yet remain to be demonstrated, as upon them, some considerable Improvements in the Doctrine of Pendulums will appear to depend. The great Lengths of a *Second Pendulum* requiring so large and cumbersome a Clock-case has long been an Objection, which the *Rotulary Pendulum* of Mr. SULLY was designed to remove; but this may be done by Methods more facile and natural, as we shall now proceed to shew.

3521. Though our Methods of shortening the *Second Pendulum* will require the Use of the *Rotula*, yet its Application will be to the *natural*, and not to an *artificial Pendulum*, as *Sully*'s was. And these Methods will admit of as much easier Constructions as they are in themselves more natural and better adapted to Use, especially at SEA, where *Sully*'s Pendulum will be much more affected by the Motion of the Ship, than any of these, which we have now to propose.

3522. The THEORY on which these Methods of reducing the Size of Second-pendulum Clocks are founded, are as follow. Let it be required to find the *Center of Oscillation* of the *Periphery* ADHC of a Circle suspended at B by an inflexible Line BH and vibrating (*in latus*) about an Axis PQ. Suppose C and D two Particles or Weights in the Periphery, and draw the Lines CB, DB, and the Diameter CD, which continue out to E, and let fall the Perpendicular BE upon it; lastly, through the Center G draw the Vertical BHA.

3523. Then (putting $CG = GD = r$, $GB = a$, and $DE = e$,) we have $CB^2 = CG^2 + BG^2 + 2CG \times GE$ (639) $= r^2 + a^2 + 2r \times \overline{r + e}$. Also we have $BD^2 =$

$BG^2 - GD^2 - 2GD \times DE = BG^2 + GD^2 - 2GD \times GE = a^2 + r^2 - 2r \times \overline{r+e}$; therefore $CB^2 + BD^2 = 2a^2 + 2r^2$; or the Sum of the Forces of the *two* Particles C and D is $\overline{a^2 + r^2} \times 2$. Therefore putting P = Periphery of the Circle, the Force of the whole Periphery will be $\overline{a^2 + r^2} \times P$.

3524. In a Circle whose Diameter is 1, the Periphery is $3{,}14159 = p$; then $1 : p :: 2r : 2rp = P$, which substituted for P in the foregoing Expression, gives $2rpa^2 + 2pr^3$, for the Force of the Periphery to resist to angular Motion from the Distance BG. When $a = r$, or the Suspension is at the Vertex H, the Force is, as $r^3 \times 2p$, or as $2rr \times P$. But when $a = 0$, or the Circle is suspended on its Center G, the Force thereof is $r^3 \times 2p$, or $r^2 \times P$.

3525. The *Momentum* of the Periphery is aP; therefore $\frac{\overline{a^2 + r^2} \times P}{aP} = \frac{a^2 + r^2}{a} = BN = C$, the Distance of the *Center of Oscillation*. When $a = r$, then $2r = HA = C$, as we before shewed from other Principles. When $a = 0$, then $\frac{a^2 + r^2}{0} = C$, infinite; or the Circle will in that Case be at rest.

3526. If we consider the Radius AG as a variable Quantity then since the Force of the Periphery is $2pra^2 + 2pr^3$, or $a^2P + Pr^2$; therefore $Pa^2\dot{r} + Pr^2\dot{r}$, or $2pr\dot{r}a^2 + 2pr^3\dot{r}$ will be the Fluxion of the Force of the circular Area ACHD (Fig. 6.) The Force then of that Area will be $Pra^2 + \frac{1}{3}Pr^3$, or $pr^2a^2 + \frac{1}{2}pr^4$. Wherefore when $a = 0$, or the Circular Area or *Rotula* ADHC moves on the Center of Gravity G, the Force is $\frac{1}{2}pr^4$ as we formerly found by another Method. (3458.)

3527. Then because the Area $ACED = \frac{1}{2}rP = rrp$, therefore apr^2 will be the Momentum thereof; consequently $\frac{a^2pr^2 + \frac{1}{2}pr^4}{apr^2} = a + \frac{r^2}{2a} = BN = C$, the Distance of the Center of Oscillation from the Point of Suspension B.

3528. What we have hitherto said is upon Supposition that the Rod BG, by which the Circle is suspended is without Weight

Weight; but if its Weight be considerable, call it (b); then its Force will be $\frac{1}{3}a^2 b$ (3306.) And the Distance of the Center of Oscillation will, in this Case, be $\frac{\frac{1}{3}a^2 b + 2a^2 + r^2}{\frac{1}{2}ab + 2a}$ $= C$. And, if $r = 0$, then $\frac{\frac{1}{3}a^2 b + 2a^2}{\frac{1}{2}ab + 2a} = C$. Which Expression is analogous to that for the Pendulum formerly described (3306.)

3529. We may now proceed to the Application of the Wheel or *Rotula* to the natural Pendulum for reducing Clocks to a *concise and portable Form*, and at the same Time to have the Advantage of a slow Vibration, and, of Course, a much greater Equability of Motion, than common short Pendulums can be expected to have. First, Let a WHEEL be made use of for this End (as in Fig. 7;) then if the Thickness of its Perimeter be inconsiderable, let the Weight thereof be called P, and the Radius $= R$, and we shall have its Force as R^2 P (3524.)

3530. Again, the Force of any heavy uniform Rod S suspended on its Middle Point or Center of Gravity will be as $\frac{1}{12}S^3$ (3286.) Also the Weight of the Rod is as S, and $\frac{1}{2}S = R$, therefore the Force of such a Rod or double Radius to resist angular Motion will be $\frac{1}{3}RRS$; therefore, if the Weight of all the Radii A, D, E, F, &c. be called W, then their Force will be $\frac{1}{3}RRW$.

3531. The central circular Part of the Wheel $(abcd)$ may be considered as a small *Rotula* whose Radius is r, and its Force is $\frac{1}{2}w\,r^2$ (3529) putting $w =$ to its Weight: For the *Weight* is as the *Area*, which is $\frac{1}{2}$ r P (3458.) $= p\,r^2 = w$. Therefore the Sum of the Forces of all the Parts of the Wheel will be $R^2\,P + \frac{1}{3}R^2\,W + \frac{1}{2}w\,r^2$.

3532. And because the Weight of the Pendulum must, in this Case be considered in every Part, and particularly expressed, we will put the Weight of the circular Area MNO $=$ p $=$ $p\,r^2$ (3527.) Whence the Center of Oscillation of the whole Pendulum will be $\frac{\frac{1}{3}a^2 b + a^2\,p + \frac{1}{2}p\,r^2}{\frac{1}{2}ab + a\,p}$.

3533. Therefore by connecting the Wheel and Pendulum

 together,

together, the Center of Oscillation resulting from both will be

$$\frac{R^2 P + \frac{1}{3} R^2 W + \frac{1}{2} w r^2 + \frac{1}{3} a^2 b + a^2 p + \frac{1}{2} p r^2}{\frac{1}{2} a b + a p} = C.$$

3534. We may hereafter see more of the Use of the foregoing Theorem in other Parts of Mechanics; but in the Business of Pendulums the *Rotula* is much preferable to the Wheel and the Theorem is more simple; for putting the Weight of the Rotula $= W$, and its Radius $= R$, then its whole Force of *Inertia* will be $\frac{1}{2} W R^2$ (3530,) and when the Pendulum is connected with it in the best Manner, its Weight will be inconsiderable, therefore $b = 0$; and the Theorem for the Center of Oscillation is $\frac{\frac{1}{2} W R^2 + a^2 p + \frac{1}{2} p r^2}{a p} = C.$

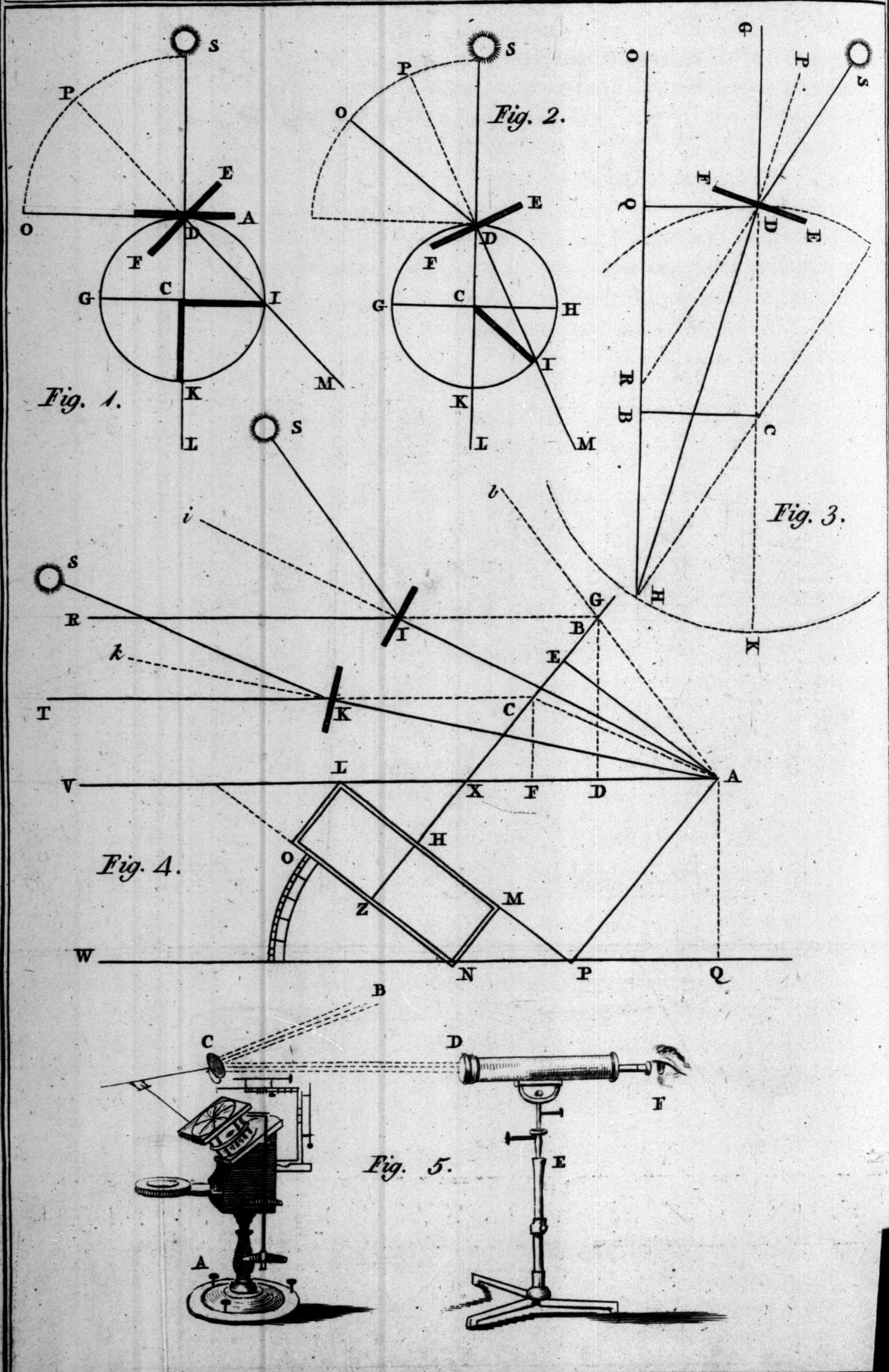
3535. If we put $a^2 + \frac{1}{2} r^2 = n$, we have $\frac{\frac{1}{2} W R^2 + n p}{a p} = C$; and therefore $\frac{1}{2} W R^2 = a p C - n p$; whence this Analogy, $p : W :: \frac{1}{2} R^2 : a C - n$.

3536. If we put $\frac{1}{2} W R^2 + \frac{1}{2} p r^2 = S$, we have $\frac{S + a^2 p}{a p} = C$, therefore $S + a^2 p = a p C$, therefore $a p C - a^2 p = S$, and $a C - a^2 = \frac{S}{P}$; and by compleating the Square, and extracting the Root, we have $a = C - \sqrt{\frac{S}{P} + \frac{1}{4} C^2}$.

3537. In the same Manner may the *Angular Pendulum* (described Chap. XVIII.) be connected with the Rotula, and its Time of Vibration thereby prolonged, or its Length very much shortned for a given Time of Vibration. For $\frac{\frac{1}{2} W R^2 + \frac{1}{3} a a b + a^2 c}{x b + x c} = n$ (3471,) or putting $\frac{1}{3} b + c = s$, and $c + b = v$; we have $\frac{\frac{1}{2} W R^2 + s a^2}{v x} = \frac{T^2}{t}$ (3477.) Whence $\overline{\frac{1}{2} W R^2 + s a^2} \times t = T^2 v x = T^2 v C g$; therefore $\frac{\overline{\frac{1}{2} W R^2 + s a^2} \times t}{T^2 v g} = C =$ Cosine of the Angle to which the Pendulum must be opened, that its Vibration may be as the given Time T (3478, 3479.) This Construction of the Pendulum and Rotula is shewn Fig. 8.

3538. Lastly,

The Principles of CELESTIAL MECHANICS.



3538. Laſtly, the uniform Pendulum, deſcribed in the laſt Chapter, may with Eaſe be adapted to the *Rotula* (as in Fig. 9.) and the Theorem (3504) may be accomodated thereto; for $2yx$ is as the Area or *Weight* of the Pendulum itſelf, whence, putting $2yx = w$, we have $\frac{1}{3}y^2 = \frac{1}{3}x^2 \times \frac{1}{4}w^2$, and putting $1 + \frac{1}{4}w^2 = s$, we ſhall have the Theorem $\frac{\frac{1}{2}WR^2 + a^2 - ax + \frac{1}{3}sx^2}{\frac{1}{2}x - a} = C$, the Length of an iſochronal Pendulum.

3539. And thus it appears how eaſy it is for the expert *Clock-Maker* to chuſe what Lengths and Forms of Pendulums he pleaſes, to vibrate in a given Time; and by this Means to render his Clocks portable and conciſe in any Degree. And further, that by ſuch compendious Conſtructions they are in the beſt Manner fitted for *Uſe at* SEA.

CHAP. XXI.

The PRINCIPLES *of* Celeſtial MECHANICS *explained, and applied to the Conſtruction of a* New HELIOSTATA, *or* Planetary CLOCK *for* fixing *the* RAYS *of* LIGHT *proceeding from the* SUN, MOON, *and* PLANETS, *by which thoſe Objects are rendered apparently at* REST *for* Aſtronomical OBSERVATIONS.

3540. HAVING ſhewn the Conſtruction of a Variety of *Planetary Pendulums*, we now proceed to conſider the Nature, Form, and Diſpoſition of a PLANETARY CLOCK, that is of a much more general Nature than the *Helioſtata* of *S'Graveſand*, which I have at large explained in my *Philoſophica Britannica*; for that is wholly confined to the *Sun*, as its Title implies; and is of a much coarſer, and more cumberſome Conſtruction, than that we have here to propoſe.

3541. It muſt be confeſſed the *Helioſtata* has been greatly improved by the learned C. G. KRATZENSTEIN in what he calls *Mechanicæ Cœleſtis Specimen Primum*, or Firſt Specimen of

Celeſtial

Celestial MECHANICS, in which he has in ſome Meaſure provided for viewing the *Moon*, and *Planets* as well as the *Sun*; but we can by no Means think the Mechaniſm of his Clock, or the Manner of applying it, ſo conciſe and delicate as the Nature of the Subject both require and will admit of; this we preſume will be readily granted by any one who will pleaſe to compare the Account we here give of the Clock of our own Conſtruction with that you find of his in the *Petropolitan* COMENTARIES, for the Years 1747 and 1748.

3542. The *Rationale* he has given of this new Kind of *Celeſtial Mechaniſm* (as he very properly calls it) is exceeding good; and as it only wants to be a little ſoftened and dilated, to ſuit it to an *Engliſh* Genius, we ſhall here undertake that Office, and, we hope, to the Satisfaction of the curious Reader.

3543. The Deſign of ſuch a *Celeſtial* MACHINE being to render the SUN, MOON, and PLANETS, apparently *quieſcent*, or *at Reſt to the View*, it cannot but be eſteemed the moſt uſeful Invention in Mechanics, and beſt adapted for aſtonomical Obſervations; where the conſtant Motion of the celeſtial Bodies and the difficult and irkſome Application of long Teleſcopes create perpetual Moleſtation to the ingenious Student; and gerally, not only diſcourage, but very often entirely defeat the Enterprizes of induſtrious Obſervers.

3544. The firſt Principle of this new Theory of *Celeſtial* MECHANICS is that of *Catoptrics*, where it is ſhewn, *that the Angles contained by the incident and reflected Ray with the Perpendicular Plane are always equal to each other* (1282.) Hence it will follow, that if the incident Ray be conſidered as fixed, and the reflecting Plane be moveable, then the reflected Ray will alſo be moveable and its angular Motion will be juſt double to that of the Plane.

3545. For ſuppoſe the Radiant Object at S, and S D a Ray of Light, incident perpendicularly upon a reflecting Plane A B; with any Radius C D deſcribe the Circle D I K G to touch the Plane in the Point of Incidence D; and let D K be the vertical Diameter, and I G the horizontal one. Then it is evident that S D will repreſent not only the incident Ray, but alſo the reflected Ray as well as the Perpendicular to the Plane, for in this Caſe they all coincide.

3546. Then

3546. Then suppose the reflecting Plane A B moveable about the fixed Point D by an inflexible strait Wire D L, and that it was thus moved about, till it came into the Position E F making an Angle A D E with its first Position, of 45°; lastly, suppose this Motion was produced by an Arm C K moveable on the Center C, and having a Fork at the End K to receive the Wire D L; then it is plain, (1.) That while the Plane is moved from the Situation A B into E F, the Wire D L is moved into the Position D M intersecting the Circle in the Point I, or End of the horizontal Diameter. (2.) The Arm C K has moved into the Position C I. (3.) That because in the Triangle D C I, the Sides D C and C I are always equal, therefore the Angles at D and I will be ever equal; and consequently (4.) The Angle K C I will ever be equal to 2 C D M, or the angular Motion of the Arm C K is twice that of the Wire D L or of the Plane A B. (5.) That the incident Ray S D, the Perpendicular P D, and the reflected Ray D O, will ever be respectively parallel to the three Sides C D, D I, and I C of the Triangle C D I. And that therefore (6.) The angular Motion of the reflected Ray O D will be double that of the Plane or Glass A B.

3547. And *(vice versa)* the same Things are equally evident with regard to the *incident Ray* being considered as *moveable*; and then, if its Motion be just twice as much as that of the Plane A B, the Position of the reflected Ray will be constant, or always the same. Thus for Example, if E F be the Position of the reflecting Plane, and O the Radiant be supposed to move through a Quadrant from O to S, then in that Time the Wire D M will move into the Position D L and the Plane into the Position A B, which angular Motion M D or E D A being but half so great as the angular Motion of the Radiant O D S (or I C K) it is evident, *the reflected Ray* D S *will be always in the same Position, and so may be considered as fixed or immoveable.**

3548. This is the fundamental Principle on which the Theory of the *Machina Cælestis*, as also that of the *Newtonian* SEA-OCTANT (usually called *Hadley*'s *Quadrant)* do entirely depend.

This,

* See Fig. 1 and 2 of the Plate entituled The PRINCIPLES of *Celestial* MECHANICS.

This, with Regard to the ſaid OCTANT, has been already ſhewn in a ſmall Treatiſe on that Subject; † and with reſpect to this CELESTIAL CLOCK, we ſhall now a little expatiate in its Illuſtration.

3549. If the SUN or PLANET were to move or circulate about us in the Horizon, it is evident, ſince the Wire or Ray D M is, in this Caſe, always in the Plane of the Horizon; and perpendicular to the reflecting Plane, therefore the reflecting Plane will always be Vertical to the Plane of the Horizon; and its Perpendicular D P will deſcribe a Quadrant while the Sun or Planet deſcribes a Semi-circle in the Horizon, at the ſame Time that it appears quieſcent or fixed, when viewed in the Reflecter E F, in the Direction S D (3547.)

3550. But if we ſuppoſe the *Sun* or *Planet* to move in a Circle parallel to the Horizon, and at a given Altitude above it, while the Eye continues in the horizontal Plane; then the Motion of the Radiant and Reflecting Plane will be in the ſame Ratio as before, but the Poſition of the ſaid reflecting Plane will now be oblique to the Plane of the Horizon, in order that the reflected Ray may be parallel to it, and continue fixed and immoveable as before.

3551. Thus if H O be the Horizon, (Fig. 3.) and the Angle S R O be that of the Sun's Altitude above it, then if we take any Diſtance R Q for Radius, and on the Point Q erect the Perpendicular Q D, interſecting the Ray S R in D, it will be the Tangent of the Sun's Altitude to the Radius Q R. Through the Point D draw G C parallel to O H; and laſtly, draw P D H to biſect the Angle of Altitude S D G. Then it is evident, if a Plane *Speculum* E F be placed with its Center on the Point D, and in a Poſition perpendicular to the Line P H, it will be that which is required for reflecting every incident Ray S D from every Part of the diurnal Circle of the Sun into one conſtant Direction D G parallel to the Horizon O H.

3552. For continue G D to K, and take D C = D R = Secant of the Altitude, and then C will be the Center of a Circle parallel to that of the Sun's Motion; and drawing C H (= C D) it will repreſent the Arm, which by its Fork at the End will always keep the Wire D H conſtantly biſecting the Angle

† The THEORY of HADLEY'S QUADRANT demonſtrated.

Angle S D G, and thereby render the reflected Ray D G permanently fixed. The above *Phænomena* belong to the *Parallel Sphere*, or to a Person under either Pole.

3553. But to the Inhabitants of an *oblique Sphere* the *Equator* itself and all its *Parallels*, make an Angle with the Horizon; and thereby cause that every Day, and every Part of every Day, the Altitude of the Sun above the Horizon will be variable; and consequently there must, in this Case, be another Figure for the Construction or Disposition of the Speculum for fixing the reflected Ray in the Plane of the Meridian, and parallel to the Horizon.

3554. Thus let A V (Fig. 4.) be an horizontal Line, and the Angle E A V be the Elevation of the Plane of the EQUATOR E A above the same. Also let the Angle E A G be the *North Declination* of the SUN or PLANET, and the Angle E A C the *Declination, South.* Then, of Course, E X A is the Latitude of the Place. Now A E being Radius, E B is the *Tangent* and A B the *Secant* of the North Declination or Elevation of the Planet above the Equator E A; therefore by drawing B R parallel to A V, and therein taking B I = A B, it will give the Point I for the Place of the Center of the *Speculum* (3551,) by which the Sun-beam S I will be reflected into I R parallel to the Horizon by a Wire I A perpendicular thereto, and which continued to *(i)* bisects the Angle of the Sun's Altitude S I R = B A V = 2 B A I.

3555. In like Manner, if through the Point C, a Line C T be drawn parallel to the Horizon A V, and in that we take C K = A C, then K will be the Place of the Speculum to reflect the Sun-beam S K parallel to C A, (in South Declination) into the Direction K T, by the Wire K A, bisecting the Angle C A V = S K T.

3556. From whence it is evident, the Center of the Circle parallel to the Sun or Planet's diurnal Motion, and in which the Speculum is placed, will always be in the Line B X parallel to the Axis of the World; and therefore if the said Line be continued each Way to G and Z indefinitely, it may represent the Axis Z H of a Clock L M N O, continued out; which Clock, being placed in a Position parallel to the Equator A E, if it be furnished with an Arm H P moving on the Center H, and carrying on

its Extremity P an upright Wire or Stem P A with a Fork at A to receive the Wire I A of the Speculum, then by the Mechanifm of the Clock the Index or Arm H P will be carried with a Motion fimilar to that of the Sun or Planet whofe Beam, S I, will thereby be always rendered permanent in the fame horizontal Direction, from the rifing to the fetting of the fame; all which is evident from the preceding Articles.

3557. It is manifeft alfo from the Conftruction of the Figure that the Pofition of the Speculum being conftantly variable, the Diftance thereof from the Point A in the horizontal Line A V, and its Altitude above the faid Line, will be variable likewife, with the different and variable Altitude of the Luminary above the Horizon of the Place; and therefore it is neceffary to calculate the horizontal Diftances and Altitudes of the Speculum from the greateft to the leaft Quantities thereof for any particular Latitude, that it may be readily adapted to any given Declination of the Sun or Planet for the Day it is ufed.

3558. With Refpect to the Firft of thefe, *viz.* the Diftance of the Foot of the Speculum I from the Point A in the horizontal Line A V, it is eafily computed thus; from the Center or Point B let fall the Perpendicular B D; then in the Triangle A B D, the Angle B A D is known, being the Sum of the Co-Latitude (E A D) and the Declination (E A B); and the Side A B (= B I) being the Secant of the given Declination to the Radius A E, from whence A D (the Bafe) is found; which added to B I, gives (B I + D A) the horizontal Diftance required, fuppofe for the greateft Declination propofed E A B. Thus in the Triangle C A F we find A F; and then A F + C K, is the horizontal Diftance for any other Declination E A C on the other Part, or below the Equator.

3559. The *Altitude* of the Speculum above the horizontal Line A V is the Perpendicular B D, for the Declination E A B; and C F for the Declination E A C; which are Parts of the given Triangles; and therefore are readily found for all Declinations. And thus the Speculum is adjufted for ready and conftant Ufe at all Times.

3560. If it be required to conftruct this *Celeftial* CLOCK univerfally, or fo as to adapt it for all Latitudes, it is only making it to move on a ftrong and well wrought Hing on the Fore-part

at

at N, and then it may be elevated to any Degree of Latitude on a graduated Arch on the back Part at O, to which it may be faſtened by a Screw.

3561. But then it muſt be conſidered, that as the Perpendicular Height A P, of the Fork A, above the Arm H P, is the Tangent of the Co-Latitude, or Elevation of the Equator, *viz.* ALP = LPW to the conſtant Radius LP; therefore that Height (AP) will be conſtantly variable with the Latitude in an inverſe Ratio, or is greater as the Latitude is leſs; and *vice verſa.* And this muſt be provided for by Calculation alſo, and conſtantly adjuſted to the Latitude by a *graduated Scale.* All which is eaſily done by the Triangle ALP, wherein LP is of a given Length, and PA is found for any given Latitude LAP = APQ. So that for every particular Latitude, the three Quantities AD, BD, and AP muſt be calculated, and the *Clock* and *Speculum* thereby adjuſted.

3562. According to the Luminary you propoſe to view or obſerve, the Pendulum of the Clock muſt be peculiarly adapted; the Methods of doing which, for the various Sorts of *Planetary Pendulums,* we have already deſcribed in their ſeveral Conſtructions, and muſt leave the Artiſt to chuſe out of them all that which he thinks beſt; for by this Time we preſume he muſt be very ſenſible the SUN, the MOON, a STAR, and each particular PLANET muſt have a Pendulum properly qualified to render it *quieſcent to the View.*

3563. In this *Celeſtial* CLOCK it will moreover be very expedient to have the Hour-circle upon the Face of the Clock *moveable,* and divided into 24 Hours, or twice XII; for by this Means the Hour and Minute of the Luminary's Culminating, or being on the Meridian, may be brought into the Vertical Line or *Meridian of the Clock*; and then the Hands being placed to the preſent Hour and Minute in that moveable Circle, will conſtantly ſhew the Time and Motion of that Luminary; for the Hour-hand will always keep Pace with it in the Heavens, and ſhew its *Right Aſcenſion,* or Difference from Solar Time, which will be ſtill *more* evident if the moveable Hour-Circle be added to that which is fixed, and both divided into twice XII Hours. (See 3325, *&c.*)

3564. Before any Obſervation is begun, the Clock muſt be very nicely placed in the Meridian Line, or ſo as that the Meridian Line of the Clock may coincide with the Meridian of the Place; to which End this Clock muſt be furniſhed with a very good *Magnetical Compaſs and Needle*, and the VARIATION thereof muſt be accurately found by Experiment for the Latitude where it is uſed. And by that the Clock may be readily adjuſted to the Meridian. Alſo two *Spirit Levels* muſt be placed in a proper Part of the Machine at right Angles to each other, by which it may be always reduced to a truly horizontal Poſition.

3565. By Means of this Machine duly adjuſted to the SUN, if the Room be darkened and a Hole made in the Window-Shutter to let in a Beam of the Sun's Rays upon the Speculum, it will continue to be reflected in one conſtant Direction all the Time it can fall on the Speculum, which may be ſo contrived as to be long enough for moſt *Optical Experiments*, either with PRISMS, or the SOLAR MICROSCOPE, *&c.* Beſides it will be no difficult Matter to follow the incident Beam with the Speculum by having *Caſtors* at the Bottom, or Foot of the Frame, ſupporting the Clock, and thereby moving and adjuſting it, as required, with the greateſt Eaſe.

3566. But the nobleſt Purpoſe to be anſwered by this *Celeſtial* MACHINE is fixing the heavenly Bodies, or rendering them ſtationary or quieſcent in the Field of the Teleſcope for aſtronomical Obſervation; an Advantage hitherto wanted (and unattainable by any other Means,) for advancing the Science of Aſtronomy to its true Summit of Perfection. The Teleſcope for this Purpoſe ſhould be of the reflecting Sort, and furniſhed with a Micrometer of different Forms, *viz.* the *Lattice*, the *parallel Wires*, the *divided Object-glaſs*, the *fine Screw*, *&c.* for meaſuring and delineating the Surfaces of the SUN, MOON, and PLANETS.

3567. I need not add, that the Foot and Stem of the Teleſcope ſhould be ſo contrived, that it may be always placed truly horizontal, exactly in the Meridian of the Clock, and juſt of the ſame Height of the Center of the Speculum, by this Means the Obſerver will ſit at Eaſe, and, without the leaſt Diſturbance, ſurvey the various and wonderful Phænomena of each celeſtial

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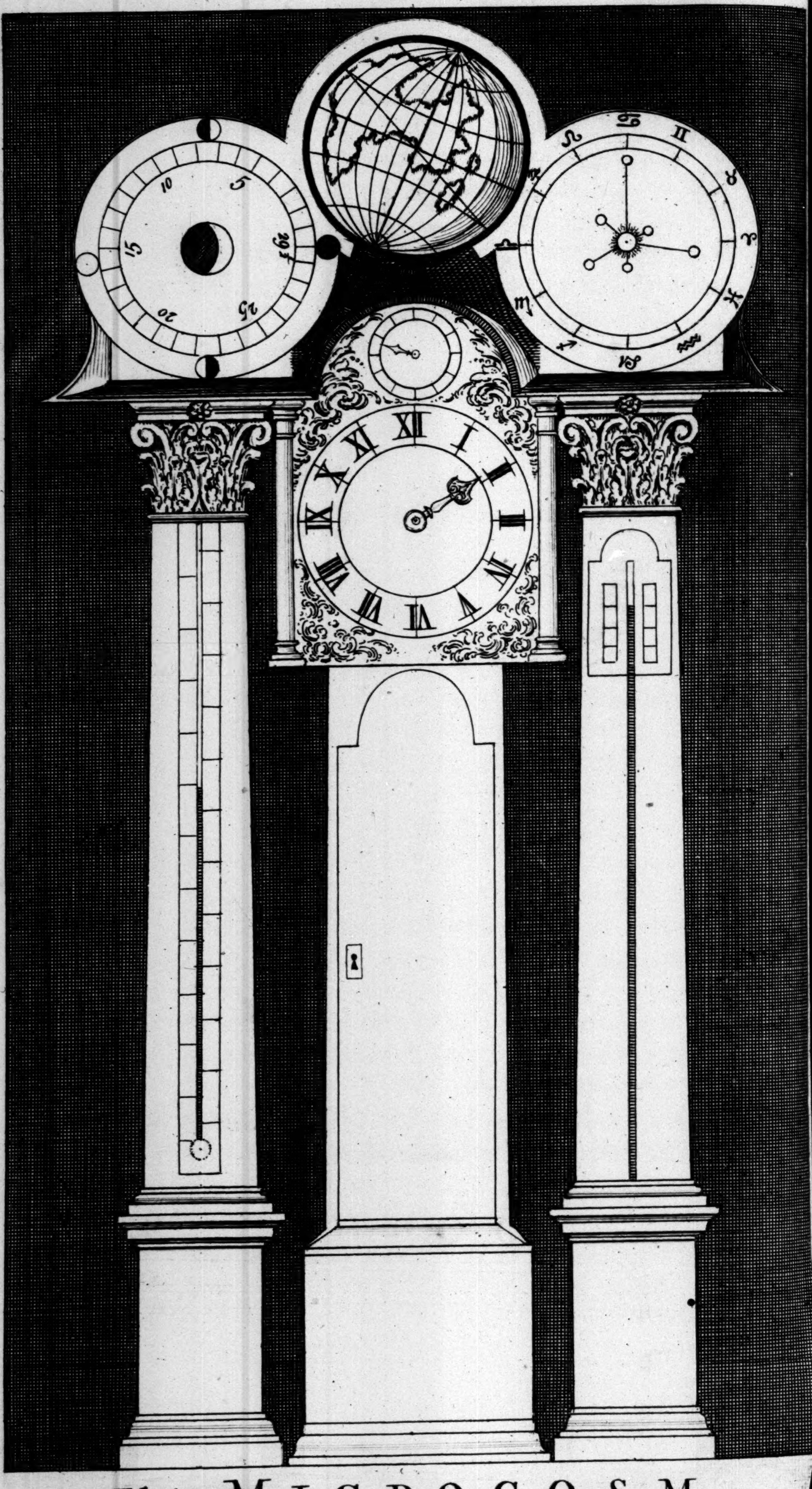


The MICROCOSM.

Body. And thus we have finished the Theory, or general Principles, of the sublimest Piece of Machinery, that the Invention of Man has yet been able to produce.*

3568. It may also be observed, that if the Speculum at I or K be placed in the Direction perpendicular to the former, or parallel to that of the Wire A I or A K then will the Ray S I, S K be reflected the contrary Way, or in the Direction I B, or K C, in the same Right Line R B, or T C as before; and in many Cases such a Reflection or Position of the Ray will be more convenient than the other on the Part towards the Sun.

3569. Lastly; The Speculum C to be used in all Observations of the heavenly Bodies must be a *plain One*, and as perfectly true as Art can make it; and indeed unless it be extremely true and well polished, it cannot answer any such Purpose. No *Concave* or *Convex Speculum* can in such a Case be used, for Reasons mentioned in the Theory of Catoptrics; though in many Experiments which require the Convergency of Rays for illumination only, a Concave Reflector may be applied to answer many useful Designs, which the Ingenuity of Artists will naturally suggest.

CHAP. XXII.

The PRINCIPLES *of* Celestial MECHANICS *applied to the Construction of a* MICROCOSM, *consisting of a* PLANETARIUM, TELLURIAN, *and* LUNARIUM, *for exhibiting the* Celestial MOTIONS.

3570. THERE is another Branch of *Clock Work*, which may be properly esteemed the *Second Part of* CELESTIAL MECHANICS; for by this we exhibit the MOTIONS of the *Heavenly Bodies*, as well as the TIME thereof, by the Clock

* See a further Account of the practical Use of this new *Heliostata*, or Planetary Clock in another Part of this Work, *viz. The Young* GENTLEMAN *and* LADY'S PHILOSOPHY, Page 302, &c.

Clock, and proper Mechanism for that Purpose, to be added thereto.

3571. The *Celestial Bodies* whose Motions are shewn by Clock-Work are the PLANETS, the EARTH and the MOON; the Motions of the *Secondary Planets*, or *Satellites* of *Jupiter* and *Saturn*, are sometimes shewn in such a System of Mechanism as is usually called an ORRERY; but the common Methods of constructing these large Instruments render them very expensive, and consequently they are but rarely made. But we shall endeavour to remove this Difficulty by proposing other Constructions and Forms of Orreries that will be less costly, and perhaps more elegant and natural, as well as much more concise.

3572. The Calculations of the *Celestial Motions* depend on considering the Times of the Revolutions of the heavenly Bodies about their proper Centers, and taking the Ratios of those Periods in such Numbers as will best answer for toothing the Wheels and Pinions of the Work by which those artificial Motions or Revolutions are to be produced; and then how or in what Manner they are to be connected with the Movement of the Clock for their Continuation.

3573. We shall first begin with the Motions of the *Primary* PLANETS about the SUN; and by the best astronomical Observations, their mean Revolutions or periodical Times are as follow, *viz.*

SATURN	revolves in	10759,3 Days.
JUPITER	———	4332,5
MARS	———	686,9
EARTH	———	365,25
VENUS	———	224,7
MERCURY	———	88.

3574. Now the Ratio of the Earth's Period to those of the Rest of the Planets is hereby given, and therefore may be expressed in lesser Numbers which may be at the same Time Integers, and conveniently adapted for the Teeth of the Wheel-Work for the intended Movement. And upon Trial those which

which are found to anſwer beſt are as in the following Table, *viz.*

As 365,25 :	88	::	83 :	20	for	MERCURY.
365,25 :	224,7	::	52 :	32	—	VENUS.
365,25 :	686,9	::	40 :	75	—	MARS.
365,25 :	4332,5	::	7 :	83	—	JUPITER.
365,25 :	10759,3	::	5 :	148	—	SATURN.

And according to theſe Numbers, the Wheel appropriated to the EARTH muſt have 50 Teeth.

3575. Then if on one Arbor you fix ſix Wheels with the following Number of Teeth, 83, 52, 50, 40, 7, 5, to drive another ſet of Wheels each moveable upon another Arbor, and with the Number of Teeth in the other Column reſpectively, 20, 32, 50, 75, 83, 148; and laſtly, if on the ſeveral moveable Wheels proper Sockets are fixed (moving within, and independent of each other,) then will round ivory Balls, at the End of ſlender Wires fixed to thoſe Sockets, be carried about the Arbor in the ſame Periods of Time, reſpectively to each other, with thoſe of the real Planets in the Heavens.

3576. If this Syſtem of Movements be properly diſpoſed in a Box; and on the Top or Cover be drawn or engraved the Calendar and Ecliptic, and a braſs or lacquered Ball be placed on the Top of the Arbor of moveable Wheels, then, by a Handle or Winch, the Whole may be put into Motion; and it will then become a *Planetarium* or *manual* ORRERY of a moſt uſeful Form, and exhibit all the *Phænomena* of the SOLAR SYSTEM, or that according to *Copernicus*.

3577. If inſtead of the lacquered Ball, you put a three-inch *terreſtrial* GLOBE on the ſaid Arbor, and a ſmall braſs Ball upon the Arm carrying the ivory Ball, repreſenting the Earth among the Planets, then will the Machine be changed into the *Ptolomaic Syſtem* of the *World*; by which all the Abſurdities of that ſenſeleſs Hypotheſis may be eaſily exhibited and refuted.

3578. Such a Syſtem of Planetary Movements may eaſily be applied to a Piece of *Clock Work* by which it may be made to go any required Time, as an Hour or two; in which the Earth may revolve two or three Times about the Sun (or the Sun

about

about the Earth,) which is a sufficient Interval for exhibiting all the considerable Appearances in either System, relative to the *Primary Planets*. But if it be required that the Motions of this artificial System should be the same with those in the real System of the World, then we must proceed as follows.

3579. Let it be required to construct a Movement for a Motion to be performed in the Space of *one Year*, or $365\frac{1}{2}$ Days; for we must not pretend to the Accuracy of an Hour or two in a Year. In $365\frac{1}{2}$ Days there are 730 Half-days of 12 Hours. This Number is produced by the Factors $18\frac{1}{4}$, 8, and 5; for $5 \times 8 \times 18\frac{1}{4} = 730$. Therefore any Number of Wheels and Pinions which will produce these Quotients, will produce the Motion desired.

For Example,

4) 73 ($18\frac{1}{4}$	4) 73 ($18\frac{1}{4}$	8) 146 ($18\frac{1}{4}$
4) 40 (10	4) 32 (8	8) 64 (8
5) 20 (4	4) 20 (5	12) 60 (5

3580. Thus suppose a Wheel (A) with a Pinion of 5 Teeth drives a Wheel (B) of 20; and this Wheel carrying a Pinion of 4 Teeth drives a Wheel (C) with 40; and this with a Pinion of 4 drives the last and largest Wheel (D) with 73 Teeth. Then it is evident (3180.) that in one Turn of B there will be 4 Turns of A; and 40 Turns of A in one Turn of C; and 730 Turns of A in one of the last Wheel D. And therefore this Wheel D moves round once in a Year, and consequently if the first Wheel A be connected with another equal Wheel fixed on the Arbor of the Great Wheel (or Fusee) of an 8 Day Clock, then by Circles on its extreme Parts the *Day of the Month*, or *Sun*'s *Place* in the Ecliptic will be very exactly shewn, through the whole Year.

3581. Now this little System of Wheels may be very easily connected with the Movement of a Clock on one Part, and with that of the *Planetarium* on the other; so that the Clock (with a proper *additional Weight*,) shall constantly keep the Planets moving about the artificial Sun just with the same angular Velocities as the real Planets themselves move with in the Heavens; and if in the Beginning of the Year the Places of the

the Planets in the Orrery be duely adjusted or rectified by an *Ephemeris*, they will nicely correspond to those they respectively represent in the Heavens, and continue so to do ever after.

3582. It will be readily allowed, that this Method of constructing a *Planetarium* is the most natural and simple that can be, as not a Wheel, or even a Tooth, need here be used more than what is absolutely necessary. By this *Planetary* Clock, it will every Day be seen, what Part of the Ecliptic the *Sun*, the *Earth*, or any *Planet* is in —— the various mutual *Aspects* of Planets —— their *Conjunctions*, *Quadratures*, and *Oppositions* —— the *apparent Direction* of their Motion, *direct*, *stationary*, or *retrograde* —— whether they are *above or below the Horizon* —— whether they *rise or set before or after the Sun:* —— In short you have by this Means the whole *System of Celestial Phænomena* constantly in View.

3583. By the same small Compages of Wheels, described (3575,) placed horizontally, it is evident the Axis of the Terrestrial Globe may be made to move about the Axis of the Ecliptic once in a Year; and thereby all the *Phænomena* of the Earth's Annual Motion, with Regard to the *Seasons* of the Year in different Latitudes and Climates, may be ready shewn, and observed. For since the Position of the Wheel D is supposed horizontal, if an Axis be fixed in its Center perpendicularly, it will represent the Axis of the Ecliptic. But if at a proper Height above the Wheel D the said Axis be bent out of the Perpendicular so as to make an Angle therewith of 66° 30′ in the Form of an *Arm*, and on this Arm another Axle be placed at right Angles, this will make, with the forementioned Axle, an Angle of 23° 30′, and consequently will represent the *Axis of the Earth*; and will be carried on round the Perpendicular Axis of the Ecliptic once in a Year, if properly connected with the Clock.

3584. Then for the *Diurnal* Motion of the Earth at the same Time, it may be easily effected thus; let the Axis of the Earth be moveable in a Socket fixed Perpendicular to the forementioned strong Arm; on the under Part of which (below the Arm,) let a Wheel (G) be fixed, suppose of 72 Teeth; then let the Teeth of this Wheel play in the Teeth of another Wheel (F) which must be but half the other Number, *viz.* 36; and this Wheel must be placed horizontally about the Stem or per-

pendicular Axis just below the Arm, and supported on a Bridge to render it independent of the Work for the *Annual Motion* below. Lastly, this Wheel F must have another equal Wheel (E) of 36 Teeth, to connect it with the Wheel A on the Arbor of the great Wheel of the Clock, and this Wheel A must also have 36 Teeth.

3585. Therefore, if upon the Terrestrial Axle, the Globe of the Earth be placed, so as to be moveable thereon, or fixed with a Nutt and Screw at the End, as Occasion requires, then it is plain, it will be carried about once in 24 Hours, because the Wheels A, E, F, each move round in 12 Hours, and the Wheel F of 36 Teeth driving the Wheel G of 72 must move it round, (and the Earth on the Axis to which it is fixed,) once in 24 Hours.

3586. With Regard to the Wheel E, as it drives the Wheels above the Bridge for the diurnal Motion, so by an Arbor going through, and carrying a Pinion of 5 Leaves, it drives the Wheels B, C, and D, below for the annual Motion at the same Time. Also by the Interposition of this Wheel E, the Globe is made to revolve the right Way, *viz.* from *West to East*, as the Earth itself really does.

3587. Then by a *Circle of Illumination*, and an *artificial Sun* at a proper Distance in the Axis thereof, with some other Apparatus, all the *Phænomena* of the *annual and diurnal Motions* of the *Earth* are most naturally exhibited for every Day throughout the Year. Thus you see the North and South Pole alternately in the illumined and dark Hemisphere; ——— the Sun rising and setting each Day upon different Parts of the Horizon; —— the variable Lengths of the *diurnal* and *nocturnal Arches* described by the City of LONDON or any other Place: —— If a *Crepuscular Circle* be added, the Time of Beginning and End of *Twilight* will be shewn for each Day; with many other important Particulars in *Geography* and *Astronomy*.

3588. 'Tis moreover easy to contrive, that this Mechanism for the Globe may be disengaged from the Clock, or be put on and taken off at Pleasure; in which Case it may be turned with a Winch so as to represent all the Phænomena of a Year or a Day, in a very short Time; and then such an *Apparatus* may be added, as will exhibit the *Solar and Lunar Eclipses*, &c. the

Manner

Manner of doing this, together with a Print of the Globe thus constructed, may be seen in the *Young* Gentleman's *and* Lady's Philosophy.

Scholium.

3589. We shall hereafter give a Construction of a *Lunarium*, to shew the *Monthly* Motion and Phases of the Moon, as also the Motion of her Apogee *and* Nodes; if then this lunar Movement be connected with that of the Clock (as it easily may) and the whole be disposed in a proper Form, it will make what may be more truly called a Microcosm, than any thing which has hitherto borne that Name; and to assist the young Artist's Imagination I have presented to his View, in a Copper-Plate, the Form of a Specimen of such a Piece of *Celestial* Architecture in which are the following distinct Parts, *viz.* (1.) A large Clock for a *Primum Mobile*. (2.) Two Pillars, (or *Pilasters*) on one is placed a *Barometer*, and on the other a *Thermometer*. These Pillars support an *Architrave* on which is placed (3.) An Orrery or *Planetarium*, such as before described. (4.) A Tellurian, or *Terrestrial* Globe with the annual and diurnal Motions. (5.) A Lunarium, shewing the Motion, Phases, and Age of the Moon. Such a superb Structure, we presume, would well become the Study or *Musæum* of the Opulent and the Great; but so little Genius, Taste, or Encouragement is now found for such Works of Art, that it is not worth while to insist further on their Use or Excellency. By Arts and Artists we, at this Time, understand only *Engraving*, *Painting*, and *Sculpture*, and those who practice them; and Learning and learned Men, mean no more than *Novels*, *Romances*, *Plays*, and their *Authors*, in the present *Tristram-Shandy* Age.

CHAP. XXIII.

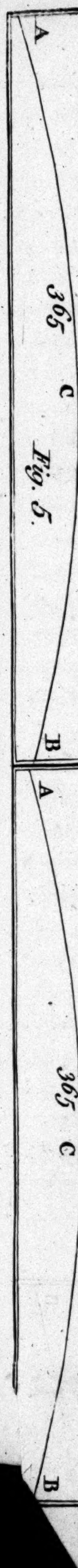
The MECHANISM *and* CONSTRUCTION *of a* TELLURIAN, *with Three* Terreſtrial GLOBES; *illuſtrating the* THEORY *of the* EARTH'S MOTION; *the* Rationale *of the* SEASONS; *and the* Inequality *of* DAYS *and* NIGHTS.

3590. IN the Conſtruction of the ORRERY deſcribed in the laſt Chapter, the Earth's *annual Motion* was ſhewn, but not the *diurnal Motion* at the ſame Time; we ſhall now explain a new Piece of *Celeſtial Mechaniſm*, which being applied to the foregoing ORRERY (now ſuppoſed in a *Horizontal Poſition)* will not only produce both theſe Motions of the Earth together, but at the ſame Time, by Means of *two other Globes*, the *Rationale* of all the various Phænomena and Affections of the Earth will at once moſt evidently appear.

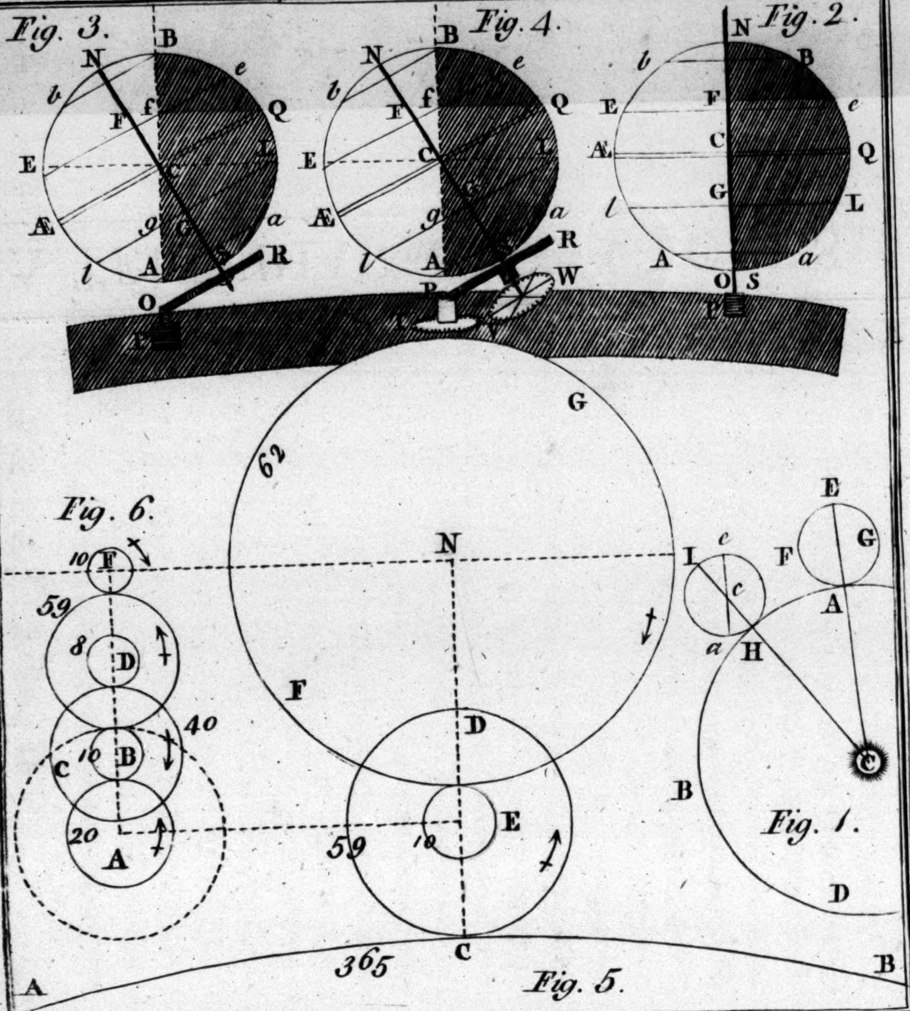
3591. I preſume it is needleſs to mention to the intelligent Reader, *that if any Body deſcribes the Circumference of a Circle, with the ſame Part always directed to the Center, it will neceſſarily be moved or turned once about its own Axis or Center in the ſame Time*; for by this Means the Body has that very Part directed to every Point in the Circumference of the Circle, which would be impoſſible without *one entire Revolution about its Axis*; this is therefore an *Axiom* or *ſelf-evident Truth*.

3592. Alſo, it is farther evident, that the Direction of the Motion of ſuch a Body, both about the Center of the Circumference it deſcribes, and about its own Center or Axis, is the ſame, or towards the ſame Part. But to illuſtrate theſe Poſitions, let A B D be the Circle deſcribed by any Plane F E G fixed upon an inflexible Rod or Arm E C, moveable about the Center C.* Let this Arm (or *Radius Vector)* revolve from the Situation E C to any other I C, then will the angular Motion

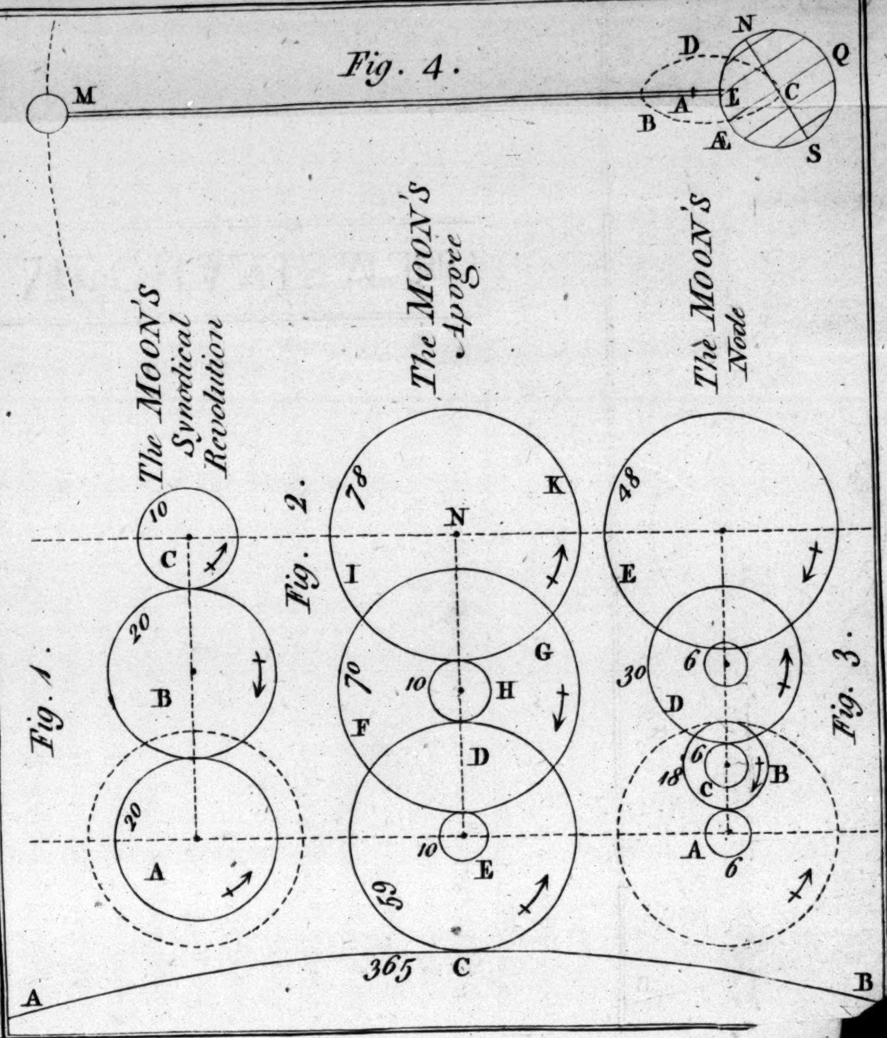
* See Fig. 1. in the Plate of the TELLURIAN with three GLOBES.



A TELLURIAN with three GLOBES.



MECHANISM for a LUNARIUM.



tion of the Body F G be = A C H about the Center C in that Time.

3593. Upon the Plane in the Situation I H, draw *a e* parallel to A E, and it will make the Angle I *c e* with the Radius I C; and thereby it appears that the Line A E (or *a e*) has deviated from its firſt to the preſent Poſition I H by the Quantity of the Angle I C E; and therefore by ſo much has the Plane moved about its own Center (*c*) in the Time of deſcribing the Part of the Circumference A H. But ſince C E and *a e* are two parallel Lines cut by a right Line I C, the Angles A C H and I *c e* will be equal; that is, the angular Motion of the Plane about the Centers C, and *c*, will be conſtantly equal; and the Direction of the Motion from *e* to I is the ſame as that from A to H.

3594. Hence it will follow, that if any luminous Body be placed in the Center C of a Circle A B D, and a Globe F G be moved in that Circle ſo as to have the ſame Part always turned to the Center, then will the ſame Hemiſphere or Part of the Globe be conſtantly enlightened, and the other Part will be always dark. ——— Thus ſuppoſe Æ N Q S (Fig. 2.) were a Planet (our *Earth*, for Inſtance) moving in the Manner abovementioned about the Sun in the Center at an immenſe Diſtance, then would the ſame Hemiſphere S Æ N be perpetually illuminated; and the other S Q N conſtantly in the Dark. Whence it is evident, a Planet in ſuch Circumſtances could never be a proper Seat, or Place for Habitation, for ſuch a Species of rational Creatures as the *Human Race*.

3595. Let ſuch a Globe, then, be ſuppoſed to move about its Axis N S placed perpendicular to the Plane of its Motion about the Sun. Such a Motion we call its *Diurnal Motion*, as it produces DAY and NIGHT alternately to all Parts of the Surface, in the Space of one Revolution which in our Globe is 24 Hours. Then it is plain, that every Part on the Surface of ſuch a Globe would be one half of the Revolution in the illumined Hemiſphere, and the other half, in the dark One; ſo that in ſuch a Condition of the Globe, there would be a *perpetual Equality of Day and Night*.

3596. Again, ſince the Axis N S of the *Diurnal* Motion is perpendicular to the Plane of the *Annual Motion*, which is ſuppoſed

posed to pass through the Centers both of the Earth and Sun; then it is evident, the Rays of the Sun will be perpendicular to the middle Part of the Globe at the Equator Æ Q, and will fall obliquely on every other Part; and the more so as the Part is nearer the Poles N, S, on either Side the Equator. Whence it will follow, that there will be *one and the same Degree of Light and Heat* to the same Part of the Surface of the Globe *constantly*; and therefore *no Variation of Seasons* throughout the Year. Hence such a Position or Constitution of the Globe of the Earth would by no Means suit with the Condition of the *Human Species*.

3597. Let us next see what will be the Case of a diurnal Motion of the Globe about an Axis in a Position inclined to the Plane of its annual Motion, such as S N (Fig. 3.) and further let us suppose that the Axis N S is constantly in the same Plane with the Line E C L which connects the Centers of the Sun and Globe. Then it will follow, that the Parts of the Globe about the upper or North Pole N, will be constantly inclined to the Sun, while the opposite Parts about the South or lower Pole S are as constantly turned from it.

3598. The Consequence of this *inclined Position* of the Globe will be, an *Inequality of Days and Nights*; and *different*, or *contrary Seasons* to Parts at an equal Distance from the Equator on either Side. This is evident by Inspection; for Æ C is the *Semidiurnal Arch* in the Equator; E f is the same in the Parallel E *e*, which is longer (in Time) than the other by the Time of describing F f. But the semidiurnal Arch (*lg*) in the same Parallel *l* L of South Latitude is shorter than Æ C by the same Quantity *g* G = F f. And the Parts about the North Pole N to the Distance of N B (which measures the *Inclination of the Axis*, or Angle N C B) are perpetually in the illumined Hemisphere and have constant Day; and on the Contrary, those Parts at the same Distance about the South Pole S have an eternal Night.

3599. Then with Respect to the Seasons, they are by this Means very different in opposite Parallels of Latitude; for now the Parts at E receive the perpendicular Rays, and have the *Summer Season*; but since the Parts within the *Arctic Circle* B *b* have an inoccidual Sun, or enjoy perpetual Day, they will have the *hottest Season*, and the North Pole N will be the hottest

Part

Part of ſuch a Globe. On the other Hand, the Parts within the *Antarctic Circle* A *a* will never feel the enlivening Beams of the Sun, and muſt be doomed to *perpetual Winter* and the intenſeſt Cold. Such a State of the Globe would therefore be very uneligible to every Sort of Inhabitants of the preſent *Terreſtrial Globe*.

3600. Now it will be readily allowed, that an *Alternation and Variety* in the *Seaſons* of the Year and *Lengths of Days and Nights*, will be the moſt deſirable and delightful State to rational Inhabitants of the Globe, and ſupply them with every Bleſſing equally and interchangeably through every revolving Year. And this wonderful and univerſal Effect is produced in our Terraqueous Globe by one ſimple Contrivance (a moſt ſtriking Inſtance of *divine Mechaniſm*,) *viz.* by giving the inclined Axis SN of the Earth's Motion an equable Movement about the Axis of the Ecliptic A B once in the Time of its annual Revolution about the Sun, and in a Direction contrary to the Earth's Motion; and this will produce the abovementioned neceſſary Effects.

3601. For we have ſhewn (3593) that while the Globe is carried about the central Sun, with the Axis SN, or the ſame Side A E B always turned towards it, the ſaid Globe and its Axis, will in that Time be alſo turned once about their proper Center C and in the ſame Direction with the Globe's annual Motion; therefore, if the ſame Motion be impreſſed on the Globe in a contrary Direction, it will have all the Parts of its Surface turned to the Sun, of Courſe, in the Space of one Revolution or Year, and its Axis S N will always keep a Direction parallel to that which it had at firſt; ſo that all Effects and Phænomena will be the very ſame as if the Earth were really at Reſt on its inclined Axis, and the Sun itſelf were to move about it (as it appears to do) in the Ecliptic Circle E C L.

3602. For then, when the Sun was in the Equinoctial Point or Beginning of the Ecliptic at C, it will, at the ſame Time, be in the Equator Æ Q, and conſequently illumine the Globe from Pole to Pole and produce an Equality of Days and Nights in every Part thereof; and this is called the SPRING-SEASON of the Year.

3603. After

3603. After the firſt Quarter of the Revolution, or the Sun has paſſed from C to E, and reached the *Tropic of Cancer* E*e*, then the Globe will be illuminated from A to B, and the North Pole N, with all northern Latitudes from E to B, will be now turned towards the Sun and receive the *greateſt Quantity of Light and Heat* they can ever have, and conſequently this will be the Midſt of their *Summer Seaſon.* The Day Ef in every northern Parallel E*e*, is longer than the Night f*e*; and all within the Arctic Circle *b*B have no Night at all on that *Midſummer Day.*

3604. Juſt the Contrary happens in ſouthern Latitudes; for there the Sun-beams reach no farther than to the Antarctic Circle A*a*; and all within it are in total Darkneſs for one whole diurnal Revolution of the Globe. And the Length of the Day *(lg)* in any Parallel *l*L, is ſhorter than the Night *g*L by the Quantity *g*G; conſequently in theſe Latitudes, the Light and Heat of the Sun is now the leaſt, or the Cold and Darkneſs the greateſt it can be, and therefore make their WINTER-SEASON.

3605. When the Sun apparently deſcends from the northern Climates to the Equator again on the other Side the Globe to the firſt Point of *Libra* oppoſite to C, it will then again make equal *Day and Night* throughout the Globe, by enlightening it from Pole to Pole, as before (3602.) The Light and Heat is now at a Mean, and make the other temperate SEASON called AUTUMN.

3606. As the Sun paſſes from the Autumnal Equinox thro' the next three Signs of the Ecliptic, the *ſouthern* Parts of the Earth will be turned gradually to the Sun; till, at Length, the Sun being arrived at L, the firſt Point of *Capricorn*, it will then be neareſt of all to the South Pole S and illuminate all that Hemiſphere A L B, which before was dark when the Sun appeared at E; and conſequently every Thing with reſpect to Day and Night, and the Seaſons, is juſt the reverſe of what it was then. All *Southern Latitudes* now enjoy the *Midſummer Seaſon*, and in the northern Latitudes it is *Winter.* What was before the Length of their Night, *viz. g*L, is now the Length of their Day, and *vice verſa.*

3607. The Sun returning again to C, renews the Spring and begins the future Year. ——— Thus by giving the Earth an annual

annual retrograde Motion about the Axis A B of the Ecliptic, all Parts of its Surface become equally expoſed to the Sun on each Side the Equator, and equally participate the Advantages which redound from ſo wiſe and beneficent a Contrivance, in the Courſe of one whole Year.

3608. Now this whole Theory (as above explained) of the Earth's *annual and diurnal Motions* may be very eaſily repreſented and illuſtrated by a Piece of Mechaniſm, which I call a TELLURIAN; conſiſting of three ſmall Terreſtrial Globes, placed in the Manner ſhewn in Figs. 2, 3, and 4. The firſt of theſe (Fig. 2.) is moveable with the Hand about an Axis N S, which is perpendicular to the Plane of its Motion about the Sun, and is fixed to the *Radius Vector*, and therefore will exhibit the Phænomena of *equal Day and Night*, and *Identity of* SEASONS through the Year, and nothing more, as before obſerved (3596.)

3609. The *ſecond Globe* (in Fig. 3.) has the Axis S N of its diurnal Motion inclined to the Axis A B of the Ecliptic in an Angle B C N = 23° 29′. But this Axis S N being fixed to a Part P O R, which is itſelf fixed to the *Radius Vector*, will indeed produce a Variety of Seaſons, and Difference in the Lengths of Days and Nights; but then as the Axis S N is conſtantly turned to the Sun, there can be no Viciſſitudes of *Seaſons* nor any Variation of the Lengths of Days and Nights, which remain conſtantly the ſame, in the ſame Place throughout the Year, agreeable to (3599.)

3610. But the *third Globe* (in Fig. 4.) is not fixed to the *Radius Vector*, but is connected with the Mechaniſm contained therein, in ſuch Sort, as to qualify it for exhibiting the real Phænomena of our own Globe, and juſt as they happen in Nature. To this End the firſt Thing to be done is to counteract or nullify the *direct annual Motion* of the Globe (in Fig. 3.) about the Axis A B of the Ecliptic, by fixing its Stem O P upon a Plate that has an equal Motion in a contrary Direction at the ſame Time.

3611. Thus; Let A C B (Fig. 5.) be a Part of the outward Rim of the Plate of the *Orrery* (3575), divided into 365 Teeth; and upon the lower Plate of the *Radius Vector*, let a ſmall Wheel C D be placed with 59 Teeth to move in thoſe of the Orrery-plate. Upon this Wheel a Pinion E muſt be fixed

with 10 Teeth to turn another Wheel F G of 62 Teeth; and then if the Shaft or Stem of the Globe (Fig. 4.) be fixed in the Center N of this Wheel or Plate, it will communicate to the Globe the *Retrograde Motion* required. For 59)365(= 6,2, or the Wheel C D is moved round its Center 6,2 Times while it revolves once round the Orrery-plate; and because 10)62(= 6,2 also; it is evident, that the Wheel F G moves once round in one Revolution of the *Radius Vector*, and because the Wheel C D moves *directly*, from West to East, 'tis plain the Wheel F G moves retrograde, or the contrary Way as the Arrows (⇅) shew in each.

3612. Hence by these two Wheels only, the annual Motion of the Globe (Fig. 4) about the Axis A B is annihilated, and a *constant Parallelism* of the Axis N S of its diurnal Motion is produced; and therefore not only the Difference of the Seasons and Days and Nights is effected, but all the Change or Variety thereof that happens to the natural Globe for every Latitude, and every Day of the Year.

3613 The next Thing to be considered, is, the System of Wheel-work for giving the same Globe its proper diurnal Motion about its Axis N S, and in the natural Direction from West to East. Here we find 365 of these diurnal Revolutions to *one* annual One; and therefore such a System of Wheels must give 59 Turns of the Globe in one Turn of the Wheel C D, which must be the *Primum Mobile*, or first Mover to that System.

3614. Therefore the Number 59 must be broke into three Quotients, because the last Wheel must have a retrograde Direction, as will appear by and by; suppose these Quotients are 10)20(= 2; 8)40(= 5; and 10)59(= 5,9; then it is evident, if upon a Bridge or Plate fixed over the Wheel-work of Fig. 5. the following Wheels are properly disposed, *viz.* a Wheel A of 20 Teeth (Fig. 6.) fixed on the Arbor of the Wheel C D (Fig. 5.) to drive a Pinion B of 10 Teeth on a Wheel C of 40, which drives a Pinion D of 8 Teeth fixed upon a Wheel of 59, and this at last drives a small Wheel F of 10 Teeth; then, I say, it is evident this last Wheel F will be turned round 59 Times while the first Mover C D turns once; and its Direction will be *retrograde*, as shewn by the Arrows.

3615. The

3615. The Wheels being so proportioned, that the Center of the last Wheel F (Fig. 6.) lies exactly over that of the Wheel FG (Fig. 5.) a Socket is to be fixed into the said Wheel F, moveable about the Stem or Fulcrum of the Globe which is fixed firmly in the Center N of the Plate F G. This Socket passing through a Plate (fastened over all the Machinery) will appear above it, as at P T. And if on this upper Part of the Socket there be fixed a Wheel T V with 10 Teeth playing in another V W of an equal Size and Number of Teeth fixed to a Part of the Axle N S below the Piece PR (in which the said Axle moves) then will the Globe (Fig. 4.) be turned round 59 Times in each Revolution of the Wheel C D of Fig. 5. and consequently it will turn upon its Axis once every Day or 24 Hours. And this Motion will be *direct*, because that of the Wheel T V (or F below) is *retrograde*.

3616. And thus the Telluran applied to the Orrery will by its three Globes together give a full Demonstration and Illustration of the *Genuine* Theory of the annual and diurnal Motions of the Earth in every Respect; *one Globe* shews (Fig. 2.) that by a *parallex Axis without Inclination*, there can be no Difference in the Seasons or Quantity of Day and Night ——— A *second Globe* shews (Fig. 3.) that an *inclined Axis without Parallelism*, can only shift the Scene of the Seasons, and produce a different Length of Day, and Night in contrary Latitudes; but can never make any Alteration in either ——— But the *third Globe* with an Axis, at the same Time, *always inclined and parallel to itself*, produces every Variety we find in Nature, and exhibits an exact Conformity to the Motions and various Phænomena of the natural Terraqueous Globe.

CHAP. XXIV.

The CONSTRUCTION *and* MECHANISM *of a* LUNARIUM, *for ſhewing the* MOTION *and* PHASES *of the* MOON, *as alſo of her* APOGEE, *and* LINE *of* NODES,

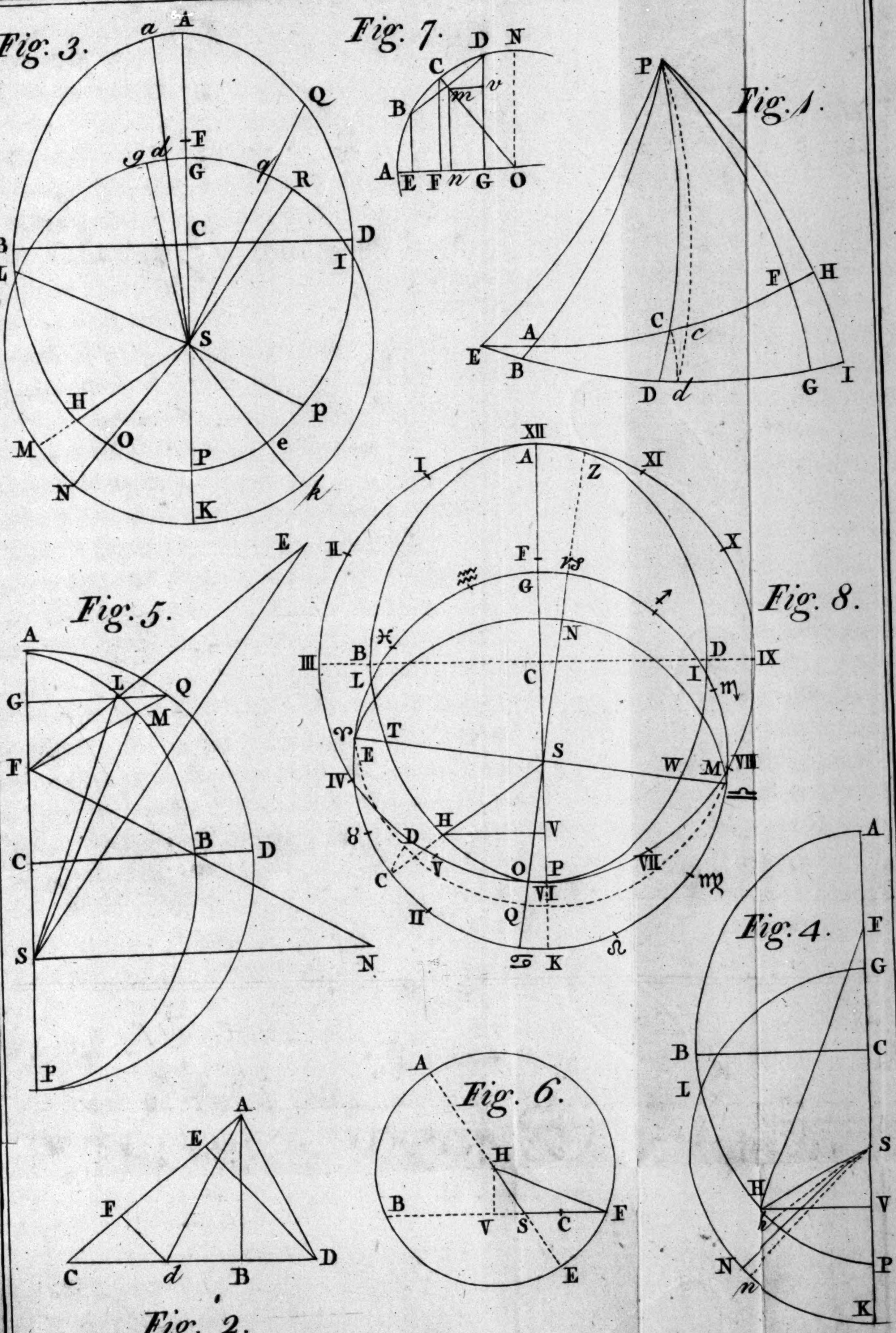
3617. THE Conſtruction of a LUNARIUM, or Inſtrument to ſhew the Phænomena of the MOON in regard to her own *Motion* and *Phaſes*, and the Motion and Poſition of the *Apſides* and *Nodes* of her *Orbit*, will be more complicated than that of the *Tellurian*; but as the periodical Revolutions are known, it will be eaſy to contrive ſuch Syſtems of Wheel-Work as ſhall repreſent them adequately, and produce Appearances every Way ſimilar, and correſponding to Nature itſelf.

3618. The LUNARIUM, to exhibit the Phænomena of Lunar Motions, may be contrived and conſtructed in the following Manner. ACB here again repreſents the dentated Rim of the Plate of the Orrery of 365 Teeth, in which the Wheel CD (Fig. 2.*) moves as before; and is now alſo to be conſidered as the *Primum Mobile*, or that which gives Motion to all the Machinery of the *Lunarium*. As this Wheel has 59 Teeth, it is evident, that if to its Arbor below, there be fixed a Wheel A of 20 Teeth, to drive another equal Wheel B of 20, and this to move a 3d Wheel C of 10 Teeth; then this laſt Wheel C will turn round once in the Time of a *Lunation*, or 29 $\frac{1}{2}$ Days; for it will turn twice in each Revolution of A, which is in 59 Days. And the Direction of its Motion will be the ſame, *viz.* from *Weſt* to *Eaſt*.

3619. In the Center of the Wheel C (Fig. 1.) let there be fixed a ſtrong perpendicular Wire to paſs through, and to a convenient Height above, the Upper Plate or Top of the *Lunarium*. Then let there be provided another Wire EM (Fig. 4.) and to one End E thereof, let therebe fixed an Ivory Ball ÆNQS, with proper Circles to repreſent the EARTH; and at the other End, ano-

* See the Plate of MECHANISM for a LUNARIUM.

The THEORY of the EQUATION of TIME.



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another Ball M to represent the MOON, one fourth Part of the Earth's Diameter. For the *Diameters of the Earth and Moon* are known to be nearly as 7964, to 2192 Miles.

3620. The Magnitude and Density of the Earth and Moon being duely considered, the common Center of Gravity A between them will be found to fall very near the Surface of the Earth; this then will be the Point on which those two Bodies will be in Equilibrio; and therefore if this Point A be fixed upon the Top of the Wire or Axle of the Plate C, both the Moon and Earth will be carried round their Center of Gravity in $29\frac{1}{2}$ Days; and in 12 Lunations or 354 Days they will be carried once about the Sun, which Revolution is called the *Lunar* YEAR, which is 11 Days shorter than the Solar Year of 365 Days; and this Deficiency of 11 Days is what the *Chronologists* call the EPACT.

3621. This Construction of a Lunarium presents us at the same Time with *another Motion of the Earth*, *viz.* its *Menstrual Motion* about the common Center of Gravity A; and hence we learn (1.) That it is not the Center of the Earth C which describes what is commonly called the *Orbis Magnus* or annual Orbit about the Sun, but the Center of Gravity A between the Earth and Moon. (2.) That the Moon also in its menstrual Motion does not regard the Center C of the Earth, but the Center of Gravity A, as the Center of her proper Motion. (3.) That the Center C of the Earth is farthest from the Sun at the *New Moon*, and nearest at the *Full Moon*. (4.) That in the Quadratures the *Menstrual Parallax* of the Earth is so sensible as to require a particular *Equation* in *Astronomical Tables*. These Points, though of so great Importance, we have not yet seen represented in *Orreries* or *Lunariums* hitherto made.

3622. Again, if a *Candle* or *Lamp* be placed in the Center of the Orrery, then will the artificial Moon, revolving about the Earth, be enlightened in the same Manner as the real Moon in the Heavens is by the Sun, and all her Phases in each Lunation with respect to the Earth will be the very same as we observe them in the Heavens.

3623. But further, if the Balls for the Earth and Moon were properly furnished with Magnets within Side, and then instead of being fixed to the Ends of the Wire E M, they were nicely suspended

suspended on fine Points of Needles placed there, with magnetical Cards shewing the several Points of the Compass, as might easily be done, then we might agreeably observe that the Earth and Moon would both constantly shew the same Part or Face to each other as they moved about the Center A; and that in turning once round, the Ends of the Wire E M would apply successively to every Point of the Card or Compass, and thereby shew that each Ball in one Revolution about the Center A, did also move once about its own Axis.

3624. But though this be the real Case of the Moon M, it is not of the Earth, which (to answer other Purposes) has another Motion communicated to it, *viz.* the *diurnal Motion* in 24 Hours, every Way independent upon the menstrual Motion about A. And therefore that the *Lunarium* may be quite similar to Nature, the Ball Æ N Q S should be still possessed of an internal Magnet, and so placed upon an Axis fixed in the End of the *Radius Vector*, as to be voluble about it at Pleasure.

3625. In this Construction, the general Phænomena of the Earth and Moon will be mutually evident; for hence it will appear (1.) That the Moon must always necessarily shew the *same Face* or Side to the Earth. (2.) That the Earth turns every Part of its Surface to the Moon once in 24 Hours: (3.) In Consequence of this, the light and dark Parts of the Moon appear permanent and fixed to us. (4.) But a great Variety of luminous and opake Parts present themselves in a constant Rotation over the Surface of the Terrestrial Ball to the Inhabitants of the Moon, if any there be. (5.) All the *Phases* of the Moon appear as in the Heavens. (6.) Also the same *Lunar* Phases will be observed in the Earth by the *Lunarians*, for we are mutually a Moon to each other. (7.) Those Phases are contrary in the Earth and Moon, for when the Moon is *new* or *dark* to us, our Earth is wholly *enlightened* or *full* to them. (8.) Our terrestrial Ball being four Times larger in Diameter, will present the *Lunarians* with a *Full moon* 16 Times as large as ours. (9.) Only one Hemisphere of the Moon being turned to the Earth, the Inhabitants of that alone can see our Globe. (10.) Our Earth appears to them always in *one Place*, or fixed in the same Part of the Heavens. (11.) The Moon has an apparent

parent diurnal Motion through the Heavens, by the Earth's real Motion on its Axis. (12.) The *Lunarians* in the opposite Hemisphere never see our Terrestrial Globe, as we can never see them.

3626. With respect to the Sun, it is evident by the *Lunarium*, (1.) That it must always enlighten one half her Globe. (2.) That every Part of the Lunar Ball is turned to the Sun in the Space of her Monthly or periodical Revolution. (3.) Therefore the Length of the Day and Night in the Moon is ever the same, and equal to $14\frac{3}{4}$ of our Days. (4.) When the *Sun sets*, the *terrestrial Moon rises* to the *Lunarians*, in the Hemisphere next the Earth. (5.) To them therefore there can never be any dark Night at all. (6) While those in the other Hemisphere can have no Light by Night but what the *Stars* afford them.

3627. Not only the Moon herself but even the very Orbit she describes about the Earth, has a sensible Motion, and is found to make one Revolution in a little less than *Nine Years*; for by the Astronomical Tables the Place of her Apogee was

			S	D	′	″
in the Year	1755	in	0 :	15 :	38 :	20
	1764	—	0 :	21 :	50 :	17
Motion in Years	9	—	12 :	6 :	11 :	57 = $366^{\circ},2$

Therefore say, as $366^{\circ},2 : 9 :: 360^{\circ} : \frac{9 \times 360}{366,2} =$ Time of one Revolution of the *Apogee*; then $\frac{9 \times 360}{366,2} \times 365 = 3229,5$ Days.

3628. Now in order to compute the System of Wheel-work for producing this Motion of the Apogee, we must divide $3229\frac{1}{2}$ Days by 59, the Teeth in the first Wheel C D (Fig. 2.) and the Quotient is 54,7; this must be broke into two other Quotients, *viz.* 7 and $7\frac{8}{10}$, both on Account of its being too large in itself, and especially because it is necessary to have two Wheels move, *viz.* F G and I K, that the latter may have a *direct Motion*, as the Apogee really has. Therefore a Pinion E of 10 Teeth on the Axis of the Wheel C D must drive a Wheel F G of 70 Teeth; and another Pinion of 10 on that Wheel must

muſt drive a Wheel IK of 78 Teeth; and then a braſs Socket fixed about the Center of this laſt Wheel, will carry about an *Ellipſis*, repreſenting the Moon's Orbit, in the Time required.

3629. Again, the Orbit of the Moon is not parallel to the Ecliptic or Earth's Orbit, but makes an Angle therewith of 5° 18′ at a Mean; now from different Forces of the Sun and Earth in different Situations, the Moon will in each Revolution be various attracted, and her Orbit agitated; being ſometimes drawn down, at other Times elevated; ſometimes it is moved forwards, and at others, backward; and thus there will be a variable Motion generated in the *Line of the Nodes*, but upon the whole the *Retrograde* Motion is greateſt, and conſequently the Nodes will move backwards in the Ecliptic, and make one Revolution in, nearly, 19 Years.

3630. For the Place of the aſcending Node ☊ in

			S	°	′	″
the Year	1746	was	11 :	27 :	1 :	4
	1765	is	11 :	19 :	30 :	32
Years,	19	——	11 :	22 :	38 :	28 = $352\frac{1}{2}$ Degrees.

Therefore ſay, as $352°,5 : 19 :: 360° : \frac{19 \times 360}{352,5}$ = Time of one Revolution in Years, or $\frac{19 \times 360}{352,5} \times 365 = 7082,5$ Days.

3631. Then 59)7082,5(= 120, very nearly; this 120 muſt be broke into *three Quotients*, that the laſt Wheel may have a *retrograde Motion*. Theſe Quotients may be 3, 5, 8; and then a Pinion A (Fig. 3.) fixed on the Axle of the Wheel CD (Fig. 2.) having 6 Teeth may drive a Wheel B of 18; this with another Pinion C of 6, drives a Wheel D of 30; and this alſo with a Pinion of 6 moves a Wheel E of 48 Teeth; and in a retrograde Direction; and therefore a Socket fixed in this Wheel will carry a Line pointing to the Place of the Nodes through all the Signs of the Ecliptic *in Antecedentia*, in the Space of $7082\frac{1}{2}$ Days as required.

3632. In the laſt Place, upon the upper Part of the Axis of the firſt Mover CD (Fig 2.) let a Pinion of 10 Leaves be fixed to drive a Wheel of 62; then a Socket (including all the

the reſt) being fixed in this Wheel will carry an *Ecliptic Circle* upon the Plate of the *Lunarium*, which may at any Time be placed in a Poſition ſimilar to the *Ecliptic* on the Plate of the Orrery, and then it will always continue ſo by Virtue of its conſtant Parallelism produced by this Mechaniſm. (3611.)

3633. If a Circle be placed about the *Line of Nodes* (3629,) and moveable upon it as an Axle, it may be occaſionally placed horizontally, or inclined to the Horizon in an Angle of 5° 18′; and then if the Moon M (Fig. 4.) be fixed to the End of a fine ſteel Wire, paſſing freely though a ſlender Socket on the End of *Radius Vector* E M, ſo that it may be over the Middle Part of the laſt mentioned Circle and conſtantly move upon it; it will be eaſy to underſtand how the Moon may be made to move either horizontally or in her proper Orbit; in order more particularly to ſhew whatever relates to the Nature, Cauſe, and various Phænomena of Eclipses.

3634. Alſo, being now furniſhed with a *Solar* and *Terreſtrial Ecliptic* it will be very eaſy to underſtand the Rationale of the general *Deceptio Viſus*, or *Illuſions of Sight* in regard to the *Motions*, *Places*, *Magnitudes* and other Affections of the heavenly Bodies, and to reſolve them all conſiſtently with Truth. Alſo from hence the *Heliocentric* and *Geocentric* Places of the Moon and Planets will readily, at any Time, appear. And thus all the Great Points and Poſitions in the *Solar and Lunar Aſtronomy* become eaſily explicable by ſuch a Conſtruction of an Orrery, Tellurian and Lunarium, as has been deſcribed.

3635. After the ſame Manner a *Jovian* Lunarium may be conſtructed with Eaſe, by knowing the Periods and Diſtances of each of *Jupiter*'s Moons, which are as in the following Table.

Satellite.	Diſtance.	Days.	H.	′	Ratio to 1 Day.	
1	$5\frac{6}{10}$	1	18	27	24	42
2	9	3	13	13	20	71
3	14,3	7	3	42	12	86
4	25,3	16	16	32	6	100

3636. The Numbers in the Ratio of theſe Periods to one Day will ſhew the Diameters and Number of Teeth in a fixed and a moveable Set of Wheels for theſe Satellites in the ſame Manner

as before in the primary Planets (3574.) That is, upon one Arbor there must be fixed four small Wheels, or Pinions, with the Numbers of Teeth 24, 20, 12, and 6, to drive four moveable Wheels of 42, 71, 86, 100 Teeth respectively. Upon these, Sockets are placed to carry the Secondaries round their Primary *Jove*, in Distances measured in Semi-diameters of his Globe expressed by the Numbers in the second Column.

3637. But now to adapt these to the dentated Edge of the Orrery, and from thence derive a proper Motion we need only consider, that if a Pinion of 8 be fitted to the Teeth of the Orrery below, and fixed to a Wheel of 64 Teeth above, playing in another Pinion of 8 fixed on the Arbor of the fixed Set of Wheels, then will that Arbor, and all its Wheels, be turned once round in every natural Day of 24 Hours; and consequently, all the Satellites will move about their Primary in such a Number of Days as they really do in the Heaven, and at proportionable Distances.

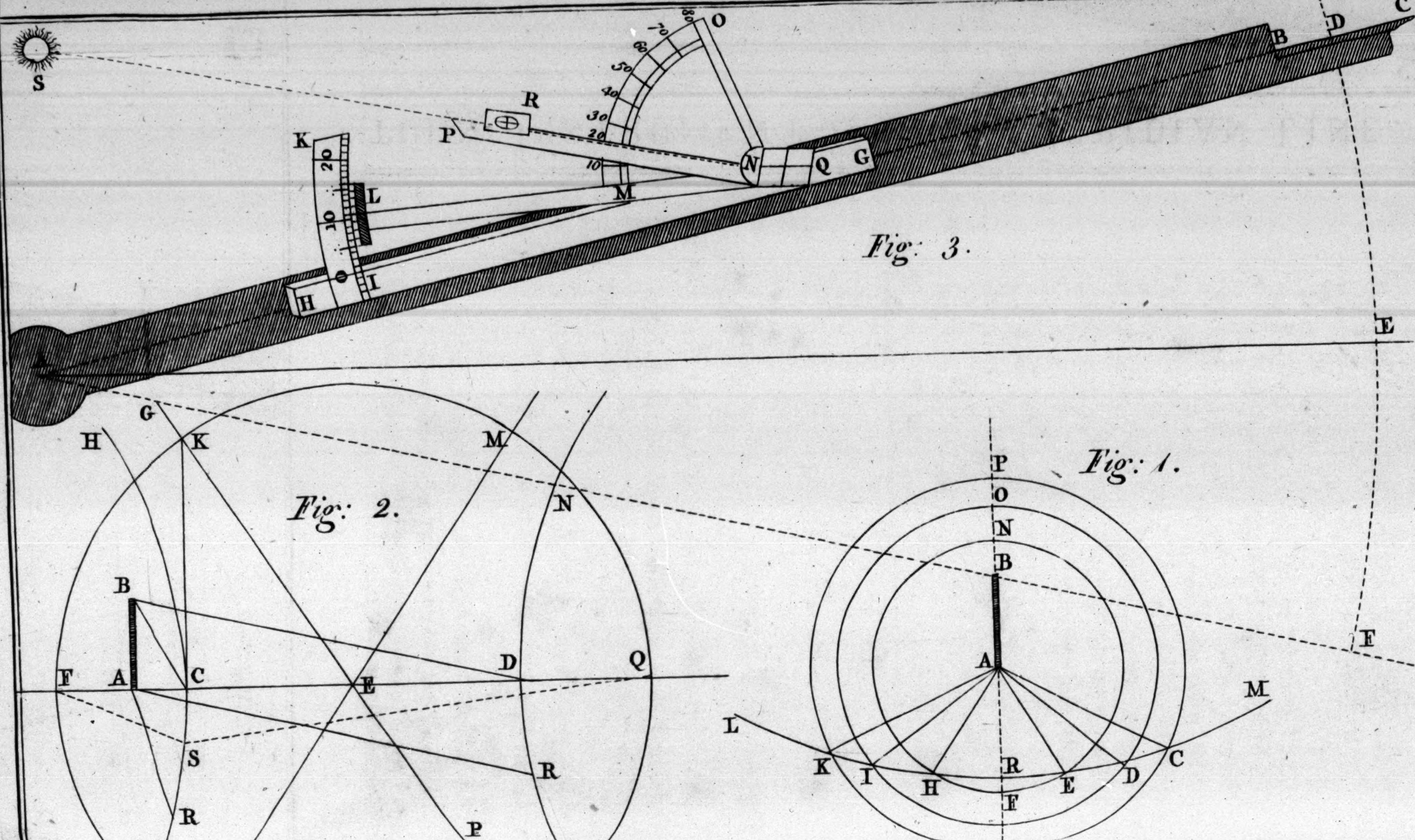
3638. If now a wax Candle be placed in the Center of the Orrery, and a Piece of white Paper on the *Radius Vector* beyond the Satellites perpendicularly; then the Motions of the Shadows of the Moons will naturally represent the apparent Motions of these Moons as seen in the Field of a Telescope, *viz.* a *Rectilineal Motion* through the Diameter of the View; as also their *direct*, *retrograde*, and *stationary* Phænomena, and lastly, those of their *Immersions* and *Emersions* in *Eclipses*; their Occultations, and Disapparition on the illumined Disk of their Primary; all which are of such Consequence in finding the Difference of Meridians, or *Longitude* of Places; and many other Points of *Astronomy*, *Navigation*, &c.

3639. A *Saturnian* LUNARIUM may be constructed in the very same Manner for 5 Moons; whose *periodical Times* are as follow, from Dr. *Halley*'s Tables.

	Days.	H.	′		Semid. of Ring.
The First revolves in	1 :	21 :	18	———	2,097
The Second ———	2 :	17 :	41	———	2,686
The Third ———	4 :	12 :	25	———	3,752
The Fourth ———	15 :	22 :	41	———	8,698
The Fifth ———	79 :	7 :	48	———	25,348

3640. The

A New Equal ALTITUDE INSTRUMENT for drawing a MERIDIAN LINE.



3640. The four inmoſt Satellites deſcribe their Orbits very nearly in the Plane of the Ring produced; which Plane makes an Angle of about 31 Degrees with the Plane of the Ecliptic; and the Nodes are in 19° 45′ of *Virgo* and *Piſces*. The Orbit of the fifth Satellite is a little wide of the Reſt. Which Particulars being obſerved in the Conſtruction of this Lunarium, it will very naturally exhibit the various Phænomena of this *Saturnian* Syſtem of Moons and Ring, for every Year of *Saturn*'s Period. *Note*, in this Lunarium, the Line of the Nodes muſt be kept always parallel to itſelf, as was ſhewn for the Line of the Earth's Nodes (3611.) And then, as *Saturn* paſſes from the Node in *Virgo* to the oppoſite one, the Sun will enlighten the northern Plane of the Ring; as it will the ſouthern Plane in paſſing through the other ſix Signs.

CHAP. XXV.

The Aſtronomical DOCTRINE *of the* EQUATION *of* TIME *explained.* A CALCULATION *of that Part ariſing from the* OBLIQUITY *of the* ECLIPTIC *to the* PLANE *of the* EQUATOR; *with a* TABLE *of this* EQUATION *for every* DEGREE *of the* ECLIPTIC.

3641. IF a Clock or Watch could be conſtructed with Mechaniſm abſolutely perfect, it would always ſhew or keep *equal Time*, or *go truly*; but as this is not the Caſe it will often want rectifying, or being ſet right. Now as we have no direct or immediate *Standard* or *Index* of equal Time, by which the Error of a Time-Piece can be pointed out and inſtantly corrected, we muſt be content, or rather, we ought to felicitate ourſelves that we have it in our Power to do this by any Means at all. For this Purpoſe we have recourſe to two Me-

 thods

thods generally; one is the *Sun-Dial* and an *Equation Table*; the other is by astronomical Observation and Calculation.

3642. The first Method is very easy, and therefore adapted to common Use, which is now to be explained; as we have before given the Theory of the 2d Method largely from Inst. 1915 to 1920 inclusive. The *Equation of Time* is a Doctrine resulting from two Principles, *viz.* (1.) The Sun's *apparent diurnal Motion* not being in the *Equator*, the only great Circle of *equal Motion* with respect to the Meridian; and (2.) The *annual apparent* Motion of the Sun not being in *a Circle*, but an *Ellipsis* which cannot admit of *Equal Motion*; as we have abundantly shewn (1236.)

3643. We shall consider each Part of the *Equation of Time* separately, and then both together. Therefore, first with respect to the Motion of the Sun being in the Ecliptic and not in the Equator; it is evident from the View of any Globe revolving upon its Axis under the Meridian, that the Motion, or rather the Velocity of the Motion, by which the several Points of the Ecliptic and Equator pass the Meridian will be very different; and that, that of the Equator will be constant and equable, and that of the Ecliptic always variable and unequal. But this Matter will be easily elucidated by a Figure.

3644. Therefore let E I (Fig. 1.) be the first Quadrant of the Equator, and E H the first of the Ecliptic; the Angle I E H being 23° 29′. Also, let P be the Pole of the Equator P H I, a Quadrant of the Solstitial Colure and P F G a Position of the Meridian very near to it; let P A B be the Meridian very nigh to the Equinoctial Point E, and P E the Meridian passing thro' that Point; then it is evident, that upon the first momentary Motion of the Globe, the very small Arches E B and E A will pass under the Meridian in the same Instant, and therefore those Arches will adequately represent the Fluxions or Ratio of the inceptive Velocities of Motion in the Equator and Ecliptic at the Equinox E.

3645. Again, as P H I is the *Solstitial Colure* and P F G a Meridian exceeding near it, the Arches G I and F H, will be as the Fluxions or ultimate Velocities with which the two Arches E G and E F become equal in the *Solstice* H. But the infinitely small Arches E B and E A may be considered

as

as right Lines, or the Angle AEB as rectilineal; and therefore EB : EA :: Co-sine of AEB : Radius :: $\dot{x}$: $\dot{y}$. And GI : FH :: Radius : Co-sine of IH (= AEB) :: $\dot{x}$: $\dot{v}$. Therefore, we have $\dot{x} : \dot{y} :: \dot{v} : \dot{x}$, or $\dot{y}\dot{v} = \dot{x}^2$, a constant Quantity; *therefore the Velocity of Motion in the Beginning of the Ecliptic* E *exceeds the constant Velocity of the Equator just as much as it falls short of it at the End of the Quadrant in* H.

3646. There is, therefore, some intermediate Point C where the Velocity in the Ecliptic is *equal* to that in Right Ascension in the Equator, which is now to be investigated; make the Arch Ed = EC, and Dd will be the Difference of the two Arches EC and ED, which, in that Case, must be a *Maximum*; for let the Arch EC = z, and ED = x; then will Dd = $z - x$, whose Fluxion, when a *Maximum*, is $\dot{z} - \dot{x} = 0$, (818,) or $\dot{z} = \dot{x}$; that is the *Fluxions or Velocities of the Arches* EC *and* ED *will be equal, when their Difference* Dd *becomes the greatest of all.*

3647. Now to determine the Quantity of the Arch EC, when Dd is a *Maximum*, we have (by *Fluxionary* Spherics 1818,) the Fluxion of EC *(ż)* to the Fluxion of ED *(ẋ)* as the Co-sine of CD to the Sine of C. Also by common Spherics (1799,) we have, the Sine of C to the Co-sine of E as Radius to the Co-sine of CD. Or thus in short, $\dot{z} : \dot{x} :: cs\,CD : s\,C$; and $s\,C : cs\,D :: R : cs\,CD$; therefore $\frac{\dot{x} \times cs\,CD}{\dot{z}} = s\,C = \frac{cs\,E \times R}{cs\,CD}$; which will give this Analogy, $cs\,CD^2 : cs\,E \times R :: \dot{z} : \dot{x}$; and therefore when C$c$ = Dd, or $\dot{z} = \dot{x}$, or when Dd is a Maximum (3646,) we have $cs\,CD^2 = cs\,E \times R$; and so $cs\,CD = \sqrt{cs\,E \times R}$. Whence EC, ED, and D$d$, are all known.

3648. The same Arch EC may be found without Fluxions by premising the following *Lemma*, viz. *The Sum of the Tangents of any two Angles* BAC, BAD, *is to their Difference, as the Sine of the Sum of those Angles is to the Sine of their Difference.* Let BC and BD (Fig. 2.) be the two proposed Tangents to the Radius AB; and take Bd = BD, join Ad, and draw DE and dF perpendicular to AC. Then it is manifest, because Bd = BD, that AD = Ad, and dAB = DAB, and consequently, that CAd is the Difference of the two Angles

BAC and BAD. Then are the Triangles CDE and CdF ſimilar, and ſo we have CD (= CB + BD) : Cd (= CB — BD) : : DE : dF; but DE and dF are Sines of DAE and dAF to the equal Radii AD and Ad, whence the Truth of the *Lemma* is evident.

3649. Then (*per* Spherics, 1789) Radius : Co-ſine E : : Tangent EC : Tangent ED (Fig. 1.) and by Compoſition and Diviſion (648) we have Radius + cs E : Radius — cs E : : Tangent EC + Tangent ED : Tangent EC — Tangent ED : : $s\overline{EC + ED}$: $s\overline{EC - ED}$ (*per Lemma.*) But the Ratio of the firſt two Terms of thoſe Analogys is conſtant becauſe the Angle E is ſo, therefore that of the two laſt Terms will be ſo likewiſe; and conſequently, $s.\overline{EC - ED}$ will be a Maximum, when the $s.\overline{EC + ED}$ is ſo, that is, when EC + ED = 90° for then the Sine thereof will be = Radius. Whence we have Radius + cs E : Radius — cs E : : Radius : $s.\overline{EC - ED}$; whence becauſe the three firſt Terms are known the fourth s.EC — ED = s.Dd, is known alſo.

3650. But there is yet a more direct and ſimple Method of coming at this Equation, becauſe when a *Maximum* it is known to be a *Third Proportional to Radius and Tangent of half the Angle* E; as will thus appear. The Radius is the Sine of 90°, and therefore is the Co-ſine of 0°. Therefore Radius + Co-ſine E : Radius — Co-ſine E : : Co-ſine 0 + Co-ſine E : Co-ſine 0 — Co-ſine E : : Co-tangent $\frac{E + 0}{2}$: Tangent $\frac{E - 0}{0}$ (1829) : : Co-tangent $\frac{1}{2}$ E : Tangent $\frac{1}{2}$ E. But it is $t : r : : r : ct$ (1831), therefore $r^2 = ct \times t$, therefore $r^2 \times t = ct \times t \times t$; whence $ct : t : : r^2 : t^2$, that is Co-tangent $\frac{1}{2}$ E : Tangent $\frac{1}{2}$ E : : Rad.^{2} : $\overline{\text{Tangent } \tfrac{1}{2} E}|^2$. Therefore Rad^2 : $\overline{t. \tfrac{1}{2} E}|^2$: : Radius : Sine of $\overline{EC - ED} = s\,Dd$, and dividing by Radius, we have Rad. Tang. $\frac{1}{2}$ E : : Tang. $\frac{1}{2}$ E : Sine of Dd, as was to be demonſtrated.

3651. The Quantity of this Equation computed is Dd = 2° 28′ 34″, which converted into *Time*, is 9 Minutes 54 Seconds; and this is the greateſt Difference of Time that would ever be found between the Sun-Dial and a Clock if the Obliquity

liquity of the Ecliptic or Sun's Path were the only Cauſe thereof. Now ſince $EC + ED = 90$, and $Dd = (EC - ED =)$ 2° 28′ 34″, therefore $EC = 46°$ 14′, therefore the Equation begins from nothing in *Aries* ♈ or in the Point E, and increaſes to its Maximum in ♉ 16° 14′ and from thence decreaſes till the Sun arrives at ♋ in the Point H, where it again becomes nothing.

3652. In this firſt Quarter of the Ecliptic, it is evident, that had the Sun moved in the Equator it would have been in *d* at the Moment it is in the Ecliptic at C, and conſequently the Sun at C is in the Meridian P D, which is before the Time the *Equinoctial Sun* arrives to it, at *(d)*; as being then *Eaſtward* of it by the Difference in Time every correſponding to the Motion in Right Aſcenſion of the ſmall Arch D *d*. Whence it appears that the *apparent Time* of the *Ecliptic Sun*, or that ſhewn by a *Dial*, is in this firſt Quadrant always *before* the *Mean or Equal Time* in the Equator by the Difference belonging to the Equation D *d*, which, therefore, when found, muſt be ſubducted from the *Solar Time* to have the mean Time for the Watch or Clock.

3653. By making the ſame Conſtruction in the ſecond Quadrant of the Ecliptic, it will appear, that the Point *(d)* is Weſtward of the Point C, and therefore comes firſt to the Meridian Arch. Whence the *mean Time* is now before the *apparent or Solar Time*, and conſequently the Equation D *d* now becomes *Addititious*, or muſt be added to the Solar Time to have the Mean, which now precedes it. Thus when the Sun is in ♌ 43° 46′ (= C H,) if we add 9′ 54″ to the Time ſhewn by the *Dial* it will give the *Mean Time* for the *Watch*; for ſo much does the *true Noon* of that Day exceed the *apparent Noon* on Account of the Sun's *oblique Motion*.

3654. It is evident the ſame Equation will be produced in the 3d and 4th Quadrants of the Ecliptic, and will accordingly be *Ablatitious* and *Addititious*, as is ſhewn in the following Table, where the Quantity of that Equation is computed for the Sun's Place in every Degree of the Ecliptic.

Subtract

Subtract from the Apparent Time.

The true Place of the Sun.

Signs	♈ ♎		♉ ♏		♊ ♐		
Deg.	′	″	′	″	′	″	
0	0	0	8	23	8	45	30
1	0	20	8	34	8	35	29
2	0	40	8	44	8	24	28
3	1	0	8	53	8	13	27
4	1	19	9	2	8	1	26
5	1	39	9	10	7	48	25
6	1	58	9	17	7	34	24
7	2	18	9	24	7	20	23
8	2	37	9	30	7	6	22
9	2	56	9	35	6	50	21
10	3	15	9	40	6	35	20
11	3	34	9	44	6	18	19
12	3	52	9	48	6	2	18
13	4	11	9	50	5	44	17
14	4	29	9	52	5	27	16
15	4	46	9	54	5	8	15
16	5	4	9	54	4	50	14
17	5	21	9	54	4	31	13
18	5	37	9	53	4	11	12
19	5	54	9	51	3	52	11
20	6	10	9	49	3	32	10
21	6	25	9	46	3	11	9
22	6	40	9	42	2	51	8
23	6	55	9	38	2	30	7
24	7	9	9	33	2	9	6
25	7	23	9	26	1	48	5
26	7	36	9	19	1	26	4
27	7	48	9	12	1	5	3
28	8	1	9	4	0	43	2
29	8	12	8	55	0	22	1
20	8	23	8	45	0	0	0
	′	″	′	″	′	″	Deg.
	♓ ♍		♒ ♌		♑ ♋		Signs

Add to the apparent Time.

CHAP. XXVI.

A CALCULATION *of that Part of the* EQUATION *of* TIME *which arises from the* Elliptic FORM *of the* EARTH'*s* ORBIT; *with a proper* TABLE *thereof for every* DEGREE *of the* EARTH'*s* ANOMALY.

3655. THE second Cause of the Inequality of Time shewn by a DIAL and a CLOCK, was said to be owing to the *Elliptic Form* of the Earth's annual Orbit (3642), and this is known to be Fact from common Observation; for through every Year the Diameter of the Sun is found to subtend a variable Angle, being sensibly greater at one Time than at another; for a few Days after the *Winter Solstice* it measures, by the Micrometer, about 32′ 43″, but the same Distance after the *Summer Solstice* it is no more than 31′ 38″.

3656. Now were the Orbit of the Earth truly *circular* it would be always at an equal Distance from the Sun, which would therefore always appear of an equal Bigness; consequently, as it does not so appear, it must be at unequal Distances from us to cause those unequal apparent Magnitudes (1451.) And because the greatest and least solar Diameters are as 32′ 42″ = 1963″, and 31′ 38″ = 1898; the greatest and least Distances of the Earth from the Sun will be inversely as the same Numbers 1963 and 1898.

3657. Let this Ellipse Orbit of the Earth be denoted by A B P D, (Fig. 3.) in whose Focus S is the Sun; and let the other Focus be F. Upon the Center S describe a Circle G I K L, whose Diameter G K is a *Mean Proportional* between the two Axis A P and B D of the Ellipsis; then will the Area of that Circle be equal to the Area of the Ellipsis (894.) And we may now compare the Motions in the Ellipsis and Circle as they would each be described in the same Time; and thus investigate an Equation by which the unequal Motion of the Earth in the former, and the equal Motion of a Point in the latter, may be always equated, or adjusted to each other.

3658. Suppose now the Earth begins its Motion from the *Aphelion* A of the Ellipse, at the same Time that the imaginary Point sets out with an equable Motion in the Circle from G; then because the Distance A S exceeds the Distance S G, the Velocity of the Earth will be less than that of the Point (1213), and therefore in the Time that the Earth describes the Arch A *a*, the Point will have described G *g*, making the Area G S *g* equal to the Area A S *a*, pursuant to the general Law of Nature. (1212.)

3659. Because the Area A S *a* = G S *g*, suduct from each Side the common Part or Area G S *d*, and there will remain the Area A G *d a* equal to that of the Triangle *d* S *g*; therefore the said Area A G *d a* will be ever proportional to the *Equation of the Orbit*, or the Arch (*g d*). Consequently when this Area becomes a *Maximum*, there the Equation will be greatest also, which is in the Point L where the Circle intersects the Ellipse, for then the said Area becomes A L G.

3660. As in the Point L the Earth in its Orbit, and the imaginary Point in the Circle have both the same Distance from the Sun, L S; they have there the same Velocity of Motion; and after they have passed that Point L, the Earth approaching constantly nearer the Sun, will have a Velocity greater and constantly gaining upon that of the Point in the Circle, but still the Point in N will be before the Earth at H, till at last the Angle H S N or Equation of the Orbit will vanish in the Line S P K, where the Earth arrives to its *Perihilion* P, and the Point in the Circle to K.

3661. The Area A S H being still equal to the Area G S N, we have at length, the Semi-Ellipsis A L P = the Semi-circle G L K; and subducting the common Area G L P, there will remain the Area A L G = the Area P L K. And as the Equation arose from nothing in the Point A to its greatest Magnitude at L, by *continual Increments*; so after passing the Point L, it must lessen by *decreasing Quantities*, which constitute the Area P L K; so that what was gained in the first Part, is lost in the latter, and the Equation becomes nothing in the Line S K.

3662. Or thus; continue the Ray S H to M; then the Eliptic Sector A S H = G S N, as being described in the same Time;

Time; from each, take the common Part G L H S, and there will remain A L G = N L O + O S H = M H L + M S N; therefore A L G — M H L = K M H P = M S N, the Measure of which Angle is the Equation M N; and is therefore nothing when M H L becomes equal to A L G or P L K.

3663. In the first half of the Ellipsis, or while the Earth descends from the higher *Apsis* or Aphelion A to the lower P, the *Mean Anomaly* will ever be greater than the true, or the Place of the imaginary Point *(g)* will be to the East of the Point *(d)*. And therefore if we suppose the Earth at rest in the Center at S, and the Circle G L K I to be the *Primum Mobile*, then *(d)* will be the Place of the Sun in the Ecliptic, and *(g)* that of the Point of *Mean Motion*; also A S P will be the *Meridian*, to which when *d* arrives, it will be Noon by the *Sun-Dial*, but when *g* comes to it it will be XII by the Watch. And since these apparent Motions of the Heavens are from East to West, it is evident, the *Time of Noon by the Dial, will preceed that by the Clock*; and consequently the Equation or Arch *g d* (turned into Time) must be *subtracted from the Time by the Dial*, to have the Mean Time by the Clock or Watch, as that is now *slower than the Dial.*

3664. But all the Time the Earth ascends from the lower *Apsis* or *Perihelion* P, to the Point of Interfection I, it will be nearer to the Sun at S than the Ecliptic or Circle, and consequently its Motion will be swifter than that of the imaginary Point; therefore its Place *p* at any Time will be before that of the said Point at *k*, and consequently will come after it to the Meridian at P, in the *apparent Motions*; therefore the Mean Time will be before the solar Time, or the *Clock will now be faster than the Dial*; and the Equation *e p* must be *added to the Time by the Dial* to have the mean Time by the *Watch*.

3665. This will also continue to be the Case till the Earth arrives at the *Aphelian* A; for though in passing from the mean Distance at I to the greatest at A, the Velocity of the Earth at Q is less than that of the Point R in the Ecliptic, and is constantly decreasing, yet it will always be before the said Point; and therefore, with Respect to the *apparent Motion*, it must come later to the Meridian than the Point of equal Motion R, and so

 the

the Equation R q is still to *be added to the solar or apparent Time by the Dial*, to have the *mean Time by the Clock*.

3666. Here R q is the *Equation*; and the Area or Sector KSR $=$ PSQ, from which take the common Area PSRI and there will remain $PKI = QIq + RSq$, and therefore $PKI - QIq = RSq$, is proportional to the Equation R q, (see 3662;) therefore when Q I q becomes AIG ($=$ PKI), the Equation of the Orbit vanishes again.

3667. Let GN be any given *mean Motion* (Fig. 4.) and AH the corresponding *true Anomaly*; draw S h, S n, indefinitely near to SH and SN; and put N $n = \dot{y}$, and H $h = \dot{z}$; then because the fluxionary Triangle HSh $=$ NSn, we have the Angle HSh : NSn :: NS^2 : HS^2 (1270) :: $\dot{z} : \dot{y}$. Therefore $\frac{NS^2 \times \dot{y}}{HS^2} = \dot{z}$, Fluxion of the true Anomaly AH.

3668. When the Fluxions of Quantities are equal, their Difference will be a *Maximum* (3646;) therefore in L, where $SH = SN = SL$ we have $\dot{z} = \dot{y}$; and consequently the *Equation of the Orbit* GN — AH, will there be the greatest possible, as we before shewed, (3659.)

3669. We have found Areas, Angles, and Arches, proportional to the Equation of the Orbit; but to compute the real Quantity of this Equation, we must first solve the so much famed Problem of KEPLER, *viz. To cut off a Sector* ASa *by the Right Line* Sa (Fig. 3.) *that shall be to the whole Area of the Ellipse, as the Time of describing the Arch* Aa, *to the Time of a whole Revolution in the Elliptic Orbit*. But as the most geometrical Way of doing this, is by *infinite Series*; and there is a much easier and very exact Method of computing the Equation invented by the late Bishop *Ward* we shall here explain that, with *Bullialdus*'s Correction thereof.

3670. This Method depends on an Hypothesis, *that a Ray* FL (Fig. 5.) *drawn from the upper Focus* F *to the Planet at* L, *describes an Angle* AFL *proportional to the Time of the Planet*'s *passing through the Arch* AL. Therefore let ABP be the Ellipse the Planet describes; AP the Line of the *Apsides*; S the Focus in which is the Sun, F the other Focus or Center of equal Motion. The Angle AFL being as the Time, the Place of the Planet will

be

be at L, and the Angle ASL is the *true* or *coequate Anomaly*.

3671. Produce FL towards E, and make FE = AP, and join ES. Then is LE = LS (769,) and the Triangle ELS being isoscles, the Angle LES = LSE, and both together equal to the Angle FLS (632.) Therefore in the Triangle EFS, having the Sides FE and FS, and the Angle of mean Anomaly AFE, or EFS, we find the Angle E, and 2E = FLS, which is the Difference between the mean Anomaly AFL, and the True ASL, and is therefore the Value of the *Equation sought*; for this taken from the mean Anomaly AFL leaves the True one ASL.

3672. Also in the Triangle FLS, having all the Angles and the Side FS, the Side SL is known, which is the Distance of the Earth from the Sun, in its Orbit at L. And thus the Equation of the Orbit, and Distance from the Sun, is found in this Hypothesis very exactly for the Earth; but for the other Planets, especially *Mars* and *Mercury*, it requires some Correction, which it received from the celebrated Astronomer *Ismael Bullialdus*, as follows.

3673. Upon the transverse Axis AP describe the Circle ADP, and let AFL be the mean Anomaly, as before. Thro' L draw the Line QLG perpendicular to the Axis, meeting the Circle in Q; join FQ, cutting the Ellipse in M, and M will be the Place of the Planet in its Orbit for the mean Anomaly AFL. Let BC be the Semi-conjugate, which continued to D, we have CB : CD :: GL : GQ, which is therefore known; join SM, and then the Angle ASM will be found in the same Manner as before the Angle ASL was found, (3671.)

3674. This Correction of *Bullialdus* accelerates the Motion of the Planets in the *first and third Quarter*; and in the *Second and Fourth retards* them a very little Matter, in Respect of *Ward*'s Hypothesis, which makes their Places by this Theory agree much better with Observations. But the late Mr. SIMPSON has given us a Construction which finds the Planet's Place still much nearer the Truth than either of the foregoing.*

3675. The general Reason of this Hypothesis is this, that the Velocity in the Orbit being every where inversely as a Perpendicular

* See his *Mathematical Essays*, p. 41.

pendicular on the Center of Force S upon a Tangent to the Orbit in a given Point or Place of the Planet (1213), it will follow that Aa, Bb, Pp be Spaces passed through in equal Times; then drawing the Lines Fa, Fb, Fp, the Angles AFa, BFb, PFp, will in the Earth's Orbit, be extremely near equal. For at A and P we have Aa : Pp :: PS : SA :: AF : FP; therefore AFa = PFp. Again, Aa : Bb :: BS : SA :: FB : FP; but in the Earth's Ellipse F is very near to C, and we shall have FB : FP :: AF : FB, very nearly; consequently the Angle BFb = AFb, extremely near. Therefore *the angular Motion about the upper Focus* F *is very near equable.**

3676. When the Planet is in B, the middle Point of its Semi-Ellipse, then FE becomes FN, and FB = BN = AC = SB; and FSN is a Right Angle, and since the Point B very nearly coincides with the Point I (Fig. 3.) where the Equation of the Orbit is a *Maximum*, therefore the Angle FBC will insensibly differ from the Half of that Equation. And because (3656) AS is as 1963, and SP as 1898, therefore AC = FB, will be as 1930,5, and AS — AC = CS = CF, will be as 32,5; therefore say As FB = 1930,5 : FC :: Radius : Sine of FBC = 58′ 9″½, which doubled, 1° 56′ 19″, = FBS, the greatest Equation of the Earth's Orbit; and such you find it in the Astronomical Tables of Dr. HALLEY. This Equation turned into Time for every Degree of the Earth's Anomaly, is as in the following Table.

* Thro' Inadvertence, the Lines, Fa, Fb, Fp, were forgot to be drawn in Fig. 5. but they can be easily supplied by the Reader, and it is therefore hoped will occasion no Obstruction to his understanding this material Point.

Subtract

Subtract from the Apparent Time.

The Mean Anomaly of the Sun.

Signs	0		1		2		3		4		5		
Deg.	′	″	′	″	′	″	′	″	′	″	′	″	
0	0	0	3	48	6	39	7	45	6	47	3	57	30
1	0	8	3	55	6	43	7	45	6	43	3	50	29
2	0	16	4	2	6	47	7	45	6	39	3	43	28
3	0	24	4	9	6	51	7	45	6	35	3	35	27
4	0	32	4	16	6	54	7	45	6	30	3	28	26
5	0	40	4	22	6	58	7	44	6	26	3	20	25
6	0	48	4	29	7	1	7	44	6	21	3	13	24
7	0	56	4	35	7	5	7	43	6	16	3	5	23
8	1	3	4	42	7	8	7	42	6	11	2	58	22
9	1	11	4	48	7	11	7	41	6	6	2	50	21
10	1	19	4	54	7	14	7	40	6	1	2	42	20
11	1	27	5	0	7	17	7	39	5	56	2	34	19
12	1	35	5	7	7	19	7	37	5	51	2	27	18
13	1	12	5	12	7	22	7	36	5	45	2	19	17
14	1	50	5	18	7	25	7	34	5	40	2	11	16
15	1	58	5	24	7	27	7	32	5	34	2	3	15
16	2	6	5	30	7	29	7	30	5	28	1	55	14
17	2	13	5	35	7	31	7	28	5	22	1	47	13
18	2	21	5	41	7	33	7	25	5	16	1	39	12
19	2	28	5	46	7	35	7	23	5	10	1	31	11
20	2	36	5	51	7	36	7	20	5	4	1	22	10
21	2	43	5	57	7	38	7	18	4	58	1	14	9
22	2	51	6	2	7	39	7	15	4	51	1	6	8
23	2	58	6	7	7	41	7	12	4	45	0	58	7
24	3	6	6	12	7	42	7	9	4	38	0	50	6
25	3	13	6	17	7	43	7	5	4	32	0	41	5
26	3	20	6	21	7	43	7	2	4	25	0	33	4
27	3	27	6	26	7	44	6	59	4	18	0	25	3
28	3	34	6	30	7	44	6	55	4	11	0	17	2
29	3	41	6	35	7	45	6	51	4	4	0	8	1
20	3	48	6	39	7	45	6	47	3	57	0	0	0
	′	″	′	″	′	″	′	″	′	″	′	″	Deg.
	11		10		9		8		7		6		Signs

Add to the apparent Time.

CHAP. XXVII.

The THEORY *of the* Compound EQUATION *of* TIME, *and the* CONSTRUCTION *of one* General TABLE *thereof depending on the* SUN'S PLACE *in the* ECLIPTIC.

3678. THE two Equations of Time we have now consider-ed must be both applied to obtain a general or absolute Equation, consisting of the Sum or Difference of these particular Ones, according to their affirmative or negative Quality; but because One requires the Knowledge of the Sun's Place in the Ecliptic, and the other, the Sign and Degree of the Earth's or Sun's mean Anomaly; therefore to render this Matter less troublesome or difficult, Astronomers have composed one general Table out of the two, making it depend entirely on the Sun's Place in the Ecliptic.

3679. But this general Table can be only a *temporary One*, as serving with any Exactness only a certain Time, because the Line of the Apsides A P, from which the Numbers in the second Table begin, is not fixed with respect to Signs of the Ecliptic, but has a direct slow Motion through the same at the Rate of one Degree in 59 Years, or $1° 41' 7''$ *per Century*. Consequently in about 30 or 40 Years this general Table will require to be renewed where any Computations of Accuracy are concerned, but for barely setting common Clocks by a Dial, it may serve much longer.

3680. By Dr. *Halley*'s Tables, the Position of this Line A P is in $8° 42' 52''$ of *Cancer* and *Capricorn* at the Beginning of the Year 1764. But as the Theory of this compound Equation is not quite so simple and evident as every young Horologist might wish, I shall endeavour to render it plain by an algebraic *Calculus*, and to illustrate the same by a proper Diagram; both which have been hitherto wanting in Books treating of this Subject. But to facilitate these Demonstrations the two following *Lemmata* are to be premised.

3681. In

3681. *In any Triangle* FHS (Fig. 6.) *we have* $\overline{FH + SH} \times \overline{FH - SH} = FS \times 2CV$; *supposing* HV *perpendicular to* FS, *and* FC = CS. For with HF as a Radius, on the Point H as a Center, describe the Circle ABF and continue the Side SH each Way to the Circle, and the Base FS to the Circle in B. Then is AH = HF, and AS = FH + SH, and SE = FH — SH; and because it is ES × SA = FS × SB (658,) and SB = FV + VS = FS + 2VS; therefore $\frac{1}{2}$SB = $\frac{1}{2}$FS + VS = CV, consequently 2CV = SB. Therefore $\overline{FH + SH} \times \overline{FH - SH} = FS \times 2CV$.

3682. *The Rectangle of the Sines of two Arches added to the Rectangle of their Co-sines, make a Sum equal to the Rectangle under the Radius and Co-sine of their Difference.* (Fig. 7.) Let the two Arches be AC, and CD (= CB); their Sum AD, and Difference AB; let CF and OF be the Sine and Co-sine of the greater Arch AC; and let mD (= mB) and Om be those of the lesser Arch CD, or OB. Also let BE and OE be those of the Difference AB. Draw mn parallel to CF; and mv parallel to AO; then it is plain the Triangles OCF, Omn, and Dmv are similar; therefore we have OC : OF :: Om : On, whence OC × On = OF × Om. Again, OC : CF :: Dm : mv, therefore OC × mv = CF × Dm. Consequently $OC \times \overline{On + mv} = OC \times OE = \overline{OF \times Om} + \overline{CF \times Dm}$. *Q. E. D.*

3683. These Lemmas premised, if we look back on Fig. 4. we shall there find the same Triangle FHS (as in Fig. 6.) by drawing the Line FH, and letting fall the Perpendicular HV. Put AC = CP = a, CB = b, CS = CF = c, SH = v, and the Co-sine of the Angle HSP = x, to the Radius SK = SN = 1. Then 1 : x :: v : vx = SV, also FH = AP — SH (769) = $2a - v$, and FS = $2c$; wherefore $\overline{FH + SH} \times \overline{FH - SH} = FS \times 2CV$; that is, $2a \times \overline{2a - 2v} = 2c \times 2 \times \overline{c + xv}$; from whence we have $v = \frac{a^2 - c^2}{a + cx} = \frac{b^2}{a + cx}$ = SH, the *Distance of the Earth from the Sun.*

3684. And becaufe $SN^2 = SK^2 = ab$ (3657), we fhall have $\frac{SN^2}{SH^2} \times \dot{y} = \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y}$, the Fluxion of the true Longitude AH, or Diftance from the Aphelion A. Having thus obtained an Expreffion of the Increment of the Longitude for the *Elliptic Anomaly*, and having before found it for the *Obliquity of the Ecliptic* (3647), we can find how far both thefe Caufes together will affect the Motion of the Earth in *Right Afcenfion*, fince in that alone confifts the whole Ground and Reafon of the *Equation of Time*; and if we thus unite both their Effects we fhall have the whole or *abfolute Equation* in one Theorem, for any Value of x, or Longitude of the Earth from either Apfis A or P.

3685 For the Fluxion of the Longitude or true Anomaly being to that of the Equator or Motion in Right Afcenfion always as $\overline{cs CD}^2 : cs E \times R :: \dot{z} : \dot{x}$, (3647.) Therefore fay, As $\overline{cs CD}^2 : cs E \times R :: \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} : \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} \times \frac{cs E \times R}{\overline{cs CD}^2}$ = the Fluxion of the Motion in the Equator, arifing from both the Caufes of Irregularity united.

3686. Now this being compared with the equable Motion in Right Afcenfion of the imaginary Point in the Circle G L P I, it will eafily appear when their Difference is the greateft or a Maximum, becaufe in that Cafe their Fluxions will be equal (3646,) *viz.* $\frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} \times \frac{cs E \times R}{\overline{cs CD}^2} = \dot{y}$; and confequently in that Cafe $\frac{a \times \overline{a + cx}^2}{b^3} = \frac{\overline{cs DC}^2}{cs E \times R}$:

3687. But as this latter Part of the Equation is not of the fame Form with the firft, it muft be reduced to algebraic Terms; therefore put m = Sine of K S ♋, the Diftance of the Perihelion P from the Solftice ♋; its Co-fine = n; then the Co-fine of the Angle P S H being = x, its Sine will be = $\sqrt{1 - xx}$; and by the Lemma (3682) we have $nx + m\sqrt{1 - xx}$ = Co-fine of H S ♋, or C S ♋ = Sine of ♈ C or Longitude of the Earth at H reduced to the Ecliptic G L K I.

3688. Let

3688. Let ENM (Fig. 8.) be the Projection of that half of the Equator which is above the Plane of the Ecliptic, and EQM of that half which is below it; and CD a Perpendicular thereto; then is the Right-angled Triangle CED the same with that in Fig. 1. Put $p =$ Sine of the constant Angle E, and $q =$ Co-sine, and we have (by *Spherics*,) Radius (1) $: s.\text{EC}\ (nx + m\sqrt{1 - x^2}) :: p : pnx + pm\sqrt{1 - x^2} =$ Sine of CD, whose Square Co-sine is therefore $1 - \overline{pnx + pm\sqrt{1 - xx}}^2 = \overline{cs.\text{CD}}^2$; and the $cs\,\text{E} = q$; these Values substituted in the Equation above, give $\frac{a \times \overline{a + cx}^2}{b^3} = \frac{1 - \overline{pnx + pm\sqrt{1 - xx}}^2}{q}$.

This Equation reduced, gives the Value of x, or Quantity of the Angle HSP.

3689. While the Angle HSP is *acute*, x will be affirmative, or $+x$; but when *obtuse* it will be negative, or $-x$. Also while H is in the Semi-ellipsis ALP, m will be affirmative or $+m$; but when H is in the other Half PIA it is negative, or $-m$; and the Equation there becomes $\frac{a \times \overline{a + cx^2}}{b^3} =$ $\frac{1 - \overline{pnx - pm\sqrt{1 - xx}}^2}{q}$.

3690. From all which it is evident, that this *Compound Equation of Time*, as it arises from the *Sums* and *Differences* of the two single Equations must be of a different Value in different Parts of the Orbit; and that there will be *four* of those *Maxima* in all, *viz.* two arising from the Sums, while x is positive; and two from the Differences of the single Equations, when x is negative, or $-x$.

3691. But to make this Matter yet plainer, let A = the Equation for the Earth's *Elliptic Orbit*, and B = that for the *Obliquity of the Eclipic*; then all the Time the Earth H is descending from A to P, the former is negative, or — A (3663.) And in ascending from P to A, it will be affirmative or + A. Also in the 1st and 3d Quarters of the Ecliptic we have — B; but in the 2d and 4th it is + B (3653, 3657.)

3692. Now 'tis evident, in the Diagram (Fig. 8.) that the first Quadrant of the Ecliptic ♈ ♋ wholly coincides with the

Semi-Ellipſe A L P, and therefore the *Maximum* Equation will here ariſe from A + B, which, as they are both *Negative*, ſhews this Equation is ſo too, or that it muſt be *taken from* the Time by the Dial, to have the mean Time by the Watch or Clock, which is then *too ſlow*. This Equation is 16′ 13″ and is when the Earth is in ♉ : 10°, or the Sun in ♏ : 10°, on *November* 2d.

3693. Again, almoſt the whole of the 2d Quadrant of the Ecliptic is in the other Semi-Ellipſe P I A; conſequently the two ſimple Equations here being both poſitive their Sum A + B will be ſo too, or muſt be *added* to the *Solar Time* to get the Mean by the *Watch*, which is here *too faſt*. This Maximum amounts to 14′ 49″ on *February* 10th. When the Earth is in ♌ : 21° : 35′, or the Sum in ♒ 21° : 35′.

3694. The third Quadrant of the Ecliptic is wholly on the ſame Side with the Semi-Ellipſe A I P, and the Earth will enter it before the third *Maximum* Equation; therefore B will be *Negative* and that Equation will be A — B = 4′ 5″, to be *ſubducted* from the ſolar Time by the Dial, becauſe now B is greater than A, and the Clocks are *too ſlow*, again. This happens on *May* 15th, when the Earth is in ♏ : 24° : 18′ or the Sun in ♉ 24° 18′.

3695. Laſtly, the laſt Quadrant of the Ecliptic is nearly the whole of it on the ſame Side of the Line of the Apſides A P with the Semi-ellipſe A L P; here B will be poſitive, or + B, and A negative, or — A, and therefore, the Equation, when greateſt, will be = B — A = 5′ 55″; and becauſe B is here alſo greater than A, therefore the Equation is to be added to the apparent Time by the Dial to have the true or equal Time by the Clock or Watch which is now *too faſt*, or before the Dial. This *Maximum* happens on *July* 26, the Earth being in 3° of ♒, and the Sun in 3° of ♌.

3696. If the Reader attentively conſiders the ſeveral particular Circumſtances and Relations of theſe two ſimple Equations he will ſee how this Compound Equation will gradually ariſe by *Addition* and *Subtraction*, and increaſe to its various *Maxima*, and then alternately decreaſe to nothing, according as the Numbers appear in the following Table, which is here adjuſted for ready Uſe to the *Signs* and *Degrees of the Ecliptic*.

A Table of the Compound Equation of Time depending on the Sun's Place.

Signs	♈		♉		♊		♋		♌		♍	
	Add.		*Sub.*		*Sub.*		*Add.*		*Add.*		*Add.*	
Deg.	′	″	′	″	′	″	′	″	′	″	′	″
0	7	40	1	9	3	56	1	6	5	49	2	14
1	7	21	1	23	3	52	1	19	5	51	1	58
2	7	2	1	36	3	47	1	33	5	52	1	41
3	6	43	1	49	3	43	1	47	5	53	1	24
4	6	24	2	1	3	38	2	0	5	53	1	7
5	6	5	2	13	3	32	2	13	5	52	0	50
6	5	46	2	23	3	25	2	26	5	51	0	32
7	5	27	2	34	3	18	2	38	5	49	0	13
8	5	8	2	44	3	11	2	51	5	47	—	5
9	4	49	2	53	3	3	3	4	5	44	0	24
10	4	30	3	2	2	54	3	16	5	40	0	44
11	4	11	3	10	2	44	3	27	5	35	1	4
12	3	53	3	17	2	35	3	38	5	30	1	24
13	3	34	3	25	2	24	3	50	5	24	1	44
14	3	16	3	32	2	15	4	1	5	17	2	4
15	2	57	3	39	2	4	4	11	5	10	2	24
16	2	39	3	45	1	53	4	21	5	2	2	45
17	2	21	3	50	1	41	4	30	4	54	3	6
18	2	3	3	54	1	29	4	39	4	45	3	27
19	1	45	3	57	1	18	4	48	4	36	3	48
20	1	27	4	0	1	6	4	57	4	26	4	9
21	1	10	4	2	0	53	5	5	4	15	4	30
22	0	53	4	4	0	41	5	12	4	4	4	51
23	0	37	4	6	0	28	5	18	3	52	5	12
24	0	20	4	6	0	15	5	24	3	39	5	34
25	0	4	4	6	0	2	5	31	3	27	5	55
26	—	11	4	5	+	12	5	36	3	13	6	17
27	0	26	4	4	0	25	5	40	2	59	6	39
28	0	41	4	2	0	39	5	43	2	45	7	0
29	0	56	3	59	0	53	5	46	2	30	7	21
30	1	9	3	56	1	6	5	49	2	14	7	42

The Equations following the Sign + to be added, and the Sign — to be subducted from the apparent Time, to get the Mean Time.

A Table of the Compound Equation of Time depending on the Sun's Place.

♎		♏		♐		♑		♒		♓	
Sub.		*Sub.*		*Sub.*		*Sub.*		*Add.*		*Add.*	
′	″	′	″	′	″	′	″	′	″	′	″
7	42	15	32	13	29	1	4	11	38	14	26
8	3	15	40	13	13	0	35	11	55	14	20
8	24	15	47	12	56	0	5	12	12	14	13
8	44	15	53	12	38	+	25	12	27	14	6
9	4	15	58	12	19	0	54	12	41	13	59
9	24	16	2	11	59	1	23	12	54	13	50
9	43	16	5	11	38	1	53	13	7	13	41
10	3	16	8	11	17	2	22	13	20	13	30
10	22	16	10	10	55	2	51	13	32	13	20
10	41	16	11	10	33	3	19	13	43	13	9
11	0	16	11	10	10	3	48	13	53	12	57
11	18	16	11	9	46	4	16	14	2	12	45
11	36	16	10	9	22	4	43	14	11	12	33
11	53	16	8	8	58	5	11	14	19	12	20
12	10	16	5	8	34	5	38	14	25	12	6
12	27	16	1	8	8	6	4	14	31	11	52
12	43	15	57	7	42	6	30	14	36	11	37
12	59	15	52	7	15	6	55	14	40	11	22
13	14	15	46	6	48	7	21	14	43	11	7
13	29	15	39	6	21	7	45	14	46	10	52
13	43	15	32	5	53	8	10	14	48	10	36
12	57	15	24	5	25	8	34	14	49	10	19
14	10	15	14	4	58	8	57	14	50	10	2
14	22	15	3	4	30	9	19	14	50	9	45
14	34	14	51	4	1	9	41	14	49	9	28
14	46	14	39	3	32	10	2	14	47	9	11
14	56	14	27	3	2	10	23	14	45	8	53
15	6	14	14	2	33	10	42	14	41	8	35
15	15	14	0	2	3	11	1	14	37	8	17
15	24	13	45	1	33	11	20	14	32	7	59
15	32	13	29	1	4	11	38	14	27	7	40

The Equations following the Sign + to be added, and the Sign — to be subducted from the apparent Time, to get the Mean Time.

CHAP. XXVIII.

Of the best Methods *of drawing a* Meridian Line *by* Concentric Circles, *by the* Hyperbola, &c. *The Theory of a* New Equal Altitude Instrument *for that Purpose.*

3698. As Clocks and Dials are of a like Nature and Uſe, ſo they are mutually ſubſervient to each other, and ſerve to correct one another, when either ſhall chance to be at Fault; but to this End it muſt be ſuppoſed that they are originally each of them properly conſtructed, and the Dial, particularly, fixed in a *true Poſition.* And as this is ſo important a Point, we here propoſe to give ſuch Directions and Precautions as are neceſſary to be obſerved in that Affair.

3699. The firſt Thing to be obſerved in fixing a *Horizontal Dial* is, that the Surface of the Pedaſtal be *truly level* or *horizontal*, and this muſt be examined and thoroughly rectified by a Plumb-line, or Spirit Level. The Reaſon of this Injunction is becauſe the Gnomon of the Dial cannot be *parallel to the Axis of the Earth* without ſuch an *horizontal Poſition*; as we have ſhewn when we treated of Dialling.

3700. The ſecond Thing neceſſary is to place the Meridian of the Dial truly *North and South*, or exactly in the *Meridian of the Place*; for this Purpoſe it will not be ſufficient to ſet it by a *Magnetical Needle* in a Rectangular Box, neither by a Clock, Watch, or any other Dial; but quite independently of any other Time-Piece, by a true Meridian Line. This is upon Suppoſition the Dial is a very good one, and intended to go very correctly.

3701. As a Meridian Line is of the moſt extenſive Uſe, we ſhall here give ſome of the principal Methods for drawing one. The moſt ſimple and practical of theſe is the following. Upon a Point A (properly choſen) as a Center, deſcribe ſeveral concentric Circles as N H E, O I D, P K C; and in the ſaid Center A fix a ſtrong Wire, as truly upright or perpendicular to the Horizon as poſſible, the Height of it ſix or eight Inches; then

then observing, in the Forenoon, very nicely where the End of the *Shadow* touches each Circle, and there make fine Marks, as suppose in the Points C, D, E; and do the same for the Afternoon Shadows, as at H, I, K. Then bisect the Arch E H in F, and the Arch C K in G; and through the Points A, F, G, draw the Right Line A F G, and it shall be the *Meridian-Line* required.

3702. It has been formerly shewn (1766,) that the Curve L R M described by the End of the Shadow is an *Hyperbola* (in any Latitude less than $66\frac{1}{2}$ Degrees,) whose Vertex is R and its Axis the *Meridian Line* P R G. Further, it is easy to conceive, that the Sun in the Horizon, projects the Shadow to an infinite Distance, and there the Curve of the Hyberbola coincides with the *Asymtote*; and that this is at a Point in the Horizon just opposite to the Sun; consequently the infinite Line joining these two opposite Points will be an *Asymptote* to the hyperbolic Curve of the Shadow described on the given Horizontal Plane, and will contain with the Meridian of the Place an Angle equal to the *Co-Amplitude* for that Day.

3703. Therefore from the given LATITUDE of the Place, and the DECLINATION and AMPLITUDE of the Sun, the *Hyperbola* for the given Day may be described, and the *Meridian Line* thereby found by a single Observation of the Shadow; and verified by any Number of them you please. In order to this Practice, it is to be considered, that on any Day in the *Summer* or *Winter* when the Sun's Declination is the same, there will be the same or equal Hyperbolas described by the Shadow, whose Vertices will be determined by the Shadow of the Meridian Altitude on each Day.

3704. Thus, for Example: Let A B be a Gnomon of a given Height, and let A C B be the Meridian Altitude of the Sun, equal to the Sum of the *Co-Latitude* and *Declination*, whence the Distance A C is known. Again, the Angle A D B, on the Winter Day, is the Meridian Altitude, equal to the Difference of the Co-Latitude and Declination; and therefore A D is known; from which take A C, and the Remainder C D is the *Transverse Axis* of the Hyperbola's.

3705 Bisect C D in E, and that will be the Center of the Hyperbola's; thro' which draw two Right Lines G P and L M

making

making with the Line A D Angles G E A and M E D equal to the *Co-Amplitude* and they will be the *Asymptotes* to the *Hyperbolas*.

3706. On the Vertical Point C erect the Perpendicular C K meeting the Asymptote E G in K, and C K will be the Semi-Conjugate of these Hyperbolas; therefore make E F and E Q equal to E K, and the Points F and Q will be the *Foci* of the *Hyperbolas*, (768.)

3707. Thus having the *Diameters* and *Foci*, the Curves H C I and N D O may be readily described, either by the Instrument in the Plate of the Fusee, or by finding a Number of Points S in the Curve which is very easy to do, because it will be every where Q S — F S = C D (769;) and so for every Point S you have Q S = C D + F S, or F S = Q S — C D, by which any Number of those Points are found, and the Curve of the Hyperbola drawn through them as required.

3708. Having thus constructed the Hyperbola H C I for the given Day, let the Paper on which it is drawn, be so placed on the Plane, that the Point A may exactly coincide with the small Hole in which the Wire or Pin was fixed, and then so moved or adjusted, that the Length of any *observed Shadow* A R taken in a Pair of fine Compasses may be applied therein, that is, when one Point of the Compasses is in A the other may fall precisely in the Curve of the Hyperbola at R; then will the Axis of the Hyperbolas F E Q be in the true Meridian of the Place. And by observing a Number of these Shadows, its Position may be verified to great Exactness.

3709. The Time most proper for this Process is at or near the *Solstices*, and the Meridian determined at or about the *Summer Solstice*, will be most exact, as the Shadows A R are then shortest and best defined at the Ends. The Reason of chusing these Times of the Year, is because the *Hyperbolas* have now the greatest Degree of Curvature, from which they degenerate gradually till, in the Equinoctial Day, they become a *Right Line*, (1764.)

3710. But when the *Meridian Line* is to be determined of a considerable Length, other Methods may be more readily applied; that by an *Equal-Altitude-Telescope* is certainly a very good one, but the Expence of such an Instrument, the Skill of the

Person, and Difficulty of the Performance, will confine this Method to very few Hands.

3711. The Method of *Tycho Brahe* was much esteemed by himself and others in his Time; it consisted in having an Instrument so placed on a Plane that he could easily thereby observe the *greatest Elongation* of any *circumpolar Star* towards the East and West; consequently the End of this Instrument would describe an Arch, the Half whereof would measure these equal Elongations from the *Meridian*, which therefore became known or determined; *viz.* by bisecting that Arch, and drawing a Line through that Bisection and the Center on which the Instrument moved. But this Method is more suited to *Astronomers* than for common Use.

3712. Dr. DERHAM's *Meridian Instrument* is well known as a very useful One for drawing a Meridian Line with tolerable Exactness. But I no not think any Method so good, (that is, so exact,) as those which have their Points and Centers *upon* (and not *above*) the Plane on which the said Line is to be drawn, and which also are entirely independent of *Shadows*, because in great Lengths Shadows always become *penumbral*, or ill defined, and therefore not fit for an Affair of such Precision.

3713. I shall therefore propose a new Method of constructing an *equal Altitude Instrument* that moves immediately on the Plane, or Floor, or Pavement where the Meridian Line is required to be drawn, and which will give as little Trouble, and as great Accuracy, as can be expected in such an Operation. This Instrument consists of a Ruler A B C made of Mahogony, or other Wood, not subject to warp; At A is a small Hole in a Piece of Brass in which the Ruler is moveable about a slender steel Pin. The central Line A C is supposed to be drawn with great Care and very strait; the Part B C is a fiducial Edge of Ivory, exactly coinciding with the said Line A C; and at D is a small black Line, which, upon the Motion of the Ruler, will describe any Arch of a Circle, as D E F.

3714. Upon this Piece of Wood (or Metal) there is placed an Instrument for taking equal Altitudes of the Sun, constructed in the following Manner. G H is the Basis of it, and from the Center N move the two *Indices*, N P and N L, upon two Arches of a Circle, *viz.* I K fixed to the immoveable Base

Base GH; and the Arch MON, fixed to the Alidade or Index LN; and consequently moveable with it. Upon the Index NP, is a Sight-Vane at R, with *Cross-Hairs* in its Perforation, and a Shadow-Vane NQ is fixed at the Center N, exactly parallel to the former.

3715. The peculiar Artifice of this Construction is, that in a *small Form*, it performs the Office of a *large Quadrant*. For the Arch MO is divided into every 10 Degrees only, and therefore is not required to be large; the Index PN is placed to that Division which is next less than the Sun's Altitude; and then it is evident, the Remainder in Degrees and Minutes is measured by the Index LN, and its *Vernier Plate* at L, upon the Arch IK, which is divided into 15 or 20 Degrees at most, and therefore but of a short Length.

3716. This Instrument properly fixed upon the large Alidade AC (as in Fig. 3.) it is plain, that if in the Forenoon it has that Situation AC when the Indices are so adjusted that the Shadows of the Cross-Hairs, in the Vane R, fall precisely on the black Lines crossing in the Center of the Vane QN, and a fine Point or Mark be made at D; and then, in the Afternoon, the said Alidade AC be brought to such a Situation, that the central Line AC be that of AF, when the Shadows of the Cross-Hairs exactly coincide again with the Lines on the Vane Q, I say then, it follows, that if the Arch DE be bisected in E, and the Line AE be drawn, it shall be the *Meridian Line* required.

3717. This is all so plain as to want no farther Explication; and I find it so easy in Practice to take any Altitudes, and to measure any Angles in general with this compendious Instrument, that I can venture to recommend it to the ingenious practical Geometrician, as worthy of his Notice. It may be furnished with a Spirit Level to place it horizontally, and if a Lens of a proper focal Distance be placed in the Vane R, it will not only shew the Height of the Sun at any Time to a Minute, but also that of any other Object by its Picture, on the Vane NQ, properly darkened. As HADLEY's Quadrants are capable of the same Improvement and concise Form, I shall give a further Account of them at another Time; only observing, they may be applied in the above Method also for drawing a *Meridian Line*.

CHAP. XXIX.

The THEORY *and* CONSTRUCTION *of an* ELLIPTICAL, CIRCULAR, *and* DIAMETRAL DIAL, *which by Means of a common* HORIZONTAL DIAL, *will find the true* MERIDIAN.

3718. WE have ſhewn the beſt Methods of drawing a *Meridian Line*, by which a Dial may be *ſet truly* upon a horizontal Plane. We ſhall next ſhew how Dials may be conſtructed in ſuch a Manner as to *ſet themſelves*, or to *find their own Meridian* by Means of a *common Horizontal Dial* only. One of this Sort is called an ELLIPTIC DIAL, the other a CIRCULAR DIAL, becauſe in the firſt the Hours are the *unequal* Diviſions of an *Ellipſis*, and in the latter they are the equal Diviſions of a Circle; in both which Reſpects they differ from common horizontal Dials, where the Hours are the unequal Diviſion of a Circle, as we have ſhewn (1772).

3719. The Theory of the *Elliptic* DIAL we ſhall here deliver with all the Perſpicuity we can from a New, and (we preſume) moſt natural Projection of the Sphere *orthographically*, in what is uſually called the *Analemma*, explained form 1703 to 1720. Therefore (in Fig. 1.*) let HZOY be the *general Meridian*; EQ, the *Equator*; PX, the *Hour Circle* of *Six*; ZY, the *prime Vertical*; and HO, the *Horizon*. AF, LK, two *Parallels of Declination*.

3720. Then in our INSTITUTES of *Gnomonical Perſpective*, it has been ſhewn (1738, &c.) that the Sun having any Declination EG, will, by its Ray GC, deſcribe that Day a Cone ACF, whoſe Baſe is the Parallel AF upon the Surface of the Sphere, whoſe *Axis* is that of the Sphere PC, its *Vertex* C, and *Height* CB. This Cone we ſhall (for Diſtinction's Sake) call the *Radial Cone*.

3721. Then if the Axis ZCY of the prime Vertical be conſidered as an opake Line, it is plain, that Point of it at C, will intercept the Ray GC, and thereby produce a *lineal Shadow* CK in the ſame Direction, which Shadow will therefore deſcribe

* In the Plate entituled the Theory of *Elliptical, Circular*, and *Diametral Dials*.

ſcribe an equal and oppoſite Cone L C K, whoſe Baſe in the Surface of the Sphere will be the oppoſite equal Parallel L K. This therefore we ſhall call the *umbral Cone*.

3722. Then as the 24 Hour-circles divide the Equator E Q, and all its Parallels A F, L K, into ſo many equal Parts, it is evident the Ray G C in applying to thoſe Circles ſeverally, will aſſign the Poſitions of 24 Lines dividing the Surface of the Radial Cone into 24 equal Parts. And alſo, of Neceſſity, the Shadow C K will at the ſame Time give the Poſition of 24 correſponding Lines on the Surface of the umbral Cone; which therefore may be conſidered as *Hour-Lines* on the Surfaces of theſe Cones, interſecting each other in the common Vertex C.

3723. As the Sun's Declination E A decreaſes, the circular Baſes A F and L K of theſe Cones will encreaſe, till at Length the Sun being the Equator at E, both thoſe Baſes coincide, and the Cones degenerate into the Plane of the Equator E Q; where thoſe 24 Hour-Lines alſo ſeverally unite and form the *polar Dial*, deſcribed (1759) whoſe Gnomon is P C the Axis of the Sphere.

3724. It is farther evident, that, in the Caſe of the Sun's being in the Equator, the Hours will be indicated by *two Shadows united into one*, projected from the two opake Gnomons P C and Z C, on the Plane and graduated Perimeter of the Equator. For whatever Hour-Circle the Sun may be upon, the Shadow of the Stile C P (as being its Axis) will be in the Plane of that Circle (1751). For the ſame Reaſon, the vertical Circle paſſing through the Sun at the ſame Time, will have the Shadow of its Axis Z C projected in its own Plane; therefore, becauſe the Planes of both of thoſe Circles interſect each other, and alſo the Plane of the Equator, in one common Line; the Shadows of both thoſe Gnomons will, of Courſe, be projected into that Line, which will be the Shadow pointing out the preſent Moment on the Dial. Indeed, properly ſpeaking, the Shadow in the Caſe now ſpecified, is only that of the Point C in the Interſection of the two Gnomons.

3725. Again, ſuppoſe the Sun has any Degree of Declination E A, the Parallel A F will be its Path for that Day; now ſuppoſe it in any Hour-Circle, as that of Six P C X, in the Point B; and let Z B Y be the Vertical paſſing through the Center of the

the Sun: Then it is eaſy to underſtand, that the Ray proceeding from the Sun's Center at B, will deſcribe the Line B C on the Side of the radial Cone A C F, becauſe that is the common Interſection of the two Planes on the Side of that Cone. Here the ſaid Ray is obſtructed by the opake Particle C in the Interſection of the two Gnomons or Wires, Z Y and P X, and by that Means the Shadow of that Point will be a Line on the Umbral Cone L C K, exactly in the ſame Direction C R, and therefore on the contrary Side.

3726. Hence then, if the Baſe L R K of the Umbral Cone were a Circle divided into 24 Hours, and the Cone itſelf were away; then it is plain, the Point R will be the Hour of Six on the Weſtern Side of that Circle, if B be the Hour of Six in the Morning, upon that Point will the Shadow of the Interſection C preciſely fall. If we ſuppoſe not only the Perimeter, but alſo the Plane of the Baſe (or Parallel) L K to be there placed, then would the two Axes P X and Z Y paſs through it in the Points R and V, of which the former is the Center. And on that Plane would be projected two Shadows, one of the Part R C, and the other of the Part C V, of the ſaid two Axes.

3727. And in this Manner the Plane of any particular Parallel L K may be conſidered as a Dial-plane, having *two Stiles or Gnomons* R C, V C, whoſe two Shadows conſtantly interſect each other on the Hour-circle in its graduated Circumference, and thereby ſhew the Hour for that particular Day.

3728. But as the Periphery of a Parallel is a variable Quantity, increaſing or decreaſing daily, it can by no Means anſwer the Intention of an *Hour-circle*, which ought to be a fixed or determinate Thing; and therefore if inſtead of the *Parallel of Declination*, we ſubſtitute a *moveable Equator*, it will anſwer the Purpoſe of an univerſal Dial for finding the Meridian in any Latitude, and conſequently, independent of the Magnetic Needle, will ſet itſelf, or ſhew the true Hour of the Day; as will thus eaſily appear.

3729. Let the given Parallel of Declination be A F, and draw I G and M N, Tangents to the Equator E Q; and then ſuppoſe a moveable Circle, juſt equal to the Equator, and divided into 24 Hours, Minutes, &c. were to ſlide up and down upon theſe Tangent Lines of the Sphere, ſo as readily to be placed in a

Situ-

Situation similar to that of the Parallel L K in the Umbral Cone, correſponding to the Parallel A F of the Radial Cone. Such a *moveable Equator* will be repreſented by the dotted Line or Circle I W N; for the Lines C L, C A, C K, C F, continued out, will cut the Tangents in I, G, and N, M; therefore E I = C W = Q N, conſequently the Plane of the moveable Equator I N, is parallel to the Plane of the Parallel L K, and the Peripheries of each are parallel Circles of the ſame Cone I C N; therefore they are divided ſimilarly by the Shadows of the two Gnomons W C, and U C, or it will be every where R V : V K : : W U : U N. And thus the Hour will be ſhewn truly, and alike in each.

3730. If therefore a Dial be conſtructed according to this *Analemma*, it will be the Plane of this moveable Equator; and ſhew both the *Hour*, and the *Meridian-Line* at the ſame Time: For the equational Hour-circle being rectified to the Tangent of the Sun's Declination (for any given Day) on the Lines G I, M N, the Hour ſhewn by the Gnomon W C (as being Part of the Axis of the Sphere) can never be true, but when H Z O is in the Plane of the Meridian; but the Hour ſhewn by the Gnomon U C, muſt alſo in that Caſe be true; therefore, when the Interſection of the Shadows of theſe two Gnomons fall preciſely on the Hour-circle, then both the *Hour and Poſition of the Meridian-Line will be given by the Dial.*

3731. The ſame Thing will be effected by a fixed Equator E Q, and a moveable Gnomon Z C; for while the Equator is carried from E Q to I N, the Index or Axis Z I, is moved through the Plane of the Equator from W to U; and therefore, if we take C S = W U, and from the Point S draw S *a* parallel to C Z, and alſo drawing G *b* M, we have the two Gnomons C *b*, and S *b* exactly equal to the former two, W C and U C, and alike ſituated to the Plane of the fixed Equator E Q, as they were to the moveable One I N. And therefore the Hour will be ſhewn in this Caſe, the ſame as in the other.

3732. Laſtly, the ſame Purpoſe will be anſwered by a Gnomon D *b*, moveable upon a horizontal Line H O, through the Space C D, ſince the Point (*b*) in the Interſection, gives the Hour, and is the ſame in both Caſes.

3733.

3733. If therefore upon the Plane of the Horizon HO in any Latitude EZ = PO, the Equator EQ be projected into an *Ellipsis*, and it be properly divided in every 15th Degree by a Table of Sines as directed (1718) these Divisions will be the Hours, and then an upright Stile D *a*, moveable in the Hour-Line of XII, or Meridian HO, will constantly shew the true Hour on this *Elliptic Equator*, as before it did on the *Circular One*. Because in the Projection there is a perfect Similarity of all the Parts in One, to all the corresponding Parts of the other, as we have shewed (1715).

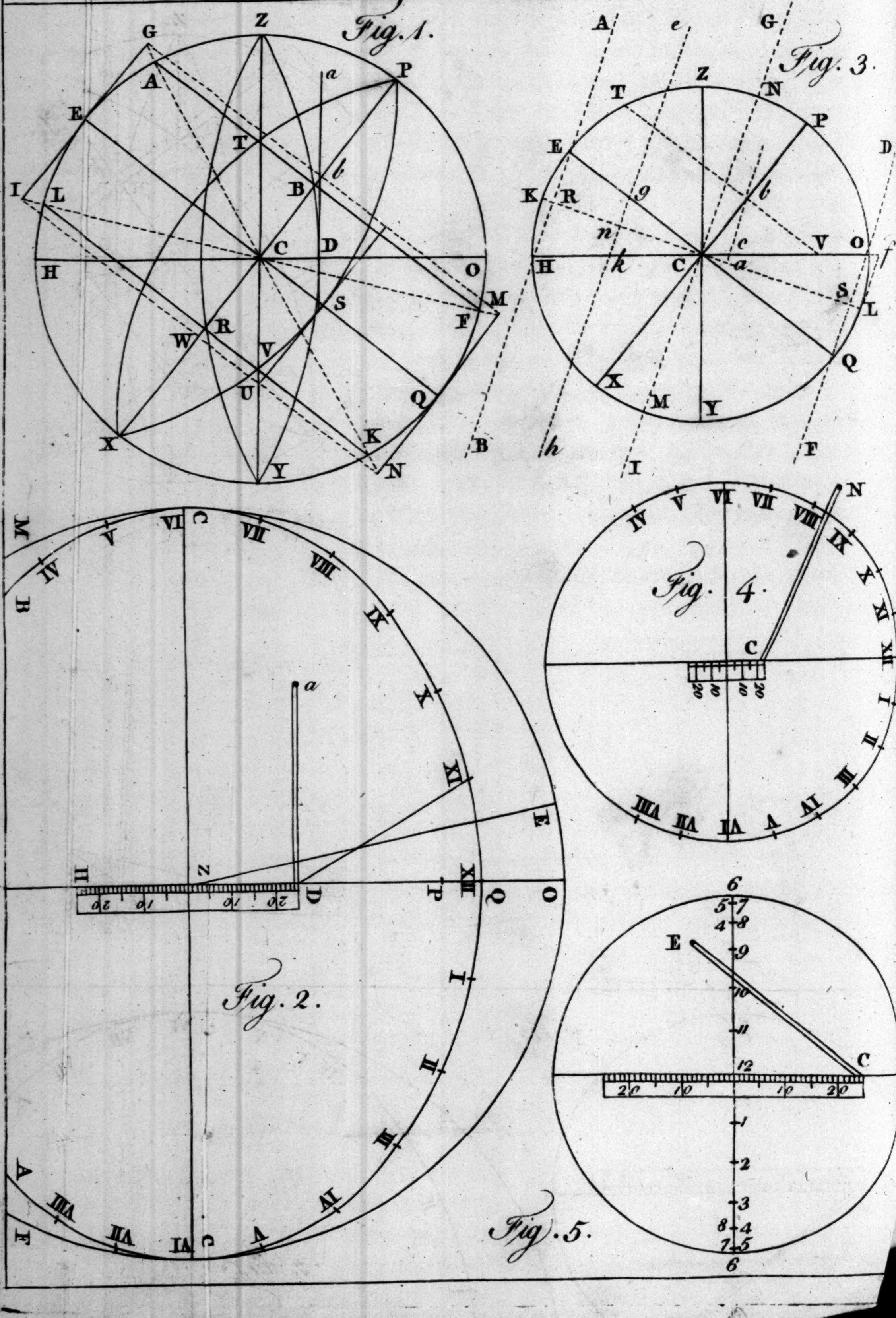
3734. In this Case, the Shadows of the two Gnomons D *b*, and C *b*, require two different Hour-circles, and so will be no longer *one Dial*, but *two of a distinct Nature*. Thus let Fig. 2. be the orthog. Projection on the Plane of the Horizon FOM; and let AQB be the Ellipsis into which the Equator is projected, and divided into its proper Hours; also let P be the projected North-pole, Z the Zenith, and D the Foot of the perpendicular Stile S *b*, for any Declination of the Sun EG in Fig. 1. Then suppose the Shadow of that Stile falls upon the Hour XI, it will be projected into the Line DXI, and this must be the Hour-Line of XI on the horizontal Plane, as is evident from the Nature of the Projection; for the Space ZD here, is equal to CD there; and D is the Projection of the Foot S of the said Stile in the Plane of the Equator.

3735. But with Respect to the Stile C *b* (in Fig. 1.) its Shadow here cannot pass through the Point (or Hour of) XI in the Ellipsis AQB; for the Foot C being in the Center Z of the horizontal Plane, and the Hour-circle of XI cutting the Horizon in E, making EO = 11° : 51′ (1772) it is plain, ZE will be the Hour-Line of XI on the horizontal Plane, or the Shadow of the Stile C *b*. And drawing the Line Z XI, there is formed the spherical right-angled Triangle XI ZQ, wherein ZQ = the Latitude, and XI Q = the Hour, being given, the Angle at Z will be found = 18° : 55′, which is 7° : 4′ greater than the Angle EZO. Consequently the two Gnomons will, in this Projection, require two different Curves for the Hours, *viz.* the *Ellipsis* AQB, and the *Circle* FOM, which therefore constitute two different Dials.

3736.

Plate 20

The THEORY of Elliptical, Circular, and Diametral DIALS.



3736. When E G (= E I) is the Tangent of the Sun's greater Declination E A, then A F and L K are the two *Tropics*; and C D = Z D (Fig. 2.) is the greateſt Space through which the perpendicular Index D *a* is to be moved in the horizontal or meridian Line H O, which is to be graduated into 23° : 30′, to correſpond to the Sun's Declination in this Manner. In the Triangle *b* C D, we have C *b* = Tangent of 23° : 30′, and the Angle at *b* = Z C P = Co-Latitude. And as Radius : *s*. C *b* D : : C *b* : C D; therefore ſince C *b* is the Tangent of 23° : 30′ to the Radius C P, C D will be the Tangent of the ſame Arch to the Sine of Z P made Radius, or applied in the Sector from 45 to 45, upon the Lines of Tangents. Thus Z D and Z H is graduated for the Summer and Winter Half-Year, by which, to ſet the Index or Gnomon for the Declination of the Sun, proper to the given Day.

3737. Therefore it is evident, that when the ſame Minute is ſhewn by the Elliptic and Horizontal Dial, the Hour Line of XII, *viz.* H O, will be in the Plane of the Meridian of the Place, and conſequently a Right Line drawn parallel to it will be a true *Meridian-Line.* And thus ſuch a double Dial will at all Times *ſet itſelf* without any Aſſiſtance of the *Magnetical Needle*, or any other *Meridian Line* than its *own.*

CHAP. XXX.

The THEORY *and* CONSTRUCTION *of* CIRCULAR *and* DIAMETRAL DIALS, *which, with a Horizontal* DIAL, *find a true* MERIDIAN LINE.

3738. BEſides the foregoing *Elliptic Dial*, there is yet another Sort which by Means of a common *Horizontal Dial*, will *ſet itſelf*, or ſhew the Poſition of the *Meridian Line* without the help of the *Needle*; and by which, of Courſe, the *Variation*

of the Needle is alſo eaſily diſcovered in the given Place. But this Dial is of a *circular Form*, and what may ſeem a little ſtrange, is, that the Hours on this Circle are all *equidiſtant*, as in the Equator itſelf. The whole Artifice or Conſtruction of this Dial depends upon a due Conſideration of the Properties and different Sections of a *Scalenous Cylinder* which will be eaſy to underſtand after what we have premiſed of the *Scalenous Cone* (1508,) and its *Subcontrary Sections* (1510, &c.)

3739. Firſt then let it be conſidered that if Tangent-lines were drawn through the ſeveral Points of the Equator, (or any Circle) and at right Angles to its Plane, they would conſtitute a *right Cylinder*, whoſe Section by a Plane perpendicular to its Axis will be a *Circle*, but in all other Caſes an *Ellipſis*, as we have ſhewn (1715.)

3740. Secondly; if theſe *Tangent Lines* are now ſuppoſed gradually to *incline* or *lean* towards the Plane of the Equator, then they form what we call a *Scalenous Cylinder*, being compreſſed by this Inclination of the Lines into a flattiſh Form; and whoſe Section by a Plane perpendicular to its Axis will be an *Ellipſis*, one Diameter of the right Cylinder being contracted gradually into a leſs Length, while the other continues the ſame; alſo all other Sections will be *Ellipſes*, except two, *viz.* one *Parallel* to the Equator (or original Circle) and the other by a Plane in a *Subcontrary Poſition*.

3741. Thirdly; theſe *Tangent Lines* conſtantly inclining to the Plane of the Equator will at laſt perfectly coincide with it, or the *Scalene Cylinder* now degenerates into a Plane coincident with that of the Equator, which, in this Caſe, is projected by theſe Lines or Rays into a *ſtrait Line*, *viz.* one of its Diameters. And this Conſideration lays the Foundation of another Species of *Dialling*, *viz.* *Rectilineal*, which we ſhall enlarge upon by-and-by.

3742. But to come directly to the Point, and for Illuſtration of what has been ſaid, let A B F D be the *Scalenous Cylinder* whoſe Axis is I G, perpendicular to which let the Line R S be drawn; this will be the ſhorteſt Diameter of the *Elliptic Section* and ſuppoſe the longeſt Diameter be H O, which alſo is conceived to paſs through the common Interſection C of the Axis, and the other Diameter R S (See Fig. 3.) Upon the Point C, with

with the Radius C H describe the Circle H Z O Y, cutting the flatted Sides of the Cylinder in the Points E, H, and O, Q; and draw the Line E C Q.

3742. Then it is evident, that since H O is equal to the longest Diameter of the Elliptic Section by a Plane through R S, it is therefore equal to the Diameter of the original Circle or Base of the inclined Cylinder A B F D, and also parallel to the Diameter through whose Extremities the Lines A B, F D pass. Consequently, the Section of the Cylinder by a Plane (through H O) parallel to the Base will be also a *Circle*, equal to the *Circular Base*.

3743. And because the Angles R E C and S Q C are equal, as also R H C = S O C, (by reason of parallel Lines A B, D F intersecting the equal Lines H O and E Q) therefore the Section of the Cylinder through E Q will be *subcontrary* to that through H O, (1509,) and consequently also a *Circle*, agreeable to (1510.)

3744. Now if F P Q X be considered as a *Meridian* of the Sphere, which is *orthographically projected* thereon, and E Q the Equator; we may then look upon H O as the Horizon of some Place Z, whose Latitude is E Z, and whose Co-latitude Z P is bisected by the Axis of the Cylinder I G, because the Section through R S is supposed to be perpendicular thereto, and consequently equally divides the Angle E C H in the Point K, or O C Q in the Point L.

3745. Then (from what we demonstrated in the last Chapter) it is plain, that if K L be considered as a Plane having a Perpendicular Stile or Gnomon C N, the Projection of the Equator E Q thereon by Rays A B, F D, parallel to C N, will be an *Ellipses*, whose shortest Diameter is R S, and longest K L. And that as the Sun advances to the Parallel T V, so the perpendicular Stile being parallelly removed from C to (*a*) will shew the Hour on that *Ellipsis*; and if T V be considered as the Tropic of *Cancer*, then C *a* will be the Length of the *Summer* Half of the *Zodiac* on the Dial-plate.

3746. Suppose now the Plane K L were taken away, but the Gnomon N C to remain; then it is evident the Rays A B, G I, D F, will project the Equator E Q into a *Circle* on the horizontal Plane H O (1510.) And the said Rays will divide the Horizontal Circle in the same Manner or Ratio, as they do

the Equator itſelf. For let *(eh)* be any one of thoſe Rays, which paſſing through the Point *(g)* of the Equator, projects it into the Point *(k)* in the horizontal Circle HO; then becauſe *eh* is parallel to AB, we have CE : C*g* :: CH : C*k*. Wherefore the Horizontal Circle will be divided into 24 equal Parts by the Rays which paſs through every 15th Degree of the Equator.

3747. Therefore 'tis evident, that if upon the Horizon HO, a Circle be drawn, and divided into 24 equal Parts; thoſe Diviſions will be the Hours truly marked out by the Shadow of a Stile or Gnomon CN elevated above the Plane thereof in the Angle NCO = PO + PN = *the Sum of the Latitude, and half the Co-Latitude of the Place* Z. So that at *London*, for Example, the Angle NCO = 51° 30′ + 19° 15′ = 70° 45′.

3748. 'Tis farther evident, that this Stile CN is a *moveable One*, as it originally and properly belongs to the *Elliptic Dial* on the Plane KL (3745;) where its parallel Motion, Northward, is C*a*; but this is now to be eſtimated in the Plane of the Horizon from C towards O, where it will be expreſſed by the Line C*c* whoſe Value is thus found. In the Triangle *b*C*a* (right-angled at *a*) we have *Rad.* (CP) : Sine of C*ba* (= OCL) :: C*b* : C*a*. Again, in the ſimilar right angled Triangles *c*C*a*, OCS, and *f*CL, we have C*a* : C*c* :: CL (Rad.) : C*f*; therefore $Ca = \frac{OS \times Cb}{Rad.} = \frac{Cc \times Rad.}{Cf}$; therefore

$$Cb : Cc :: Rad.^2 : OS \times Cf :: Rad. : \frac{OS \times Cf}{Rad.} = \text{Tan-}$$

gent of LC*f*. Whence the Value of C*c* is known.

3749. But becauſe it will be moſt convenient to ſet the Index CN by the Sun's Declination, therefore the Semi-zodiac C*c* muſt be conſidered as a Tangent of 23° 30′ to ſome Radius, which from the Analogy above is evidently the Line L*f*, *viz. the natural Tangent of the Half Co-Latitude of the Place* z. Therefore by applying the Line L*f* from 45 to 45 in Lines of Tangents on the Sector for Radius; you may then take the parallel Diſtance from 23° 30′ to 23° 30′, and that will give the Length of C*c* from the Center C each Way in the Meridian Line for the Summer and Winter half Year. And thus your *Circular Dial* will be compleated, as in Fig. 4.

3750. There

3750. There are many other Particulars obſervable in this Sort of Dials, but we muſt paſs them by at preſent as not neceſſary to our Purpoſe, which is only to ſhew *how by two Dials of a different Conſtruction, and placed upon one Right Line, a true Meridian Line may be drawn, and the Hour ſhewn, without any other Aſſiſtance.*—This we have ſhewn may be done by a *common horizontal Dial* combined with the *Elliptic*, and *Circular* Ones; we ſhall next ſhew the Nature and Conſtruction of a *Rectilineal* or DIAMETRAL DIAL for the ſame Purpoſe.

3751. Suppoſe (Fig. 3.) that the *Tangent Rays* A B, G I, D F, were (in the firſt Place) all at right Angles to the Plane of the Equator E Q; an infinite Number of them would, in this Caſe, conſtitute a *perfect Cylinder* whoſe Section, perpendicular to the Axis would every where be a *Circle*, as being parallel to the circular Plane of the Equator.

3752. Let the Rays now gradually deviate from that rectangular Poſitition to one more and more inclined to the ſaid Plane of the Equator in a Direction from P towards E. Then it is evident, all that Time the *Radial Cylinder* will become more and more flatted or ſcalenous; and every perpendicular Section R S will become of a leſs and leſs Diameter; and, of Courſe, the *Hour-Circle* in the Equator degenerates into an *Ellipſis*, and becomes more and more ſo; 'till at length, the Rays all arrive at the Plane of the Equator, and thence the *ſcalenous Cylinder* itſelf becomes a *Plane*, and coincides with that of the Equator; alſo the horary Ellipſis now Collapſes, and both Sides unite in a Right Line which is the Diameter of a greater Circle P X paſſing through the Poles.

3753. During this ſuppoſed Motion of the Tangent Rays the Dial Plane K L has deſcribed the Quadrant E X, and its Gnomon or Index C N the Quadrant P E, and its Hour-Circle has paſſed thro' every Degree of Elliptic Curvature from a Circle to that of a Right Line; and the Poſition of its Gnomon, which at firſt was in the Axis of the World C P, is now C E in the Plane of the Equator.

3754. This Poſition of the Dial-Plane P X and its Gnomon C E fits it for *An univerſal Right-lined* DIAL *in every Latitude*; for from the Nature of the *Analemma*, it is evident, that the 24 Hour-Points of the Equator are now projected into the *El-*

liptic

liptic Line (as it may be called) or the East and West Diameter of the Plane D X. Which Line is equally a Diameter in the Plane of any great Circle, as that, for Instance, of the Horizon H O, because it is the *Common Section* of them all, and so is equally as much in one as in another.

3755. Therefore since the Rays which pass through the Beginning of each 15th Degree of the Equator, will project those Points into the said Common Section or Hour-line of the Dial, this Line will be thereby divided in the Manner of a *Line of Sines* each Way from the Center, as represented in Fig. 5.

3756. The Stile of this we have shewn (3753) must be precisely elevated to the Plane of the Equator, *viz.* the Angle E C H (Fig. 3.) for any proposed Horizon H O; and as this Dial is of the *Elliptic Kind*, its Stile or Gnomon C E is a *moveable One.* And therefore if T V be the Parallel of the Sun's greatest Declination, it will also represent the Position of the said Gnomon for the tropical Day, as being parallel to C E, and intersecting the Axis C P in the Point *(b)* making C *b* = Tangent of 23° 30′. Therefore, since in the Triangle C *b* V, it is C *b* : C V :: Radius : Secant of the Angle P C O, the Distance C V is known for any Latitude P O or E Z.

3757. And since C *b* is the Tangent of 23° 30′ to the Radius C P; therefore C V will be the Tangent of 23° 30′ to a *Radius equal to the Secant of the Latitude*, and hence C V the Semi-Zodiac for this Dial, is easily graduated for the duly adjusting the Index to the Sun's Declination, as directed (3736,) and as shewn in Fig. 5.

3758. But this *Rectilineal Dial* is not so conveniently combined with the *common horizontal* Dial for finding the true Hour or Meridian Line as the *Elliptical Dial* is; because in this the Hours at the extreme Parts of the Line run so near together, as not to admit of sufficient Accuracy in observing and comparing the Time in each. Nor is the *circular* Dial (Fig. 4.) so fit for this Purpose as the *Elliptic One*, because of the *small Length of the Zodiac* for setting the Stile C N (Fig. 3.) not allowing a sufficient Estimation of the Sun's Declination.

3759. 'Tis very observable that both the *circular* and *right-lined Dials* are *universal Ones*, or will serve for *all Latitudes*, because

cauſe the *horary* Diviſions of each always remain the ſame, and the Indices or Stiles only require to be properly rectified, *viz.* that of the *Circular Dial* as directed (3749,) and that of the *Rectilineal One* to the *Co-Latitude* HE, be it what it will.

3760. Laſtly, in the *Right-lined Dial* (Fig. 5.) the Reader will obſerve that the Shadow of the Stile muſt go *backward* and *forward twice a Day* in the Summer Half-year, *viz. Morning* and *Evening*, as is evident by Inſpection. What remains on this Subject of Dialling will be delivered in the Third Volume; wherein it is propoſed to give the Theory and Construction of all the uſeful Mathematical and Philoſophical Instruments, as alſo all the neceſſary Tables uſed in thoſe Sciences, with the *Rationale* and particular *Uſes* of each.

End *of the* Second Volume.